

YEAR 2 INACCURATE BUT FLEXIBLE MENTAL COMPUTERS: TEACHER ACTIONS SUPPORTING GROWTH

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The Year 1-10 Mathematics Syllabus, recently implemented in Queensland's schools introduces mental computation as the main form of computation with written computation emphasising students' self developed strategies (QSA, 2004). To facilitate the incorporation of mental computation into the curriculum a teaching experiment that adopted a case study design was conducted. A Year 2 teacher was provided with a series of professional development (PD) sessions that incorporated the mathematics of mental computation and the use of support materials. She then used this knowledge to develop (with the assistance of the researcher) a series of 8 half hour lessons delivered over an eight week period. This paper reports on the pre- and post-study interview results of two of her students. Their selection was based on responses to questions that probed for accuracy and flexibility of strategy. During the pre-study interview both students demonstrated inflexible and inaccurate mental computation. On post-interviewing the students remained inaccurate; however, their repertoire of strategies had developed such that they were categorised as flexible mental calculators. Close examination of teacher actions, including engaging students in mathematical discussions, the use of representations and teacher consideration of the mental computation process of proficient mental computers all appear to have supported student growth. Further growth in accuracy as well as flexibility, although seen in other students in the class was not evident with these two students. Close examination of error patterns will further support teacher acquisition of content and pedagogical knowledge essential for the teaching of mental computation.

The new *Year 1-10 Mathematics Syllabus* (Queensland Studies Authority [QSA], 2004) is being implemented throughout Queensland. This syllabus introduces many changes, one of which is the introduction of mental computation, emphasising students' self developed strategies. This change in the syllabus is well supported by the literature where researchers and educators have stressed the importance of including mental computation in number strands of mathematics curricula (e.g., McIntosh, 1996; Sowder, 1990; Willis, 1990). In effect, mental computation promotes number sense (National Council of Teachers of Mathematics, 1989; Sowder, 1990).

In terms of computational efficiency, Thompson and Smith (1999) classified mental computation strategies so that *aggregation* ($28+35: 28+5=33, 33+30=63$) and *wholistic* ($28+33: 30+35=65, 65-2=63$) were the most sophisticated. Similarly, Heirdsfield and Cooper (1997) argued that *separation right to left* ($28+35: 8+5=13, 20+30=50, 63$), *separation left to right* ($28+35: 20+30=50, 8+5=13, 63$), *aggregation* and *wholistic* represented increasing levels of strategy sophistication.

Proficiency in mental computation has been the focus of several research projects (e.g., Beishuizen, 1993; McIntosh & Dole, 2000; Reys, Reys, Nohda, & Emori, 1995). The research showed that weaker students tended to use less efficient *separation* strategies (Beishuizen, 1993). In contrast,

skilled mental computers employed a variety of strategies that reflected understanding of number and operations.

Research reported by the first author investigated mental computers and the cognitive, metacognitive and affective factors that supported proficiency (Heirdsfield, 2001, Heirdsfield & Cooper, 2002). That study investigated the part played by number sense knowledge (e.g., number facts, estimation, numeration, and effect of operation on number), metacognition (metacognitive knowledge, strategies and beliefs), affects (e.g., beliefs, attitudes), and memory (working memory and long term memory) in mental computation. Flexibility in mental computation was defined as employment of a variety of efficient mental strategies, taking into account the number combinations to inform the mental strategy choice. The research showed that students proficient in mental computation (accurate and flexible) possessed integrated understandings of number facts (speed, accuracy, and efficient number facts strategies), numeration (including multiplicative understanding, e.g., ten tens are the same as one hundred; canonical understanding of number, e.g., $54=5$ tens and 4 ones; and noncanonical understanding of number, e.g., $54=4$ tens and 14 ones), and effect of operation on number (e.g., the effect of changing the addend and subtrahend). These proficient students also exhibited some metacognitive strategies and beliefs, and affects (e.g., beliefs about self and teaching) that supported their mental computation. Further, proficient mental computers had reasonable short-term recall to hold interim calculations and recall number facts (*phonological loop* – see Baddeley, 1986), and well developed *central executive* (Baddeley, 1986) to attend to the demanding task of mental computation and retrieve strategies and facts from a well-connected knowledge base in long term memory. Proficient mental computers chose from a variety of efficient strategies, as they possessed extensive and connected knowledge bases to support these strategies. Thus, there was evidence of the importance of connected knowledge, including domain specific knowledge, and metacognitive strategies, affects and memory for proficient mental computation. As a result of that study, a conceptual flowchart representing the mental computation *process* of proficient mental computers was developed (see Figure 1). See Heirdsfield and Cooper (2004) for further details.

With a less connected knowledge base, students would compensate in different ways, depending on their beliefs and what knowledge they possessed. One choice was to employ teacher taught strategies (pen and paper algorithms were taught at that time) in which strong beliefs were held, as long as the procedures could be followed, and if they were supported by fast and accurate number facts and some numeration understanding. Further, working memory (slave systems and central executive) had been sufficient. Thus, one method used to compensate for a less-connected knowledge base was to employ an automatic strategy. These students were identified as inflexible and accurate.

Another form of compensation was inventing strategies when the teacher-taught strategies (pen and paper algorithms) could not be followed. Although working memory was sufficient, the knowledge base was minimal and disconnected, thus compensation strategies were not efficient, and resulted in errors. Further, the knowledge base did not support high-level strategies. These students were identified as flexible and inaccurate. Heirdsfield (2001) posited that these students were unable to use teacher-taught strategies; so, out of necessity, they attempted to formulate another strategy. In order to be able to do this, some supporting factors were required. These were number facts strategies (extension of these strategies to mental strategies), numeration understanding (canonical, noncanonical, multiplicative, and proximity of number) and metacognitive strategies (choosing a strategy). However, all these factors were evident at a limited level, so advanced mental strategies could not be selected. Further, number and operation understanding was not present. Heirdsfield (2001) suggested that both numeration understanding and number and operation understanding were necessary for employment of advanced mental computation strategies (e.g., *wholistic compensation*: $246+199$: $246+200=446$; $446-1=445$). Thus, the difference between proficient

mental computers and the flexible and inaccurate mental computers was that proficient mental computers *could* choose alternative and efficient strategies, whereas, the flexible and inaccurate computers *had* to choose alternative strategies.

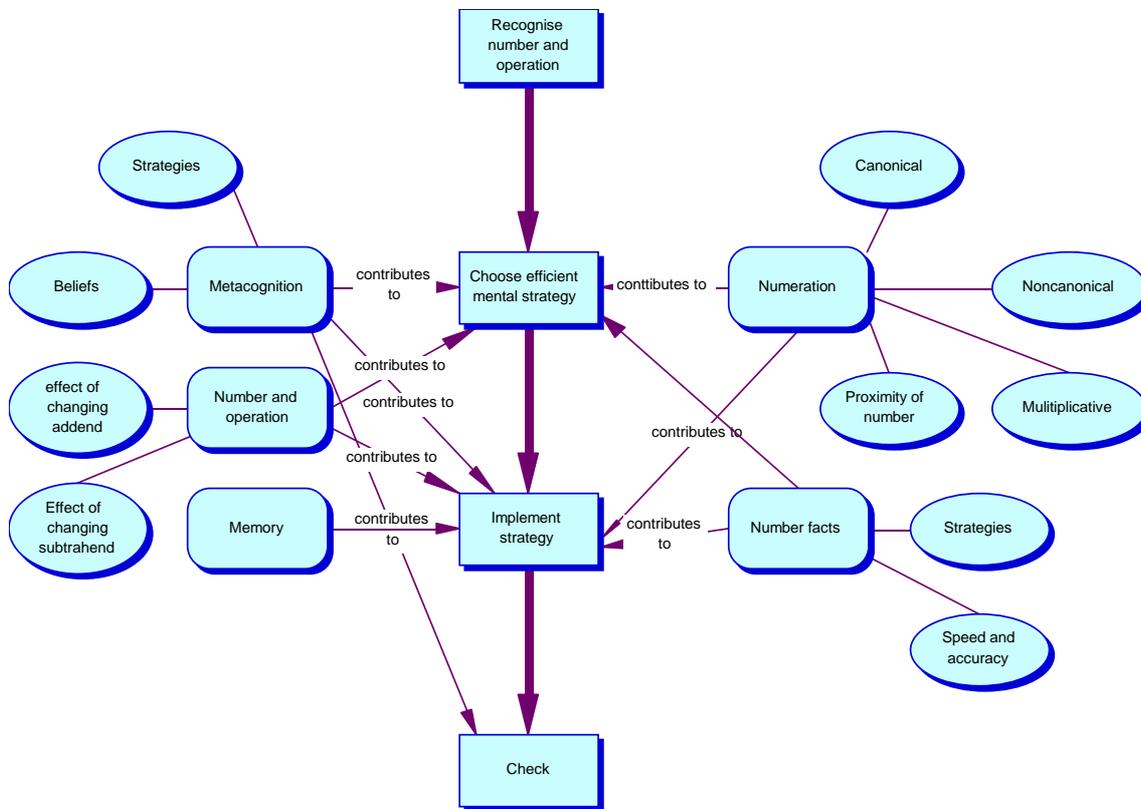


Figure 1. Flowchart representing the mental computation process of proficient mental computers.

Finally, a deficient and disconnected knowledge base and deficient working memory could not support mental computation, or an attempt to employ alternative strategies. To compensate, they attempted to use an automatic strategy (pen and paper algorithms), but their knowledge base and impoverished working memory would not support this. These students were identified as inflexible and inaccurate.

Flowcharts were subsequently developed by Heirdsfield (2001) to represent mental computation processes of the following types of mental computers: (a) inflexible and accurate; and (b) flexible and inaccurate. No flowchart was formulated for the inflexible and inaccurate mental computer as they exhibited little understanding or skills. Questions were raised as to whether instructional programs aimed at building on students' existing mental strategies and focusing on connected knowledge would improve students' access to mental computation strategies.

The purpose of this paper is to use the framework for the proficient (flexible and accurate) mental computer to support identification of where the breakdown in the structures (as outlined in Figure 1) occurs; thereby, providing an avenue for appropriate response in teaching practices. While there were several students in the class of 21 Year 2 students who were considered poor mental computers (inaccurate and inflexible) before the instructional program, two students are discussed here. They did not exhibit the most startling improvements in accuracy, but they reflected general changes exhibited by many of the students who employed more sophisticated strategies after the instructional program, compared with the strategies they employed before the instructional program.

THEORETICAL FRAMEWORK

The theoretical perspective that has guided the study being reported here has been the role of mental models in assisting students to both construct and co-construct specific mathematical concepts such as number and operation, numeration and number facts. These concepts are essential for the promotion of mental computation. Cheng (2000) suggests that effective representations can contribute to significant conceptual learning. Using effective representations is also recommended by National Council of Teachers of Mathematics (NCTM). Further literature argues that the model/representation chosen must (a) represent the relations and principles of the domain, (b) engage various modalities (e.g., kinaesthetic and visual), and (c) be unambiguous (English, 1997). Teacher actions that support the appropriate use of these models are critical to the process of student construction of understanding. It is argued that (a) the use of concrete materials must directly relate to the mathematical concept being studied, (b) recognise student potential as well as pre existing constructions, and (c) engage students in active participation (Davis & Maher, 1997).

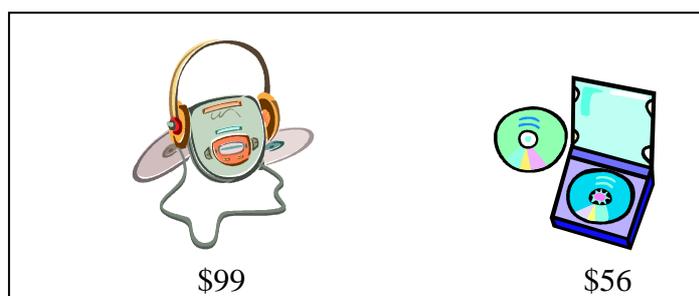
Some researchers consider the number line to be an important model in teaching aspects of number, including computation (Fueyo & Bushell, 1998; Klein, Beishuizen, & Treffers, 1998; Selter, 1998). Gravemeijer (1994) suggests that the empty number line is well suited to the development of computation as it reflects informal methods that children develop. While other models include blocks (Dienes Multibase Arithmetic Blocks (MAB)) and hundred square they are possibly not efficient models to support the development of mental computation (e.g., Beishuizen, 1999).

THE STUDY

This study adopted a case study approach (Lesh & Kelly, 2000) in which a teaching experiment was conducted in a Year 2 classroom (n=21). The new Queensland *Mathematics Years 1 to 10 Syllabus* (QSA, 2004) introduces mental computation, emphasising students' self generated strategies. With this in mind the first author worked collaboratively with the classroom teacher to develop a series of eight, half-hour lessons delivered over an eight week period. These lessons experimented with teaching ideas by incorporating a range of representations to support the construction of varying levels of strategy sophistication. Prior to the introduction of the lessons the students were interviewed, the series of lessons were then given, and the students were interviewed again on completion of the lessons.

Participants: Two Year 2 students, Jan and Claire, and their classroom teacher are the subject of this paper. The students' selection was dependent on responses to the pre- and post-study interview questions that probed for both accuracy and flexibility in strategy choice while calculating addition and subtraction questions that had been presented verbally.

Instruments: The students were individually interviewed where they were asked 10 addition and 10 subtraction word problems. The first four questions for each operation were categorised as number facts (e.g., $4+4$, $10-5$) and the remaining six questions were categorised as mental computation (e.g., $23+20$, $25+23$, $46-10$, $30-19$). These problems included one and two digit numbers and increased in difficulty for each operation. As the interviewer asked the questions the students were presented with stimulus cards that included both pictures and numerals (see Figure 2). The interviewer utilised the Van der Heijden (1994), predetermined scaffolding by asking specific questions of students who experienced difficulty. The interviews did not exceed twenty minutes.



What is the total cost of the CD player and the CD?

Figure 2. Example of addition mental computation item.

Data Collection: Data collection included *observation* where each of the eight lessons were video recorded and transcribed, *field notes* were taken by the first author during lesson observation, and *pre- and post-study interviews* were conducted on a one to one basis with each student. *Teacher and researcher reflections* were also discussed and recorded after each teaching episode. These discussions resulted in a responsive and intuitive interaction in the instructional program, guiding the preparation of the subsequent lesson the following week. This is consistent with the methodology of Steffe and Thompson (2000).

Data Analysis: The transcribed lessons were combined with the field notes to develop a ‘rich’ picture of teacher actions and researcher analysis of classroom interactions. These notes were then analysed to enable the classroom discourse to be categorised allowing conjectures about teacher actions that promoted student generated mental computation strategies to be made. Once video analysis was completed the researchers then turned to the pre- and post-study interviews to categorise the strategies used by the students and to make judgements about the flexibility of the students’ strategies. The students could then be classified as being flexible or inflexible and accurate or inaccurate mental computers. This scrutiny allowed the researchers to make conjectures about possible pathways of student learning.

The two groups of questions asked during the interviews, number fact and mental computation questions, each have a range of possible strategies. The categorisation of strategies used to answer the eight number fact questions was identified using the strategies listed in Table 1. The remaining 12 questions that focused on mental computation were analysed for strategy choice, flexibility and accuracy. The strategies listed in Table 2 guided this categorisation.

Table 1
Number Fact Strategies

Strategy	Example
<i>Count using fingers</i>	4+3: 5, 6, 7 (using fingers)
<i>Count, for example</i>	<i>count all</i> 3+2: 1, 2, 3, 4, 5.
	<i>count on</i> 3+2: 4, 5.
	<i>count back</i> 8-3: 7, 6, 5.
<i>Derived facts strategies (DFS), for example</i>	<i>use doubles</i> 8+7: 7+7=14, 14+1=15.
	<i>through 10</i> 8+5: (8+2)+3=13.
	<i>use addition (for subtraction)</i> 15-8: 8+7=15, ∴ 15-8=7.
<i>Immediate fact recall</i>	<i>use another fact</i> 9+3: I know that 9+2=11, so 9+3=12.
	8+4: answer 12.

Table 2

Mental strategies for addition and subtraction (based on Beishuizen, 1993; Cooper, Heirdsfield & Irons, 1996; Reys et al., 1995; Thompson & Smith, 1990)

Strategy	Example
<i>Separation</i>	<i>Right to left (u-1010)</i> 28+35: 8+5=13, 20+30=50, 63 52-24: 12-4=8, 40-20=20, 28 (subtractive) : 4+8=12, 20+20=40, 28 (additive)
	<i>Left to right (1010)</i> 28+35: 20+30=50, 8+5=13, 63 52-24: 40-20=20, 12-4=8, 28 (subtractive) : 20+20=40, 4+8=12, 28 (additive)
	<i>Cumulative sum or difference</i> 28+35: 20+30=50, 50+8=58, 58+5=63 52-24: 50-20=30, 30+2=32, 32-4=28
<i>Aggregation</i>	<i>Right to left (u-N10)</i> 28+35: 28+5=33, 33+30=63 52-24: 52-4=48, 48-20=28 (subtractive) : 24+8=32, 32+20=52, 28 (additive)
	<i>Left to right (N10)</i> 28+35: 28+30=58, 58+5=63 52-24: 52-20=32, 32-4=28 (subtractive) : 24+20=44, 44+8=52, 28 (additive)
<i>Wholistic</i>	<i>Compensation (N10C)</i> 28+35: 30+35=65, 65-2=63 52-24: 52-30=22, 22+6=28(subtractive) : 24+26=50, 50+2=52, 26+2=28 (additive)
	<i>Levelling</i> 28+35: 30+33=63, 52-24: 58-30=28 (subtractive) : 22+28=50, 28 (additive)
<i>Mental image of pen and paper algorithm</i>	Child reports using the method taught in class, placing numbers under each other, as on paper, and carrying out the operation, right to left.

RESULTS AND DISCUSSION

During the series of eight lessons four models were used to support student learning. The models used were (1) hundred chart, (2) bundling sticks, (3) number ladder and (4) number line incorporating number lines with graduations of 10 marked from 0 – 100 and number lines with graduations but no numbers marked. The number line was selected as a model as it has successfully been used for the development of mental computation strategies (e.g., Gravemeijer, 1994; Klein, Beishuizen, & Treffers, 1998). The teacher chose to use the hundred chart as she had already introduced it to the students as an introduction to 2-digit number study. Bundling sticks have been typically used in Year 2 in Queensland to develop place value concepts (MAB were usually used in Year 3). While there was no intention to develop place value concepts, it was decided that the bundling sticks might be used to develop *aggregation* strategies. The number ladder was the teacher's "invention", the purpose of which was to present jumping forwards and backwards in multiples of ten (an additional model to the number line). The use of these models allowed the teacher to engage her students in mathematical discussions as all students used the models to support their growing understanding of mental computation. This teacher having participated in several professional learning episodes with the first author also considered her actions and response to student discussion in light of the process that proficient mental computers employ when calculating (Figure 1).

Teacher Actions

The teacher spent considerable time discussing with her students and questioning them on the patterns inherent in the numbers they worked with while using the various representations. Lesson one immediately orientated the students to this way of thinking with the teacher having the students identify the pattern in numbers and how counting in tens starting at any position can be achieved. Each child, working with their own hundred board, was asked to identify a pattern. The teacher's main interest was to establish that there were multiple ways of calculating.

- Teacher: We counted on ten from nine and we got to nineteen. Let's count on ten more. Where will that take us? Look for the pattern. Let's start at nine.
- Whole Class: 19, 29 ... 99.
- Teacher: What is happening with this pattern?
- Student A: They are all in the same row. (Student means column)
- Student B: They all end in nine.
- Student C: They are all counting in tens.
- Teacher: Yes, all good answers. Well done.

Opportunities to discover separation strategies were offered by the teacher. The accurate use of these strategies relies on developing a secure numeration knowledge encompassing multiplicative, canonical and noncanonical aspects of numeration. The hundred board, number ladder and number line were each an effective modality (visual and kinaesthetic) providing unambiguous models for the students to use. For example;

- Teacher: Put your marker on the number ten more than twenty-four. Student D?
- Student D: Thirty-four.
- Teacher: How did you find ten more than twenty-four?
- Student D: I just went straight under twenty-four.
- Teacher: Why did you go under twenty-four?
- Student D: Because that is the same as counting on ten.

The teacher also wanted to provide the students with examples of where wholistic compensation would be an effective strategy, but in order to do this, she focused on developing her students' number and operation sense by looking at the effect of changing the addend and subtrahend. At first the teacher ascertained her students' existing strategies to add on 9 or take away 9. Realising her students were using inefficient methods she demonstrated another method.

- Teacher: I am on ten but I only want to jump forward nine spaces. Who can think of a really fast way to do that?
- Student E: Go diagonally. (See below for further discussion)
- Teacher: Does anyone have another way?
- Student F: I counted in three's.
- Teacher: You were very clever to do that Student F. Now I am going to show you my way. I could add on ten and that will get me to twenty but I only want to add on nine so I just go to the number before twenty and that is nineteen.

The students quickly adopted this method as one child, Student G, demonstrated.

- Teacher: This time I want you to add on nineteen to seventeen.
- Student G: Thirty-six.
- Teacher: What did you do Student G?

Student G: I went down and then down and then back one.
 Teacher: What does down and down mean?
 Student G: Adding on two tens – which is twenty. Then you go back one.
 Teacher: Student D, can you go diagonally when you add on nineteen?
 Student E: No.

During the course of the eight week intervention the teacher provided opportunities for her students to develop a range of mental computation strategies. Most opportunities focused on addition with considerably less time being spent on subtraction. The questioning techniques employed by the teacher enhanced the students' opportunities to learn as she used a range of techniques. Firstly she used a technique that was designed to orientate the students where general questions were asked and students encouraged to make a contribution to the class discussion. The second technique involved direct questioning where selected students were to contribute their computation strategy while the third technique involved the teacher combining questioning and modeling. This allowed her to share her own thinking process and model that process with the representation being used at the time.

Jan and Claire's Responses to the Number Fact Questions

The first four questions for addition and subtraction were classified as number fact questions. While there was no emphasis on number facts strategies in the teaching experiment, the teacher reported focusing on the development of these strategies in other mathematics lessons. However a baseline of student performance was achieved by including these questions in the pre- and post-study interview schedule. The results achieved by Jan and Claire are listed in Table 3 below.

Table 3
Pre- and Post-study Interview Results: Number Fact Questions

Factors	Number Fact Accuracy		Number Fact Strategy	
	Pre	Post	Pre	Post
Jan	Addition: 100%	100%	Count on using fingers, immediate fact recall	Count on /back, use doubles, through 10, immediate fact recall
	Subtraction: 25%	75%		
Claire	Addition: 100%	100%	Count on/back, doubles, immediate fact recall	Count back, use another fact, use doubles, immediate fact recall.
	Subtraction: 75%	75%		

Both Jan and Claire achieved 100% accuracy for the addition number fact questions on the pre- and the post-study interviews. By the post-study interview, Jan had eliminated the use of her fingers when counting on and Claire only used count back when subtracting. Examination of the transcripts reveals progress in the sophistication of the strategy chosen for the number fact questions. When asked the addition question 6+5 Jan responded:

Jan: Five plus five equals ten and I plus one. That equals eleven.

Jan had successfully used the doubles strategy. When Jan calculated 5+9 she said:

Jan: Ten plus five equals fifteen but I have to take one. That equals fourteen.

In this instance Jan had changed her strategy from pre-study interview of counting on by ones to utilising the *through ten* strategy. Claire too was able to analyse her metacognitive process as she calculated her answers. For the question 7+2 she changed from giving an immediate response to being able to describe the use of through 10.

Claire: Seven plus three equals ten. Less one equals nine.

While for 6+5 she applied her knowledge of doubles to calculate;

Claire: Double five is ten. Add one more is eleven.

These students had become very strategy focused. Jan and Claire’s ability to verbalise their metacognitive processing allowed the interviewer access to that information. Prior to the eight intervention lessons the students were unable to discuss their strategy choice. For more information, the interviewer had to rely on her observations of the students’ actions, such as using their fingers while *counting on*.

When the number fact responses involving subtraction were examined, the same mental dexterity that provided accuracy for addition was not evident; however, a range of strategies was utilised. On the pre-study interview Jan was unable to *count back* using her fingers to support the calculation. For 6-2 and 13-4 she considered the starting number as the first one subtracted. So for each of the above questions she calculated 5 and 10 respectively. Yet, when calculating 10-5 on the pre-study interview, she used a *DFS – use doubles*, to assist her correct calculation. By the post-study interview Jan had correctly ascertained the *count back* strategy giving her the correct answer for each and she again used the doubles strategy for 10-5. However, for 15-9, Jan was able to verbalise that 9 was close to 10. Unfortunately she was unable to make progress beyond that point. In the end she said, “I can’t do it.”

Claire, on the other hand, achieved 75% accuracy on her pre-study interview and did not improve on her post-study interview. However the strategies she used indicated a growing depth in understanding. Claire used the inverse operation to calculate 6-2, and for 15-9 she too verbalised that 9 was close to 10 but like Jan failed to implement the strategy:

Claire: It is close to ten. Fourteen, thirteen, twelve... (counting in ones backwards arriving at the incorrect answer).

Jan and Claire’s Responses to the Mental Computation Questions

Both students improved significantly in accuracy from pre- to post-study interview for the mental computation questions (Table 4). However their accuracy rate was still not of a level that could be categorised as accurate mental computations. On the other hand the range of strategies they employed to calculate their answers from the pre-study interview to the post-study interview indicated considerable growth, sufficient to categorise these students as flexible mental computers. Their critical awareness of strategy choice as well as knowledge of the range of strategies from which to choose was fundamental in allowing the students to transfer this knowledge to larger number domains. This was evident when the students used addition in the mental computation questions. However these particular students were not able to transfer this knowledge as easily to the operation of subtraction, as their transcripts demonstrate.

Table 4

Pre- and Post-study Interview Results: Mental Computation

Factors	Mental Computation Accuracy		Mental Computation Strategy	
	Pre	Post	Pre	Post
Jan	Addition: 16.6%	83%	Guessed, Counted on with fingers	Count on/back by ones; Separation L to R; Wholistic compensation
	Subtraction: 0%	33.3%		
Claire	Addition: 16.6%	83%	Counted on/back by ones	Count on/back by ones; Separation R to L and L to R; Cumulative sum; Wholistic compensation
	Subtraction: 0%	50%		

Again both girls demonstrated a large discrepancy between the accuracy rates for addition and subtraction. The mental computation questions as a whole were very difficult for the students during the pre-study interview with both students counting on by ones, guessing or not attempting to perform a calculation as they said that the questions were too difficult. Although Jan counted on by tens for the question $20+30$ and correctly calculated the answer, she became confused when counting on by fives for the question $23+20$, stating that the question was too hard. This situation changed significantly by the time the students were interviewed at the conclusion of the study.

The change in accuracy for the addition mental computation questions from the pre-study interview, where the students only got one question correct, to the post-study interview where they got all but one question correct displayed a growing capacity for mental computation. However of greater interest was the range and sophistication of method that the students employed. Jan did not change her method of calculation for $20+30$ counting in tens as she had on the pre-study interview. However Claire identified the number pattern and its multiplicative nature and utilised it well. This type of reasoning is fundamental to the progression of sophisticated mental computation.

Claire: Well two plus three equals five so twenty plus thirty equals fifty.

For $23+20$, the students used jumping forward by tens. Jan articulated this nicely saying,

Jan: Ten plus twenty three is thirty-three and another ten is forty - three.

However she became confused with $25+23$ and ultimately decided she could not answer the question. Claire, on the other hand, demonstrated her canonical understanding as she used *separation right to left* articulating:

Claire: Five plus three equals eight. Double two equals four. Four tens. That's forty-eight.

An appreciation of the effect of changing the addend was highly developed for these students, with Jan using this understanding to support her calculations to $23+19$, $26+9$ and $36+99$ each time using wholistic compensation. For example:

Jan: Ninety-nine, that's one hundred. One hundred plus thirty-six is one hundred and thirty-six. One less is thirty-five.

Interviewer: Is it thirty-five?

Jan: No, one hundred and thirty-five.

Claire used wholistic compensation but she also utilised the strategy of cumulative sum to assist with the calculation of $26+9$:

Claire: Nine is close to ten. Add ten to twenty, that's thirty. And six equals thirty-six.

The students had experience using *wholistic compensation* in the classroom; however, it was mainly focused on addition. Therefore, difficulties were encountered when attempting to translate this strategy across to subtraction during post-study interviewing. Both Jan and Claire experienced difficulty with the subtraction questions. Jan only correctly answered one of these questions where she easily counted by ten. Three of the questions she decided she could not do even with prompting from the interviewer. But when she tried to use a strategy, she became confused, as the following dialogue shows: $46-19$

Jan: Twenty take forty-six equals 26. Less one is 25.

Interviewer: Do you take one or add one.

Jan: Um, you take one.

An understanding of the effect of changing the subtrahend appeared to be the major factor impacting on Claire's number and operation sense; although, two questions earlier, Claire was able to accurately calculate $38-14$, as the numbers allowed her to choose an appropriate strategy. This strategy relied on her canonical understanding:

Claire: Take away one ten; that equals twenty. Eight take away four equals four left. The answer is twenty four.

This response may indicate that Claire exhibited signs of developing an understanding of number and operation. However, when her responses to the questions that followed are examined, it would indicate that she reverted to a “buggy algorithm”. This would signify that she did not have a secure understanding of numeration. For example when Claire calculated 30-19 she said:

Claire: Thirty take one ten is twenty. Um nine ones left, but there is no number behind the thirty. I don’t know what to do.

Relating Results to Flowchart of Proficient Mental Computers

Conjectures about possible pathways of student mental processing when mentally computing can now be made by relating these students’ strategy choice to the flowchart of the proficient mental computers developed by Heirdsfield (2001) for students who were flexible and accurate mental computers (Figure1). However, this flowchart was based on results of interviews conducted with Year 3 students; consequently, the flowchart was modified to accommodate younger students. Results of the current study were compared with the results of Heirdsfield (2001). It was evident that the aspects of accurate and flexible mental computation that were *not* present in the process of mental computation of these inaccurate but flexible mental computers were:

1. a knowledge of number and operation (which impacts on the student’s understanding of the effect of changing the addend and the subtrahend);
2. the ability to compute number facts with speed and accuracy (these students possessed some number fact strategies, but could not quickly and accurately solve number facts); and
3. some aspects of numeration (noncanonical and multiplicative understandings).

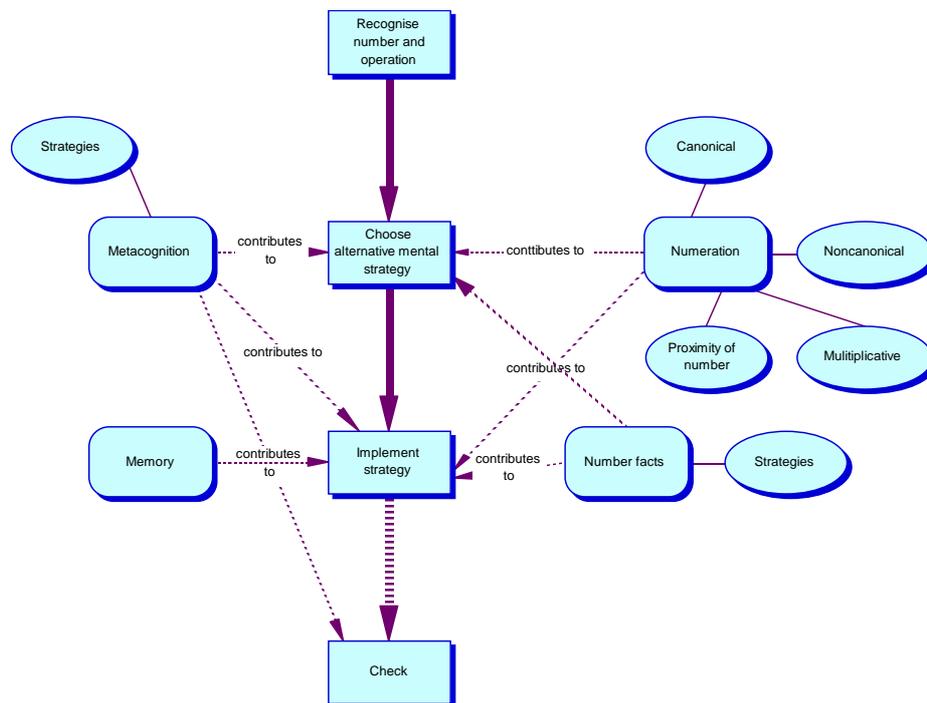


Figure 3. Flowchart indicating factors affecting the mental computation process of flexible but inaccurate mental computers (dotted lines indicate weak connections)

The pathways taken by these students can therefore be represented by the flowchart above (Figure 3) that indicates the possible factors impacting on students’ ability to accurately mentally compute one and two digit addition and subtraction questions. The dotted lines indicate weak or developing

association between concepts. These weak links therefore highlight to the teacher possible avenues for future teacher action.

CONCLUSION

This study has been able to both confirm findings from earlier research as well as contribute new ideas to understanding how young children develop mental computation strategies in the process of becoming both flexible and accurate mental computers. These two Year 2 students used lower order strategies such as *counting on* using their fingers inaccurately when asked to mentally compute one and two digit numbers prior to intervention, confirming the finding that weaker students use lower order strategies (e.g., Beishuizen, 1993; McIntosh & Dole, 2000; Reys, Reys, Nohda, & Emori, 1995). However they were able to use a range of sophisticated strategies on post-study interviewing for addition, but reverted to *counting on/back* when asked to subtract as their links to number and the effects of changing the subtrahend were weak. This study found that addition was easier for the children to grasp than subtraction, supporting the finding by Fuson (1984) and Thornton (1990). The benefits of teacher actions that would see developing an understanding of the association between the effects of changing the addend and the subtrahend when discussing strategy options may assist in further developing this otherwise weak link that is preventing these two students from developing more advanced strategies for subtraction.

In the eight, half hour lessons, conducted over an eight week period these students enthusiastically embraced the concepts of mental computation and have been able to successfully select from and implement a range of strategies. This success can in part be attributed to the representations used (Cheng, 2000). This teacher keenly engaged her students in active discussions where representations (e.g., hundred chart, number lines, number ladder) chosen directly related to the mathematical concepts being studied. These representations utilised both visual and kinesthetic modalities as recommended by English (1997) and Davis & Maher (1997). Further achievement in accuracy will be possible with ongoing participation in this environment, which encourages discussion, uses a range of representations, and where the teacher is informed of the connections made by proficient mental computers.

The inclusion of mental computation as the main form of computation is new to the *Year 1-10 Mathematics Syllabus* (QSA, 2004). Consequently Queensland's teachers will have to access professional development in the content and pedagogy of mental computation. It will be essential for this PD to alert teachers to the connected web of concepts that are outlined in Figure 1. With this knowledge teachers will then be well placed to intervene with appropriate teacher actions if students struggle to develop strong links between concepts, as these links are fundamental to both efficient and accurate mental computation.

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