Appendix

Proof of Theorem 1

We first show that for any given \( \eta > 0 \), there exists a large constant \( C \) such that

\[
P \left( \inf_{||u||=C} L(\beta_0 + n^{-1/2}u) > L(\beta_0) \right) > 1 - \eta. \tag{A.1}
\]

By Taylor’s expansion and condition (C1), we have

\[
L(\beta_0 + n^{-1/2}u) - L(\beta_0) = \sum_{i=1}^{n} \rho_{\tau,\lambda} \left( \frac{Y_i - X_i^T \beta_0}{S_n} \right) u_i - \sum_{i=1}^{n} \rho_{\tau,\lambda} \left( \frac{Y_i - X_i^T \beta_0}{S_n} \right).
\]

Therefore, by choosing a sufficiently large \( C \), \( I_2 \) dominates \( I_1 \) in \( ||u|| = C \). Since \( E \left[ \psi'_{\tau,\lambda}(\epsilon/\sigma) \right] > 0 \), this completes the proof of Equation (A.1). Equation (A.1) implies with probability at least \( 1 - \eta \) that exists a local minimum of \( L(\beta) \) in the ball \( \{ \beta_0 + n^{-1/2}u : ||u|| \leq C \} \). The proof of Theorem 1 is completed.

Proof of Theorem 2

Let

\[
\phi_n(\theta) = \frac{1}{nS_n} \sum_{i=1}^{n} \psi_{\tau,\lambda} \left( \frac{Y_i - X_i^T \theta}{S_n} \right) X_i.
\]

By Taylor’s expansion, there exists a vector \( \hat{\beta}_n \) on the line segment between \( \beta_0 \) and \( \hat{\beta}_n \) such that

\[
\phi_n(\hat{\beta}_n) = \phi_n(\beta_0) + \hat{\phi}_n(\beta_0)(\hat{\beta}_n - \beta_0) + \frac{1}{2}(\hat{\beta}_n - \beta_0)^T \phi''_n(\beta_0)(\hat{\beta}_n - \beta_0),
\]

where \( \hat{\phi}_n(\cdot) \) and \( \phi''_n(\cdot) \) are the first-order derivative and the second-order derivatives of \( \phi_n(\cdot) \). From (2.2), we have \( \phi_n(\hat{\beta}_n) = 0 \). By condition (C1) and Theorem 1, we have

\[
\frac{S_n}{\sqrt{n}} \sum_{i=1}^{n} \psi_{\tau,\lambda} \left( \frac{Y_i - X_i^T \beta_0}{S_n} \right) X_i = \sqrt{n}(\hat{\beta}_n - \beta_0) \frac{1}{n} \sum_{i=1}^{n} \psi'_{\tau,\lambda} \left( \frac{Y_i - X_i^T \beta_0}{S_n} \right) X_i + o_p(1).
\]
Since $S_n \xrightarrow{D} \sigma$ as $n \to \infty$, thus,

$$\sqrt{n}(\hat{\beta}_n - \beta_0) \xrightarrow{D} N\left(0, \frac{E\psi_{\tau,\lambda}'(\epsilon/\sigma)}{(E\psi_{\tau,\lambda}'(\epsilon/\sigma))^2} \sigma^2(EX^TX)^{-1}\right).$$

**Proof of Theorem 3**

According to the definition of $c_{n-m}$, for all $||\beta|| = 1$, we can obtain

$$\#\{i : m + 1 \leq i \leq n, \text{ and } |X_i^\top \theta| > 0\}/(n - m) \geq 1 - c_{n-m}.$$

By the assumption of Theorem 3, there exists $c_{n1} > c_{n-m}$ and $c_{n2} > c_{n-m}$ such that

$$\epsilon < (1 - 2c_{n1})/(2 - 2c_{n1}), a(\tau, \lambda) < (1 - \epsilon)(2 - 2c_{n2}).$$

We take $c_n^* = \min\{c_{n1}, c_{n2}\}$, then $c_n^* > c_{n-m}$, and have

$$\epsilon < (1 - 2c_n^*)/(2 - 2c_n^*), a(\tau, \lambda) < (1 - \epsilon)(2 - 2c_n^*). \quad (A.2)$$

By using a compacity argument (Yohai, 1987), we can find $\delta > 0$ such that

$$\inf_{||\beta|| = 1} \#\{i : m + 1 \leq i \leq n, \text{ and } |X_i^\top \beta| > \delta\}/(n - m) \geq 1 - c_n^*.$$

According to Equation (A.2), we can obtain $1 - \epsilon > 1/(2 - 2c_n^*)$. Therefore, we can find $\zeta$ such that $(1 - \epsilon)(1 - c_n^*) > 1 - \zeta > 1/2$. Take $\Delta > 0$ which satisfies

$$1 < 1 + \Delta < \min\left\{\frac{(1 - \epsilon)(1 - c_n^*)}{1 - \zeta}, \frac{(1 - \epsilon)(2 - 2c_n^*)}{a(\tau, \lambda)}\right\}.$$

Denote

$$a_0 = \frac{(1 - \zeta)a(\tau, \lambda)(1 + \Delta)}{(1 - \epsilon)(2 - 2c_n^*)}.$$

Then, we have $a_0 < \min\{1 - \zeta, a(\tau, \lambda)/2\}$. Therefore, $m/n \leq \epsilon$ implies

$$a_0(n - m)/n \geq (1 - \epsilon)a_0 > (1 - \zeta)a(\tau, \lambda)/(2 - 2c_n^*).$$

Since $\rho_{\tau,\lambda}(t)$ is a bounded, continuous, and even function, there exists $k_2 \geq 0$ such that $\rho_{\tau,\lambda}(k_2) = a_0/(1 - \zeta)$. Let $C = (k_2S_n + \max_{m+1 \leq i \leq n}|Y_i|)/\delta$. Hence, $m/n \leq \epsilon$ implies

$$\inf_{||\theta|| \geq C} \sum_{i=1}^n \rho_{\tau,\lambda}(r_i(\beta)) \geq \inf_{||\theta|| = 1} \sum_{i \in A} \rho_{\tau,\lambda}\left(\frac{|Y_i| - C|X_i^\top \theta|}{S_n}\right) \geq (n - m)(1 - c_n^*)\rho_{\tau,\lambda}(k_2)$$

$$= (n - m)(1 - c_n^*)a_0/(1 - \zeta) > na(\tau, \lambda)/2 \geq \sum_{i=1}^n \rho_{\tau,\lambda}(r_i(\hat{\beta}_n)),$$

where $A = \{i : m + 1 \leq i \leq n \text{ and } |X_i^\top \theta| > \delta\}$.

For a contaminated sample $D_n$ with $m/n \leq \epsilon$, if $||\hat{\beta}_n|| \geq C$, we have

$$\sum_{i=1}^n \rho_{\tau,\lambda}(r_i(\hat{\beta}_n)) > \sum_{i=1}^n \rho_{\tau,\lambda}(r_i(\hat{\beta}_n)).$$
This is a contradiction with the fact that $\hat{\beta}_n$ minimizes $\sum_{i=1}^{n} \rho_{\tau,\lambda}(r_i(\beta))$ for $\beta \in \mathbb{R}^p$. Note that $m/n \leq \epsilon$, $\epsilon < (1 - 2c_{n-m})/(2 - 2c_{n-m})$, and $a(\tau, \lambda) < (1 - \epsilon)/(2 - 2c_{n-m})$. Therefore, we obtain

$$BP(\hat{\beta}_n; D_{n-m}, \tau, \lambda) \geq \min \left\{ BP(\bar{\beta}_n; D_{n-m}), \frac{1 - 2c_{n-m}}{2 - 2c_{n-m}}, 1 - \frac{a(\tau, \lambda)}{2 - 2c_{n-m}} \right\}.$$  

Since the sample $D_n$ is in general position, we have $c_{n-m} = (p - 1)/(n - m)$. Because $\bar{\beta}_n$ is a robust estimator with asymptotic breakdown point $1/2$, therefore,

$$BP(\hat{\beta}_n; D_{n-m}, \tau, \lambda) \geq \min \left\{ 1 - \frac{\kappa}{2}, 1 - \frac{\kappa}{2}a(\tau, \lambda) \right\}.$$