What do Error Patterns tell us about Hong Kong Chinese and Australian Students' Understanding of Decimal Numbers?

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Abstract

Mathematics educators have had a long standing interest in students' understanding of decimal numbers. Most studies of students' understanding of decimals have been conducted within Western cultural settings. Similar research in other countries is important for a number of reasons, perhaps most importantly because it can provide insights that may benefit the mathematical learning of all students. The present study sought to gain insight into Chinese Hong Kong students' and regional Australian students' general performance on a variety of decimals tasks. These tasks included: comprehending place value after the decimal point; the proper use of algorithms for computation of decimals; and the application of decimals to "real life" problems. More specifically, it aimed to investigate students' error patterns. The study also aimed to investigate Chinese Hong Kong students' and Australian students' procedural and conceptual understanding of decimals. The results indicated that for many of the questions, Hong Kong and Australian students shared similar misconceptions. For others, Hong Kong students exhibited stronger procedural and conceptual understanding of the task.

Introduction

Mathematics educators have had a long standing interest in students' understanding of decimal numbers. This is an important area of research because of our base-ten number system and the emphasis on the metric measurement system, and the growing use of calculators in the classroom (Thipkong & Davis, 1991; Ubuz & Yayan, 2010). Misunderstanding of decimal numbers and misconceptions about the meaning of decimal number notation have been documented in many parts of the world (see for example: Graeber & Tirosh, 1990; Moloney & Stacey, 1997; Muir & Livy, 2012; Okazaki & Koyama, 2005;

Resnick, Nesher, Leonard, Manogne, Omanson, & Peled, 1989; Steinle & Stacey, 2003). Most studies of elementary school students have focused on one of two types of decimal number tasks: comparison of decimal numbers (comprehending place value after the decimal), or decimal computation (addition, subtraction, multiplication and division of decimals). Relatively few studies have investigated students′ understanding of these related concepts in a single study. Arguably, there is a need to understand students′ conceptions of decimal numbers as a holistic set of knowledge, ranging from comprehending place value after the decimal to the proper use of algorithms for computation of decimals. Such research enables comprehension of how students operate with decimals dynamically in a manner that demonstrates that they understand the conceptual meaning of decimals (Irwin & Britt, 2004).

The identification and analysis of students' errors is a powerful mechanism for understanding students' mathematical thinking (An & Wu, 2012). Radatz (1980) points out that students' errors are not always due to carelessness but may be the result of failure to master concepts. Radatz argues further that students' errors are very often systematic and can be analyzed and described as "error techniques" Building on this idea, studies by Borasi (1987) and Lannin, Barker and Townsend (2007) show that students' error patterns can be a powerful tool to diagnose learning difficulties and misconceptions.

In general, many misconceptions that students hold are the result of inappropriate generalization from whole numbers and from concepts that are appropriate for fractions but not for decimals (Irwin & Britt, 2004). Previous studies consistently identify three erroneous "rules" that many children use in comparing decimal numbers (Desment, Gregoire & Mussolin, 2010; Nesher & Peled, 1986; Peled & Shahbari, 2003; Pierce, Steinle, Stacey & Widjaja, 2008; Sackur-Grisvard & Leonard, 1985; Stacey & Steinle, 1999; Steinle & Stacey, 2010). In summary, Resnick et al. (1989) named these rules the whole-number rule, fraction rule and zero rule. The whole-number (or "longer-is-larger") rule is the selection of the number with more decimal places as the larger of two decimals; for example, 4.38 would be considered larger than 4.6 because 4.38 has two decimal places whilst 4.6 has only one. This error is thought to stem from an overgeneralization of an impoverished method for comparing whole numbers (Baturo & Cooper, 1995; Resnick et al, 1989; Stacey, 2005; Steinle & Stacey, 1998; Steinle, 2004a; Steinle, 2004b). It reflects the fact that students transfer, unaltered, all the relationships within the whole number system to the decimal number system (Moskal & Manogne, 2000). The fraction (or "shorter-is larger") rule involves the selection of the number with fewer decimal places as the larger of two decimals; for example, 2.4 would be considered larger than 2.64. This error may stem from an overgeneralization of the principle for comparing common fractions; that is, the larger the denominator, the smaller the fraction (Baturo & Cooper, 1995; Desment, Gregoire & Mussolin, 2010; Resnick et al, 1989; Stacey, Helme & Steinle, 2001; Steinle, 2004a; Steinle, 2004b; Steinle & Stacey, 1998). Hiebert and Wearne (1983) point out that students who do not recognize the differences between fractions and decimals may treat the decimal value as if it were a fraction, with a decimal point replacing the fraction bar. The zero rule is employed when students select the decimal with zero(s) to the immediate right of the decimal point as the smaller decimal; for example, 4.08 is assessed as smaller than 4.8 because there is one zero to the immediate right of the decimal point in 4.08 whilst there is no zero in 4.8. Baturo and Cooper (1995) and Steinle and Stacey (2001) report that the zero rule always produces a correct result but for an inappropriate reason.

Decimal computation tasks (addition, subtraction, multiplication and division) also reveal misunderstandings amongst elementary and secondary school students (see for example, Baturo, 1997; Bonotto, 2005). Some scholars have concluded that elementary school students extend their understanding of multiplication and division of whole number to decimals, and therefore hold misconceptions such as “multiplication always makes bigger”, “division always makes smaller”, “the dividend must be larger than the divisor”, “the divisor must be a whole number” and “the quotient must be smaller than the dividend” (Bell, Fischbein & Greer, 1984; Okazaki & Koyama, 2005). Graeber, Tirosh and Glover (1989), and Graeber and Tirosh (1990) identify similar misconceptions in their study and report that, because of their reliance on such misconceptions, students are unable to choose a correct operation for a word problem.

Some researchers have concluded that many of the misconceptions held by students arise because of students’ reliance on memorizing the procedures with little understanding of the associated concepts that underlie them (Hiebert, 1992). Procedural understanding involves knowledge of the rules and procedures (Hiebert, & Wearne, 1986; Skemp, 1976) which describes a process, procedure, or the steps taken to complete a mathematics task (Fuchs, Fuchs, Hamlett, Phillips, Karns & Dutka, 1997). Wearne and Hiebert (1988) describe procedural understanding as syntactic processes which involve symbol-manipulation and routinizing the rules for symbols. Conceptual understanding involves knowing the relationship between related concepts (Wearne & Hiebert, 1988), an understanding of why a procedure works (Hiebert & Wearne, 1986) and whether a procedure is legitimate (Bisanz & LeFevre, 1992). Hiebert (1992) concludes that conceptual knowledge is knowledge that is rich in relationships but not rich in techniques for completing tasks, while procedural knowledge is rich in rules and strategies but not rich in relationships.

In Western mathematics education, there has been tension between procedural knowledge and conceptual knowledge (Lai & Murray, 2012) and animated discussion of their respective roles in student's learning of mathematics (Star, 2005). Western educators often emphasize the need for students to construct a conceptual understanding of mathematical symbols and rules before they practise the rules (Li, 2006). On the other hand, Chinese learners tend to be oriented towards rote learning and memorization (Marton, Watkins & Tang, 1997). Chinese learners have been criticized for relying solely on procedural knowledge but arguably this criticism overlooks the relationship between procedural and conceptual understanding in Chinese teaching and learning (Lai & Murray, 2012). In Chinese scholarship, these types of knowledge are viewed as intertwined components, such that developing one's procedural knowledge in a domain is crucial for improving one's conceptual knowledge in that domain and vice versa (Rittle-Johnson, Siegler & Alibali, 2001).

Most studies of students′ understanding of decimals have been conducted within Western cultural settings. Studies of students′ understanding of decimal numbers in other countries are important for a number of reasons, perhaps most importantly because they can provide insights that may benefit the mathematical learning of all students. The present research was conducted in Hong Kong and regional Australia. Chinese Hong Kong students consistently outperform their Western counterparts in many international comparative studies on mathematics achievement such as TIMSS (Beaton, Mullis, Martin, Gonzalez, Kelly & Smith, 1997; Mullis, Martin & Foy, 2008). These tests include tasks specifically concerned with decimal numbers (see for example, Mullis, Martin & Foy, 2008, p.119 and p.123). A comparative study of Chinese Hong Kong students and Australian students′ performance on a range of decimal number tasks can cast light on the conceptions and misconceptions of students from different cultural backgrounds.

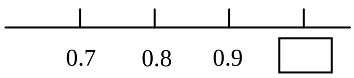
An overarching rationale for the current study was to contribute to an increased understanding of students' conceptions and misconceptions about decimals. The present study had several aims. Broadly, it sought to gain insight into Chinese Hong Kong students′ and Australian students′ general performance on a variety of decimals tasks. These tasks including comprehending place value after the decimal point; the proper use of algorithms for computation of decimals; and the application of decimals to "real life" problems. More specifically, it aimed to investigate students' error patterns. Analyzing students' errors can reveal faulty arithmetic and problem-solving processes (Radatz, 1980); and perhaps highlight deficits in classroom instruction (Borasi, 1987). In addition, the study aimed to investigate Chinese Hong Kong students' and Australian students' procedural and conceptual understanding of decimals.

Method

Participants. Three hundred and eighty-four Hong Kong Grade six students from six elementary schools located in different districts with varied socio-economic status completed a written test on decimal numbers in Hong Kong. One hundred and three Australian students from four regional elementary schools located in Central West of New South Wales completed the same written test in Australia. The students in the Hong Kong and Australian samples had an average age of 12 years, and students in both samples had attended elementary school for six or seven years.

Instrument – written test. A written test (refer to Table 1 for sample test items) on decimal numbers was constructed with reference to the Hong Kong primary mathematics curriculum (The Education Department HK, 2000) and the Mathematics K-6 Syllabus (Board of Studies NSW, 2006). A content validity panel was established which included three Hong Kong elementary mathematics educators, one Australian secondary mathematics educator, one Australian elementary mathematics educator, ten Hong Kong elementary mathematics teachers, and two Australian elementary mathematics teachers. The test items were modified in response to advice provided by the panel. A pilot study was undertaken to establish test-retest reliability and to check for clarity of language for the Chinese written test in Hong Kong. A class of grade 7 students (in their first year of secondary school in Hong Kong) was given the test on two occasions, with a three month time period in between each. Cronbach's alpha was 0.86, indicating a high degree of reliability. A similar test-retest was not undertaken in Australia. However, the pilot test was administered to an Australian Grade 6 class which did not participate in the formal study. The students reported that the language was easy to understand and accorded with language in their textbook and worksheets.

Table 1: Sample test items

Examined content area	Number of items of this type	Illustrative test item
A. Comparison of decimals	6	Circle the larger decimal in the pair of six pairs. For example, 4.8 and 4.63.
B. Convert fractions with denominators either 10 or 100 to decimals	2	Convert $3\frac{500}{1000}$ to a decimal.
C. The representation of place value in decimals	3	In 0.723, the digit "7" represents seven lots of a) 1000; b) 100 c) 10 d) 0.1 e) 0.01 or f) 0.001
D. The concept of place value in decimals (number line)	1	What is next to 0.9 in the following number line? Write it into the box as shown. 
E. The concept of continuous quantity in decimals	1	Give a decimal between 0.75 and 0.8
F. The addition and subtraction of decimals	2	11.05 – 3.8
G. Multiplication of decimals	1	23 × 0.12
H. Division of decimals	1	0.12 ÷ 3
I. Translating a word problem into an equation (Addition)	1	There are two containers with 1 L capacity each. One container has water 0.238L and the other 0.53L. How much water altogether? (Give a mathematical expression only, no answer is required.)
J. Translating a word problem into an equation (Division)	1	0.96 L of orange juice was shared among 8 children. How much orange juice a child had? (Give a mathematical expression only, no answer is required.)

The test contained a mixture of fixed choice and free-response items. Fixed choice items have been used in much previous research (see for example, Peled & Shahbari, 2003; Stacey &

Steinle, 1999; Steinle, 2004a; Steinle, 2004b) and have the advantages of being easy to read and score. However, free response items (where students can respond with any answer they choose) enable a more detailed exploration of students' reasoning processes from their choice of rules and procedures. The tasks were presented in horizontal form because this is how problems commonly appear in textbooks in both countries. All of the items allowed students to use methods with which they were familiar.

Procedure. The written test was conducted in students' first language (that is, Chinese for the Hong Kong students and English for the Australian students) and was administered collectively to the students in their classrooms one to three months before the end of the semester. Each student worked individually, in silence and without time constraints. All students were able to complete the test within 35 minutes. The test was marked and scored immediately after students' work was collected.

Results and discussion

Overview of interpretive framework for analysis of students' responses

An interpretive framework was used to analyse students' responses to the written tests; in particular, students' procedural and conceptual understanding of decimal numbers. This framework is based largely on Hiebert and Wearne's work (Hiebert, 1992; Hiebert & Wearne, 1986) and is supported by other research findings (Bell, Swan & Taylor, 1981; Greer, 1987; Hiebert & Wearne, 1985; Moss & Case, 1999; Resnick, Nesher, Leonard, Magone, Omanson & Peled, 1989; Stacey, 2005). This framework was chosen because it incorporates categories of knowledge specific to the topic of decimal numbers and embeds related types of "sub-knowledge". The analysis of students' written test focused on four types of knowledge: (1) understanding decimal numerical symbols, (2) interpreting operational symbols (e.g., +, −, ×, ÷), (3) knowledge of the symbol rules, and (4) knowledge of quantities. Each of these aspects will be discussed in turn, exploring their relationship to procedural and conceptual understanding.

Understanding decimal numerical symbols refers to the "knowledge of the symbols that are used to write decimal fractions and knowledge of the form that constrains how the symbols are positioned on paper" (Hiebert, 1992; p.290). This type of understanding does not require knowledge of what the symbols mean and what quantities they represent. For the task of comparison of decimals, for example, Stacey (2005) comments that some students provide correct answers by comparing digits from left to right until one digit is found to be larger than the corresponding one. It can be argued that this knowledge is procedural only, in that it does not necessarily involve connections between the written symbols and the actual quantities they represent.

Interpreting operational symbols refers to making connections between operational symbols and an appropriate action on quantities (Hiebert, 1992). Using an example from the test (one container has water 0.238L and the other 0.53L, how much water is there altogether?), the "+" symbol takes on meaning when it is connected to the action of combining water in two separate containers. In another test item, the symbol "÷" takes on meaning when 0.96L orange juice is shared among 8 children. This type of task involves knowledge of *interpreting operational symbols*, which is best understood as knowledge for conceptual understanding.

Knowledge of the symbol rules denotes the knowledge for "the rules that prescribe how to manipulate the written symbols to produce correct answers" (Hiebert, 1992; p.290); for example, at the syntactic level, the rule for adding and subtracting decimals can be described in terms of lining up decimal points (Hiebert, 1992). The multiplication rule for decimals (i.e., counting the decimal places in multiplicand and multiplier for locating the decimal point in the answer) provides another example. These rules will produce correct answers if followed precisely step-wise even without an understanding the underlying concepts. This level of knowledge can be considered procedural understanding.

Knowledge of quantities refers to knowledge about the quantities the decimals represent and the underlying concepts for the symbol rules. Those tasks concerned with representation of place value, decimals on number line and concept of continuous quantity in decimals require a basic concept of decimal notation: "the value of a particular position is determined by beginning with the unit and, if moving to the right, dividing the previous value by 10 and, if moving to the left, multiplying the previous value by 10; and the ones position is marked with a decimal point on its immediate right" (Hiebert, 1992; p.286). Without a reasonable understanding of this principle, one is not able to understand the actual quantities the decimal symbols represent; for example, "2" represents two lots of 0.01 in 0.723, 1 instead of 0.10 is next to 0.9 on a number line, and 0.76 is a decimal between 0.75 and 0.8. This knowledge is best understood as knowledge for conceptual understanding.

Results

The results from the written test are summarised in Table 2. Throughout the results section, students' written answers are reproduced, as they appeared on the test papers. These figures provide insight into students' reasoning, and in particular, shed some light on students' understanding of decimal numerical symbols, interpreting operational symbols, knowledge of the symbol rules, and knowledge of quantities.

Table 2: Percentage of scores for each of the question types for Hong Kong group (N=384) and Australian group (N=103)

			Score							
Type of question (number of items of that type)	**Full mark**	**Country**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**Missing data**
A. Comparison of decimals (6)	6	HK	2.3	1.8	1.8	2.1	7.3	6.5	**77.6**	0.5
		AU	1.0	1.0	2.9	1.0	13.6	6.8	**72.8**	1.0
B. Convert fractions with denominators either 10 or 100 to decimals (2)	2	HK	4.9	19.5	**75.0**	*N/A*	*N/A*	*N/A*	*N/A*	0.5
		AU	8.7	27.2	**63.1**					1.0
C. The representation of place value in decimals (3)	3	HK	8.1	4.2	2.6	**85.2**	*N/A*	*N/A*	*N/A*	0
		AU	34	10.7	1.9	**52.4**				1.0
D. The concept of place value in decimals on a number line (1)	1	HK	11.5	**88.5**	*N/A*	*N/A*	*N/A*	*N/A*	*N/A*	0
		AU	11.7	**87.4**						1.0
E. The concept of continuous quantity in decimals (1)	1	HK	17.7	**82.3**	*N/A*	*N/A*	*N/A*	*N/A*	*N/A*	0
		AU	15.5	**83.5**						1.0
F. The addition and subtraction of decimals (2)	2	HK	1.3	14.6	**83.6**	*N/A*	*N/A*	*N/A*	*N/A*	0.5
		AU	20.4	16.5	**62.1**					1.0
G. Multiplication of decimals (1)	1	HK	19.8	**80.2**	*N/A*	*N/A*	*N/A*	*N/A*	*N/A*	0
		AU	56.3	**47.2**						1.0
H. Division of decimals (1)	1	HK	14.8	**84.6**	*N/A*	*N/A*	*N/A*	*N/A*	*N/A*	0.6
		AU	47.6	**51.5**						1.0
I. Translating a word problem into an equation – Addition (1)	1	HK	8.9	**90.6**	*N/A*	*N/A*	*N/A*	*N/A*	*N/A*	0.5
		AU	21.4	**78.6**						0

J. Translating a word problem into an equation – Division (1)	1	HK	9.6	**90.4**	*N/A*	*N/A*	*N/A*	*N/A*	*N/A*	0
		AU	46.6	**48.5**						4.9

*missing data: responses did not relate to the task

Table 3 shows the mean scores of each of the question types of the two language groups and *t*-test between language groups.

Table3: Mean, SD, *t*-test and *p*-value for each of the question types for Hong Kong group (N=384) and Australian group (N=103)

Type of question (number of items of that type)	Full marks	Language group	Mean	SD	Comparison	
					t	***p***
A. Comparison of decimals (6)	6	HK	5.42	1.347	0.123	0.902
		AU	5.40	1.196		
B. Convert fractions with denominators either 10 or 100 to decimals (2)	2	HK	1.70	0.556	2.193	0.030
		AU	1.55	0.654		
C. The representation of place value in decimals (3)	3	HK	2.65	0.893	6.259	0.000*
		AU	1.74	1.400		
D. The concept of place value in decimals on a number line (1)	1	HK	0.89	0.319	0.085	0.932
		AU	0.88	0.324		
E. The concept of continuous quantity in decimals (1)	1	HK	0.82	0.382	-0.492	0.623
		AU	0.84	0.365		
F. The addition and subtraction of decimals (2)	2	HK	1.82	0.423	4.780	0.000*
		AU	1.42	0.814		
G. Multiplication of decimals (1)	1	HK	0.75	0.402	8.753	0.000*
		AU	0.43	0.498		
H. Division of decimals (1)	1	HK	0.85	0.357	10.452	0.000*
		AU	0.52	0.502		
I.. Translating a word problem into an equation – Addition (1)	1	HK	0.91	0.285	2.889	0.05
		AU	0.79	0.412		
J. Translating a word problem into an equation – Division (1)	1	HK	0.90	0.295	7.431	0.000*
		AU	0.51	0.502		

* Significant difference between groups. Tests were conducted using Bonferroni-adjusted alpha levels of .0026 per test (0.05/19).

Tables 2 and 3 show that Hong Kong Chinese students significantly outperformed their Australian counterparts on many of the tasks. The following section will discuss student performance and the pattern of errors in detail, using the interpretive framework as a guide**.**

A. Comparison of decimals

For the six questions dealing with comparison of decimals (Type A), students were required to choose a larger decimal from a pair of incongruent length decimals for six pairs. The task was designed to investigate students' understanding of decimal numerical symbols. The performance of the two groups of students was similar. Nearly 78% of Hong Kong students and 73% of Australian students made no mistakes and 22% of Hong Kong students and 25% of Australian students made at least one error. Two per cent of Hong Kong students and less than one per cent of Australian students got all six questions wrong.

An analysis of the students' responses was undertaken, looking for the application of the three types of "rules" outlined earlier: the longer-is-larger rule, the shorter-is-larger rule and the zero rule. About 5% of Hong Kong students and 11% of Australian students revealed a shorter-is-larger misconception. These students may have attempted to connect decimals with fractions, thinking that the shorter the decimal, the bigger the value. Figures 1 and 2 illustrate student responses of this type.

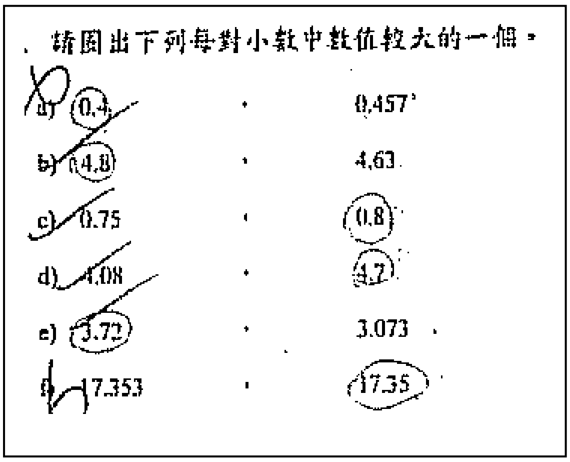


Figure 1

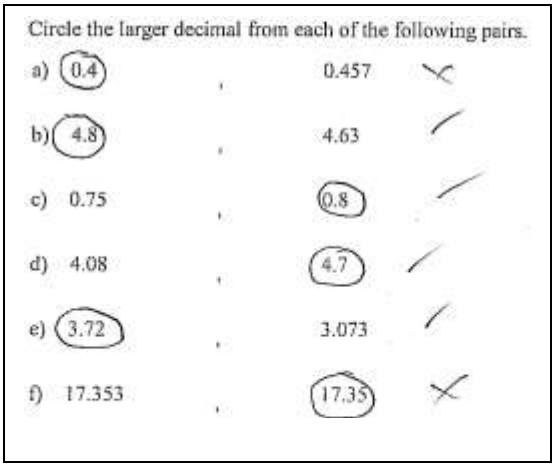


Figure 2

About 4% of Hong Kong students and 3% of Australian students applied the longer-is-larger rule and the zero rule in conjunction. This group of students generally believed that a longer decimal is larger than a shorter decimal but when there was a zero to the immediate right of the decimal point in one of a pair of decimals of incongruent length, they recognised that the decimal with zero at the tenths position was smaller, regardless of the length of the pair of decimals. Figure 3 illustrates this pattern of errors.

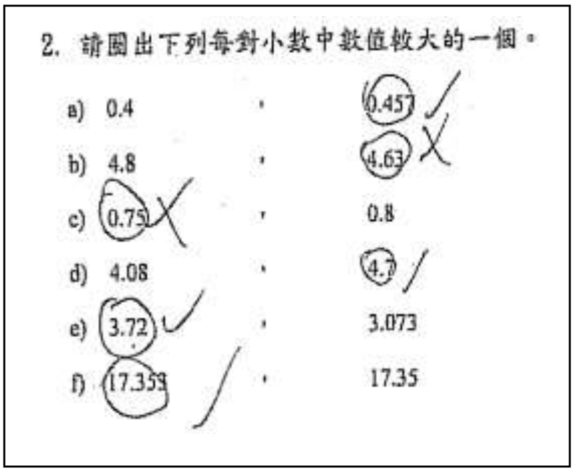


Figure 3

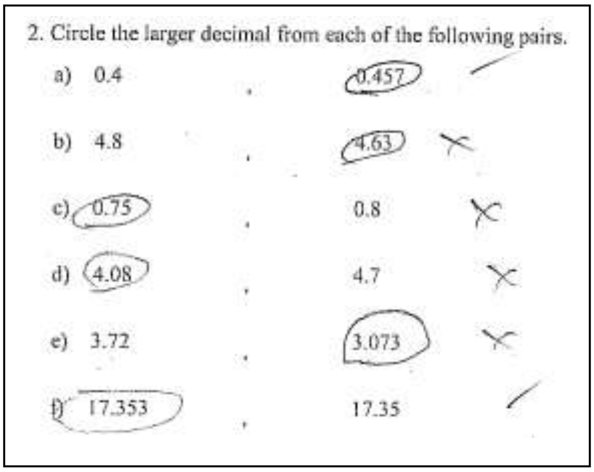


Figure 4

About 5% of Australian students indicated that they believe that a longer decimal is larger than a shorter decimal, regardless of the presence of a zero to the immediate right of the decimal point. Figure 4 shows this error. The last group displayed error types of many kinds throughout this task.

The results reveal that about three-quarters of Hong Kong and Australian students exhibited sound knowledge of decimal notation. Many of them correctly decoded the numerical symbols of decimals. However, as Steinle and Stacey (2001) argue, students who are able to provide correct answers for decimals comparison tasks are not necessarily true 'experts' because they may follow the rules without understanding them. Correct performance on this type of task does not require understanding of what symbols mean and the quantities represented. However, both groups' strong performance indicates their sound procedural knowledge of decimal numerical symbols.

B. Convert fractions with denominators either 10 or 100 to decimals

For the two questions of type B, students were required to translate fractions with a denominator expressed as power of 10 to decimal numbers. These test items assessed students' knowledge of quantities, in particular the connections between decimal symbols and fraction quantities (Hiebert, 1992). About 75% of Hong Kong students and 63% of Australian students provided correct answers for all items of this type. About 20% of Hong Kong students and 27% of Australian students provided one correct answer and one incorrect answer. About 5% of Hong Kong students and 9% of Australian students provided incorrect answers for both questions. In both groups, the most frequent errors on the task were 0.05 or 0.005 for $\frac{500}{1000}$ and 0.9 or 0.009 for $\frac{9}{100}$. The results indicate that the majority of students in both groups had established a good understanding of the relationship between decimal symbols and fraction quantities, in particular fractions with the denominators expressed in power of 10.

C. The representation of place value in decimals

For the three questions of type C, students were required to indicate what a digit in a certain position of a decimal represents. These items assessed students' understanding of place value in terms of how to decode the digits after the decimal points; for example, what does 7 mean in 0.723? Over 85% of Hong Kong students and 53% of Australian provided correct answers for all of the items of this type, and about 8% of Hong Kong students and 34% of Australian students were not able to provide one correct answer. The Hong Kong students significantly outperformed the Australian sample in this task (p=0.000). In both groups, the most frequent errors (60 % of the incorrect responses) were "seven 100s, two 10s and three ones" and the second most frequent errors (40 % of the incorrect responses) were "seven 10s, two 100s and three 1000s" for 0.723. The results reflect the fact that some Hong Kong students but almost half the Australian did not have a well-developed understanding of knowledge of decimal notation - they may have interpreted place value of digits after the decimal point by "mirroring" the concept of a whole number (Hiebert & Wearne, 1983; 1986). Students who gave "seven 100s, two 10s and three ones" for 0.723, appeared to have decoded the number after the decimal point as a whole number. Students who gave "seven 10s, two 100s and three 1000s" for 0.723, have assumed that the major difference between whole numbers and decimal numbers is the direction of progression of place value, multiplying the previous value by 10, when moving to the left for whole numbers but moving to the right for decimals. This pattern of errors is further illustrated in the analysis of addition and subtraction of decimals, discussed later in the paper. Overall, the majority of Hong Kong students displayed sound conceptual knowledge of quantities – the actual quantities that each digit (after the decimal point) represents.

D. The concept of place value in decimals on a number line

For questions in category D, students were required to indicate the number next to 0.9 on a number line. Both groups of students performed similarly, and well, on this task. It is not surprising that the most frequent error for both groups was 0.10. More interestingly, no one gave 0.1 as an answer. The results showed that most students had a good conceptual understanding of decimal quantities, in this case - the quantities between 0.7 and 0.8, 0.8 and 0.9, and then inferred the “decimal” after 0.9. Just over 10% of students did not have a sound knowledge of quantities and incorrectly interpreted the numerical symbols. They decoded the number after the decimal point as a whole number: 10, but not 1, comes after to 9 in the whole number system.

E. The concept of continuous quantity in decimals

For question E, students were required to give a decimal between 0.75 and 0.8. This task was designed to test students’ knowledge of the continuous quantity of decimal notation and in particular “the property of being dense”; that is, there are an infinite number of decimals between any two decimals (Hiebert, 1992). Over 80% of students in both groups provided the correct answer. About 10% of Hong Kong students and 5% of Australian students provided a decimal smaller than 0.75. About 6% of Hong Kong students and 2% of Australian students gave a decimal bigger than 0.8. No other pattern could be observed in the incorrect responses. In general, the errors may reflect two phenomena. Firstly, students may not have a well-established understanding of place value; secondly, they may not understand decimal notation as a representation of continuous quantity. The results reveal that the majority of Hong Kong and Australian students displayed sound conceptual understanding of decimal quantities – “the decreasing rate of increase in the size of the number as digits are adjoined to the right” (Hiebert, 1992; p.289).

F. The addition and subtraction of decimals

The questions in set F were addition and subtraction computation tasks. Although use of column form was not the only means to arrive at the answer, almost all the Hong Kong students and 90% of Australian students worked out their answer using algorithms in column form. Over 83% of Hong Kong students and 62% of Australian students provided correct answers for all of the items. The Hong Kong students significantly outperformed their Australian counterparts in this task (p=0.000). Figures 5 and 6 show Hong Kong students' common arithmetic errors. They illustrate the general finding that many Hong Kong students had mastered the basic concept of place value in decimals and addition or subtraction of digits of the same positional values but failed to regroup correctly. The results reveal that most of the Hong Kong students had sound procedural knowledge of the symbol rules for addition and subtraction of decimals.

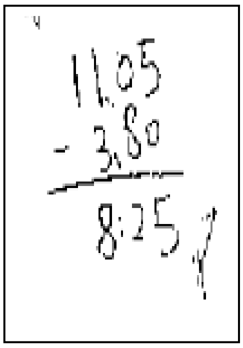


Figure 5

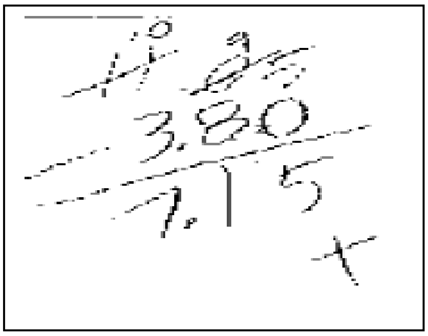


Figure 6

Interestingly, about half of the incorrect responses of the Australian students were due to failure to line up the decimal points, as show in Figure 7. The results indicate that some Australian students had not mastered the knowledge of symbol rules for addition and subtraction of decimals, nor had they understood the basic concept of addition: adding the digits of the same place values. Another half made arithmetic errors in the algorithms.

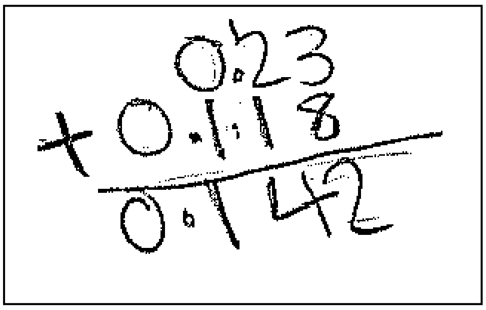


Figure 7

G. Multiplication of decimals

Question G was a multiplication computation task presented in horizontal form and as a free-response item. Hong Kong students significantly outperformed their Australian counterparts (*p*=0.000**; *d*=0.701). Over 80% of Hong Kong students and 47 % of the Australian students provided the correct answer for this item. Almost all the Hong Kong students who provided correct answers used a multiplication ‘rule’ as follows: lining up the most right digit of the decimals, doing the multiplication as whole number then counting the number of decimals places in both the multiplier and multiplicand and finally putting the decimal point accordingly in the answer (Hiebert & Wearne, 1985). Among those Hong Kong students who scored no marks, nearly 50% used the “multiplication ‘rule’” correctly (that is, lined up the most right digit) but made some general arithmetic errors in calculation, as illustrated in Figure 8. These students generally gave the correct location of the decimal points in incorrect numerals. These results indicate good procedural knowledge of symbol rules for multiplication of decimal numbers but minor arithmetic errors in the process of multiplication. Another 50% undertook the arithmetic correctly but located the decimal points in wrong places in their answers, as shown in Figure 9. This type of error reveals conceptual misunderstanding; that is: “multiplication always produces a bigger number”.

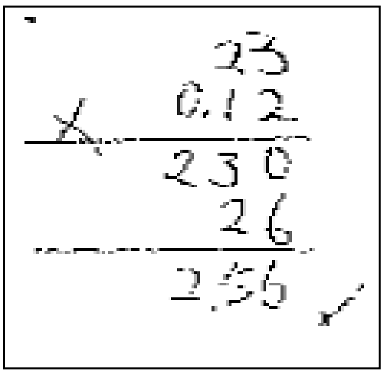


Figure 8

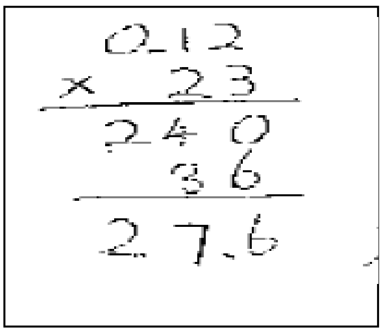


Figure 9

Unlike the Hong Kong students, over 13% of Australian students did not follow the “multiplication rule” but rather lined up the decimal points instead of the most right digits, as shown in Figure 10. Lining up the decimal points in an algorithmic column form could still produce correct answers if correct multiplication procedures were performed. However, students appeared to lose track in executing the multiplication and wrongly placed the decimal points in their final answers. These errors suggest fragile understanding of the place values after the decimal point, and the meaning of tenths of tenths, tenths of hundredths and hundredths of hundredths (see figures 10 to 12). Among those Australian students who employed the “multiplication rule”, over 32% made arithmetic errors in their calculations. An example of this type of error is shown in Figure 13. These students had developed some knowledge of the symbol rules for decimal number multiplication, but their knowledge of whole number multiplication was not sound.

Over 12% of Australian students ignored the decimal points and calculated the question as a whole number multiplication, as shown in Figure 14. Interestingly, this category of students did not put the decimal point in their algorithm and therefore, did not place a decimal point in their answers. They treated the multiplication of decimals in exactly the same manner as whole number multiplication.

Figures 11, 12, 13 and 14 exemplify a general trend across the multiplication tasks; that is, most of the Australian students’ errors represent major mathematical errors. They highlight the fact that many Australian students did not have a good understanding of decimal notation and whole number multiplication. Many Australian students did not have fully developed knowledge of numerical symbols nor well-established knowledge of the symbol rules for multiplication of decimals.

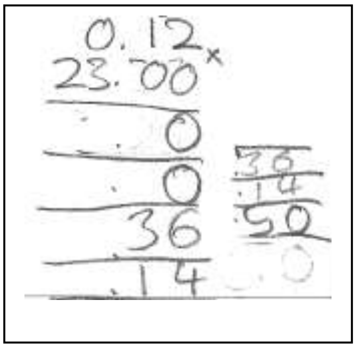


Figure 10

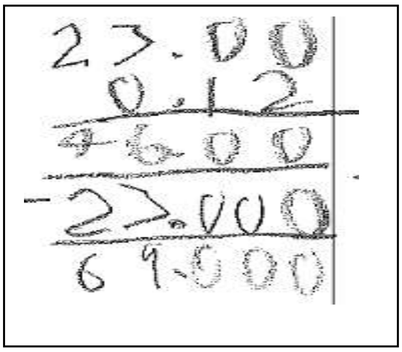
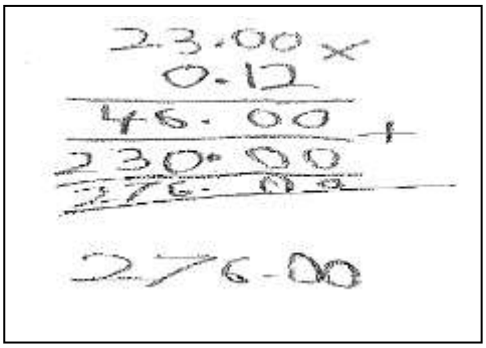


Figure 11



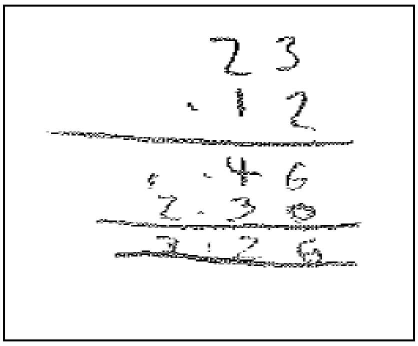


Figure 12 **Figure 13**

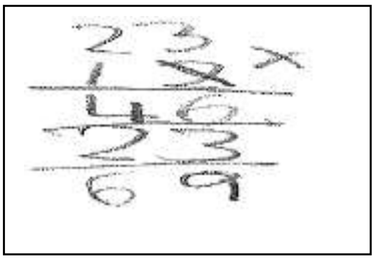


Figure 14

H. Division of decimals

Question H was a division computation, presented in horizontal form and as a free-response item. Over 84% of Hong Kong students and 51% of Australian students provided correct answers for this item. In this task, the Hong Kong students significantly outperformed their Australian counterparts (p=0.000). Over 90% of Hong Kong students but less than 2% of Australian students completed the question by applying the division rule for decimals: “multiplying both the dividend and division by the same multiple of 10 until they become whole numbers and then doing the division as whole number”. Interestingly, most of the Australian students who supplied correct answers did not provide any written computation on the algorithm as shown in Figure 15.

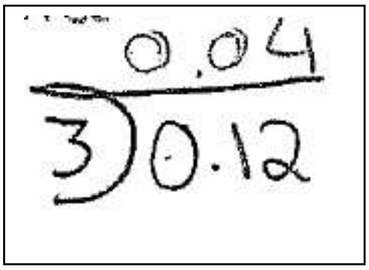


Figure 15

Of those students who got this question wrong, 60% of Hong Kong students and 50% of Australian students located the decimal points in the wrong places, and provided either 4 or 0.4 as their answers. Figure 16 shows typical errors of this type. Among those Hong Kong students who did not provide correct answers for this item, over 30% failed to multiply by 10 correctly the divisors and dividends as shown in Figure 17 (the original equation were 0.12÷3). No Australian student made this type of error. About 20% of incorrect responses for the Australian group involved mixtures of addition and subtraction, and over 10% of the incorrect responses of Australian group used the multiplication algorithm. A few Hong Kong students and 5% of Australian students made general whole number division errors in algorithms, as shown in Figure 18. Less than 1% of Hong Kong students and 2% of Australian students reversed the positions of divisor and dividend in the long division algorithms. Interestingly, those Australian students who swapped the division and dividend in the algorithm used repeated subtraction to compute the answer as shown in Figure 19. Overall, 84% of the Hong Kong students demonstrated their sound procedural knowledge of symbol rules for division of decimals. In contrast, half of Australian students had not yet mastered this knowledge.

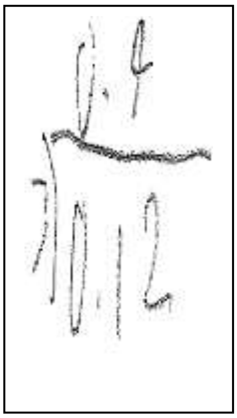


Figure 16

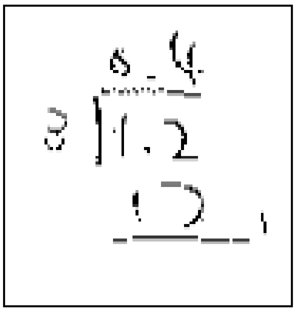


Figure 17

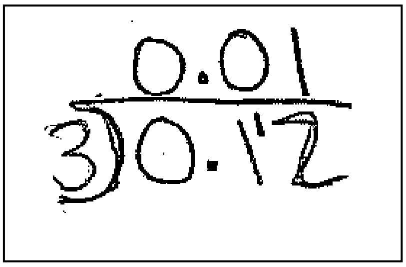


Figure 18

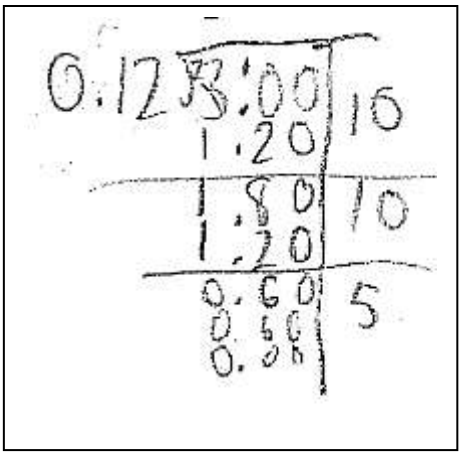


Figure 19

I. Translating a word problem into an equation (Addition)

Question I was a word problem that involved one-step addition. In this task, students were required to translate the word problem into an addition equation (answer: 0.238L+0.53L or 238ml + 530ml). The purpose of this task was to explore students' choice of operation for the solution of a single-operation 'real life' word problem; students were asked to supply an equation but not compute the answer. Over 90% of Hong Kong students and over 78% of Australian students provided the correct equation for this word problem. In this task, the Hong Kong students significantly outperformed the Australian sample (p=0.05*; d=0.339). Of the 10% of Hong Kong students and 22% of Australian students who did not give correct equations, most of the Hong Kong students (over 80%) and nearly 45% of Australian students were able to recognise that the problem was an addition problem. The cue "together" in the question stem provided a strong association with the addition operation. The results indicate that the students of both groups correctly interpreted the operational symbol for this one-step addition task.

Half of the Australian errors were made by students who did not supply an answer at all. Of those Australian students who gave an incorrect equation, about 21% offered 238+53, 4% of them wrote 0.238+53 and 4% of them gave 238+0.53 as their answers. The results suggest that this category of students had difficulty in conceptualizing the metric measurement system. The question involved the concepts of changing a large unit to a small unit (0.53 L to 530 mL instead of 53mL) and adding metric units (0.238L + 0.53L or 238ml + 530ml but not 238ml + 0.53L). A few Australian students gave a multiplication equation for this task.

Interestingly, no one particular type of error was observed among the Hong Kong students. Less than 10% of Hong Kong students did not provide correct equations but 17 different types of errors were recorded. Although the problem involved one-step addition and was straightforward, those Hong Kong students who provided incorrect answers gave complicated equations involving subtraction, multiplication or division. For example, 5 students (1.4%) gave (1+1) – (0.238+0.53), 6 students (1.7%) gave 0.53x2, 1 student (0.26%) gave (1000+0.238)+(1000+0.53) and 1 student (0.26%) gave 0.238+0.53÷1 as their answers. Their errors might be due to superfluous information in the question stem: "There are two containers with 1 L capacity each". The results perhaps indicate that some of the Hong Kong students had difficulties in selecting relevant information from a situation which contained superfluity of data (Bell, Swan & Taylor, 1981). No such error pattern was observed in the Australian group.

J. Translating a word problem into an equation (Division)

Question J was a word problem that involved one-step division. In this task, students were required to translate the word problem into a division equation (Answer: 0.96L÷8 or 960ml÷8). Similar to Question I, only an equation was required for this item. Over 90% of Hong Kong students and about 48% of Australian students provided correct equations for this word problem. In this task, the Hong Kong students significantly outperformed their Australian counterparts (p=0.000). Again, the Hong Kong students displayed their good understanding of operational symbols. Among those Australian students who did not provide correct equations, over 77% did not write anything, 16% gave 8÷0.96 as their answer and about 7% gave 96÷8. Similar to question type I, students had difficulty in conceptualizing the conversion of metric measurement units (0.96L to 960 ml instead of 96ml). Of those Hong Kong students who did not provide correct equations, half did not write anything, 17% appeared to make simple mistakes (e.g., 0.96÷9), 21% offered 8÷0.96 and 9% gave 0.96×8 The results of this task further illustrate the error of reversing the positions of the divisor and dividend that was apparent in the long division algorithms.

Discussion

The broad aim of the current study was to compare Hong Kong Chinese and Australian students' performance on a comprehensive set of decimal number tasks. More specifically, the study aimed to analyse the errors that students made in order to understand more fully students' thinking about decimals. A related aim was to explore students' procedural and conceptual understanding of decimal numbers. The current research revealed interesting similarities and differences between the two samples on each of these dimensions.

As noted earlier, much of the research on children's understanding of decimal numbers has been conducted in Western settings. The present study found significant differences between the Hong Kong and Australian students on many of the tasks, with the Hong Kong group consistently performing better than the Australian sample. The overall direction of these findings is not surprising, given the consistent trend for Hong Kong Chinese students to outperform Australian students in international mathematics assessments such as TIMSS (Beaton, Mullis, Martin, Gonzalez, Kelly & Smith; 1997; Mullis, Martin, & Foy; 2008). While a difference between the two groups was perhaps predictable, the magnitude of some of the differences was surprising.

The test items ranged from comprehending place value after the decimal point, the appropriate use of algorithms for computation of decimals, and the application of decimals to "real life" problems. The question of whether connections existed between students' conceptual and procedural knowledge could not be answered unequivocally for all test questions. However, the analysis of errors captured some crucial features of students' thinking and understanding of decimals.

Both groups performed equally well on the task of comparison of decimals. The analysis of errors of tasks in which both groups performed reasonably well illuminated important aspects of their thinking about decimal numbers – aspects which were shared by both groups. Students exhibited many of the errors that have been reported in previous studies. The 'longer is larger' and the 'shorter is larger' misconceptions (see, for example, Resnick et al., 1989) were exhibited by a small percentage of students in each group. The results revealed that the students had a good procedural understanding of numerical symbols for decimals. As noted earlier, it is not possible to determine whether or not the students had an accompanying conceptual knowledge of quantities that are represented by the numerical symbols.

In the questions requiring conversion of fractions to decimals, the two groups both performed moderately well but the Hong Kong students performed significantly better than the Australian group. Again, the Hong Kong students revealed their good understanding of numerical symbols in relation to fractions with denominators expressed in power of 10. About one quarter of the Hong Kong group and over one third of the Australian sample made at least one error on this type of question. The pattern of errors indicated that some students did not have a well-established conceptual understanding of the relationship between decimal symbols and fraction quantities (Moss & Case, 1999).

While the Hong Kong students performed much better on the questions involving the representation of place value, the two groups made similar types of errors, with a significant minority of students in both groups mistakenly adopting concepts of whole numbers in their understanding of decimal numbers. This inappropriate 'mirroring' of whole number concepts to decimal numbers has been reported by other researchers (Hiebert & Wearne, 1983; 1986). Likewise, both groups performed equally well on tasks associated with understanding the concept of place values in decimals on a number line, and the related concept of continuous quantity in decimals. Over 80% of both groups answered these questions correctly. Thus, the results indicate that the students had acquired a good understanding of numerical symbols as well as knowledge of quantities that the symbols represent.

The addition and subtraction problem also yielded significant differences between two groups and revealed the Hong Kong students' relatively strong procedural knowledge. In contrast, over one-fifth of the Australian sample did not understand that aligning the most right digits of decimals violated the syntactic conventions, and one-tenth misinterpreted the place value of digits after the decimal point as whole number. It would be interesting in future research to explore students' mathematical reasons for the use and misuse of the algorithmic procedure.

One of the most stark and statistically significant differences in the performance of the two groups occurred for the items involving multiplication and division computation. The analysis of errors was particularly valuable in understanding some of these highly significant differences in performance between the groups. The Hong Kong students reliably used procedures and algorithms to solve the multiplication and division problems. Indeed it is fair to say that overall the Hong Kong group exhibited strong knowledge of symbol rules. Their errors were generally simple arithmetic ones. In contrast, a far lower percentage of Australian students used algorithms or standard procedures to solve these questions, and generally this group committed more procedurally-flawed errors. One example of this was the tendency for Australian students to reverse the divisor and dividend in the division algorithmic column form. The results for the Australian students reflect findings of other studies that have reported students have difficulty in conceptualizing division beyond certain restricted classes of numbers (see, for example, Bell, Swan & Taylor, 1981; Bell, Fischbein & Greer, 1984; Greer, 1987).

This difference in type of errors may also reflect the way in which the two groups had been taught. As discussed earlier, Hong Kong and other Chinese students are taught with an emphasis on procedural knowledge. Recent mathematics education in Australia has placed emphasis on conceptual knowledge (Van de Walle, Karp & Bay-Williams, 2010; Li, 2006).

Some researchers might argue that the Hong Kong students' reliance on procedures and algorithms does not necessarily indicate a real understanding of decimal numbers (Hiebert, & Wearne, 1985). In fact, as noted earlier, some researchers have argued that many misconceptions arise precisely because of students' reliance on procedural knowledge without a real understanding of the associated concepts that underlie them (Hiebert & Wearne, 1986; Lachance & Confrey, 2002). So, while it is clear that the Hong Kong students exhibited far stronger knowledge of symbol rules than the Australian sample, what are we able to conclude about each group's conceptual understanding?

Once again, the error analysis proved fruitful in exploring this question, particularly for the items which required translation of a word problem into numbers. Arguably, these questions require both procedural knowledge and conceptual knowledge of interpreting operational symbols for their successful completion. Students needed to understand the question being posed and translate the question into a numerical equation. Both groups performed well at the most simple translation problems involving addition of decimals. However, a highly statistically significant difference in the performance of the two groups occurred for the items involving translation of word problems into an equation involving decimal division. Many of the Australian students were unable to provide any answer at all for this question. The results support an observation from Greer's study (1987) that "the conceptualization of the operations is strongly dependent on the types of numbers involved, the situation being modelled, and the interactions between them" (p.38). The crucial feature here is that the decimal (0.96) made it more difficult to recognise the operation and relationship involved in the problem. For this problem, although the context of the problem was familiar, the abstraction and transfer of the mathematical structure proved difficult for some students, but particularly in the Australian sample (Bell, Swan & Taylor, 1981). Some Hong Kong students appear to be genuine "experts" with a solid conceptual understanding of decimal division and an integration of procedural and conceptual understanding of decimal division.

As discussed earlier, there has been tension between procedural knowledge and conceptual knowledge in mathematics education (Lai & Murray, 2012). In Western countries procedural knowledge is associated with rote learning and is criticised for leading to poor learning outcomes (Watkins & Biggs, 2001). In fact, Bosse and Bahr's study (2008) revealed that the USA in-service and pre-service teachers devalued procedural knowledge and considered that it was tantamount to no understanding at all. Conceptual and procedural knowledge tend to be dichotomised in Western mathematics education. In contrast, procedural and conceptual knowledge are closely linked when conceptualizing mathematics teaching and learning in the Chinese context. Scholars who have investigated learning in Chinese cultures argue that the two types of knowledge do not develop independently but develop iteratively, with gains in one leading to gains in the other, which in turn trigger new gains in the first (Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler & Alibali, 2001). The results of the current study have provided some evidence to this claim. For some of the tasks, Hong Kong students exhibited stronger procedural and conceptual understanding than the Australian students. More importantly, for many items, Hong Kong students' strong procedural knowledge appeared to underpin their conceptual understanding.

Conclusion

The current study had some limitations arising from the use of a written test only. The degree to which students' reasoning processes could be assessed was limited to some extent by the fixed-choice nature of some of the questions. The free-response questions illuminated students' thinking to a greater extent, but even then, some interpretation of students' written responses was required. Further research in which students are interviewed about their answers and the thinking behind them would be valuable. This would enable the students to articulate the processes they used to answer a question, and describe their reasons for doing so. Further questioning could elicit whether students understood the mathematical concepts behind the algorithms they had chosen. This type of research with a particular focus on student errors and the reasoning behind these would be useful.

Despite these limitations the present study provided useful insights into students' conceptions and misconceptions about decimal numbers. The analysis of errors provided a fine-grained picture of students' procedural and conceptual knowledge of decimal concepts ranging from comprehending place value after the decimal point, the proper use of algorithms for computation of decimals, and the application of decimals to real life problems. For many of the questions, Hong Kong and Australian students shared similar misconceptions. For others, Hong Kong students exhibited stronger procedural and conceptual understanding of the task.

In summary, although it was not a focus for the current research, the analysis of errors points to important issues for the effective teaching of decimal numbers in highlighting the way that conceptual understanding for rules and procedures are built into the students' knowledge system. Procedural and conceptual knowledge should be viewed as two intertwined components such that one is the complement to the other (Lai & Murray, 2012) and such that they develop iteratively. While we believe that the results of the current study have provided some evidence for this claim, further research exploring the links between students' procedural and conceptual understanding of decimal numbers employing diagnostic interviews and open-ended tasks would be valuable.

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