Title Page

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Evidence of *Implemented Anticipation* in Mathematising by Beginning Modellers

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Abstract Data from open modelling sessions for Year 10 and 11 students at an extra-curricular modelling event and from a Year 9 class participating in a program of structured modelling of real situations were analysed for evidence of Niss's[1] theoretical construct, *implemented anticipation*, during mathematisation. Evidence was found for all three proposed aspects. With respect to Niss' s enablers of ideal mathematisation explaining unsuccessful mathematisations, flaws in the modelling of the Year 10-11 students were related to the required mathematics being beyond the knowledge of the group members or poor choice of the particular mathematics to use in the modelling context; whilst unsuccessful attempts at mathematisations in the Year 9 class were related to inability to use relevant mathematical knowledge in the modelling context. The necessity of these enablers as requisites for modelling, particularly in a classroom context, needs further investigation.

Keywords *Mathematisation . modelling . implemented anticipation . secondary*

Mathematical applications and modelling of real world situations are receiving increased emphasis in several curricular and educational standards documents internationally currently in countries as diverse as USA (Common Core State Standards Initiative 2010), Ireland (National Council for Curriculum and Assessment 2012), Switzerland (CIIP 2010) and Singapore (Ministry of Education 2006). In the Common Core Standards, for example, modelling is identified as a standard for mathematical practice that teachers should seek to develop in students thus enabling them to access and use existing mathematical knowledge in solving real problems. Despite changes in the popularity of applications and modelling in various curricula, the teaching and learning of applications and modelling has been a vibrant research field for many years as is evident from the 14th ICMI study (Blum et al. 2007) and overviews in books produced by the International Community of Teachers of Mathematical Modelling and Applications (ICTMA) (e.g., Kaiser et al. 2011; Niss 2001; Stillman et al. 2013) and research handbooks (Williams & Goos 2013). Two research areas receiving on-going attention (e.g., Brown 2013; Grigoras et al. 2011; Schwarzkopf 2007) are mathematisation (i.e., translation into mathematics) of the idealised problem formulated from the real situation and the reverse process, de-mathematisation (i.e., interpretation of mathematical outputs of modelling in the real situation). In 2010, Niss added to the theoretical models informing this research within the context of mathematical modelling. The authors acknowledge that other conceptualisations of mathematisation and de-mathematisation exist and

[1] All references to Niss refer to Mogens Niss except where explicitly stated otherwise.

are used in mathematics education literature, for example, as a didactical principle (e.g., Treffers 1991) or as a means to critique the “social availability of mathematical knowledge” (Jablonka and Gellert 2007, p. 1). These are not the focus in this article.

Mathematising has been promoted by the mathematics frameworks for PISA 2003-2012 assessing the mathematical literacy of 15 year olds ostensibly at the end of compulsory schooling. In the earlier frameworks what was presented as mathematisation was the entire mathematical modelling cycle (see OECD 2009, p. 105). For PISA 2012 the mathematical modelling cycle was described as a key feature (OECD 2010) but mathematising was less prominent. It is one of seven fundamental mathematical capabilities that underpin the framework. Mathematising is taken to mean the fundamental mathematical activities that are involved in “transforming a problem defined in the real world to a strictly mathematical form…or interpreting or evaluating a mathematical outcome or a mathematical model in relation to the original problem” (OECD 2010, p. 18). The latter is termed “de-mathematisation” by Niss (2010). PISA views mathematising as a cognitive capability that can be learnt through schooling enabling students to understand and engage with the world mathematically. This view of mathematisation is more in keeping with the work of Blum, Galbraith, Kaiser, Maaß, Stillman and members of ICTMA than its use by others such as de Lange (1989). The purpose of this article is to demonstrate whether or not there is empirical evidence for the main explanatory construct of Niss’s model of mathematisation processes (Niss 2010, p. 57), namely, what he calls “implemented anticipation” (p. 56).

Theoretical Framework

Diagrams of the “so-called” modelling cycle (e.g., Niss 2010) are often used by researchers to discuss what is happening at a task and mental level during modelling. These are mere simplifications but are meaningful communication tools amongst international researchers. They can also scaffold the modelling activity of beginner modellers. Fig. 1 reproduces the Niss diagram. This representation of modelling shows two disparate domains: the amoeba-like, ill-defined extra-mathematical domain (including the real world situation of interest that is cut and trimmed to the modeller’s idealisation of this) and the sharply defined mathematical domain. Doerr and Pratt (2008) suggest that this separation of “the experienced world of phenomena from the constructed world of the model” is a result of the epistemological stance that is at the heart of modelling, namely, “that the world of phenomena and the model world co-construct each other” (p. 260). Others such as Stillman (1998) argue for the importance, particularly in the schooling context, of also including a representation of the blending of the real world and mathematical world supporting students in mathematisation and de-mathematisation.

In Fig. 1 idealisation (formulation of an ideal problem from the real situation) occurs through making assumptions and identifying essential elements or features in the situation which are of interest that are then formulated, that is, specified into a problem statement which may take the form of, or include, a question or questions. The idealised situation is mathematised through translation into mathematics into the mathematical domain. The mathematical domain includes the mathematical model(s) that has been made of the situation, mathematical questions posed and mathematical artefacts (such as graphs and tables) that might be used in solving the mathematical model. Mathematical outputs (i.e., answers) need then to be de-mathematised, that is, interpreted in terms of the idealised situation and the real situation which sparked the modelling in the first place (i.e., back into the extra-mathematical domain). These can then provide answers to questions posed about the real situation in the extra-mathematical domain or stimulate a further cycle of modelling if inadequate for this purpose.

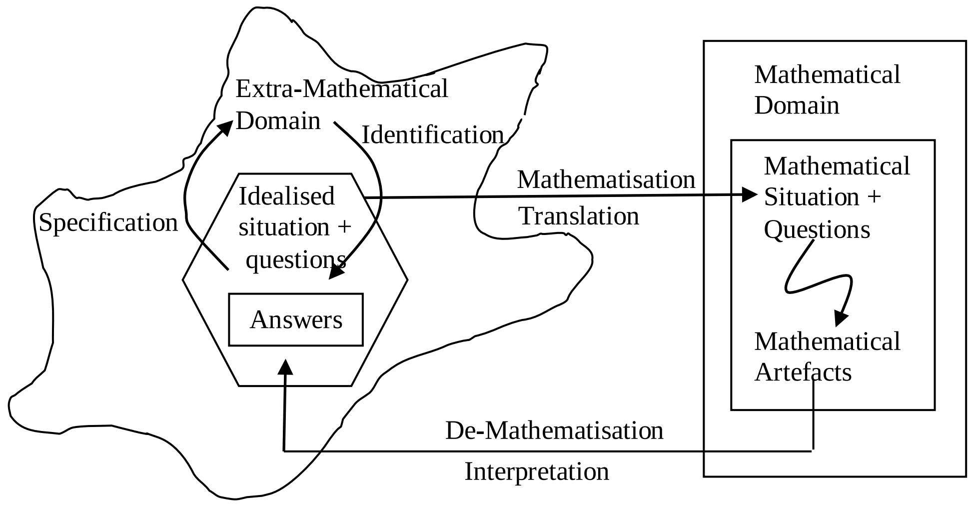

Fig. 1 Modelling processes (after Niss 2010, p. 44)

Implemented anticipation

The notion of anticipating as a characteristic thought form in mathematical contexts has been proposed in several areas of mathematics such as in the development of algebraic thinking (e.g., Boero 2001, p. 99). In order to produce a theoretical model of the mathematisation process, Niss (2010) uses this idea but coins the term, “implemented anticipation” (p. 54). The use of the past tense in “implemented” is deliberate as successful mathematisation, from Niss’s perspective, involves not only anticipating what will be useful mathematically in subsequent steps of the cycle (see Fig. 2) but also implementing that anticipation in decision making and carrying through of actions that bring to fruition those next steps.

- Firstly, the idealisation and specification of the real situation from the extra-mathematical domain involves implementing decisions about what elements or features are essential as well as posing any related question or statement of the problem in light of their anticipated usefulness in mathematising.
- Secondly, when mathematising this formulation of the problem situation the modeller needs to do this by anticipating mathematical representations and mathematical questions that, from previous experience, have been successful when put to similar use.
- Thirdly, when anticipating these mathematical representations, the modeller has to be cognisant of the utility of the selected mathematisation and the resulting model in future solution processes to provide mathematical answers to the mathematical questions posed

by the mathematisation. This thus involves anticipating mathematical procedures and strategies to be used in problem solving after mathematisation is complete.

The arrowed path in Fig. 2 represents this three step foreshadowing and feedback that is captured in successful implemented anticipation. There is an obvious correspondence between this foreshadowing of the results of future actions being "projected back onto current actions" (Niss 2010, p. 55) and Treilibs' emphasis on a "sense of direction" as being of crucial importance in modelling (1979, p. 142). Maaß (2006) also concluded from her work with Year 7 students that a sense of direction during modelling is one of the factors influencing modelling competencies.

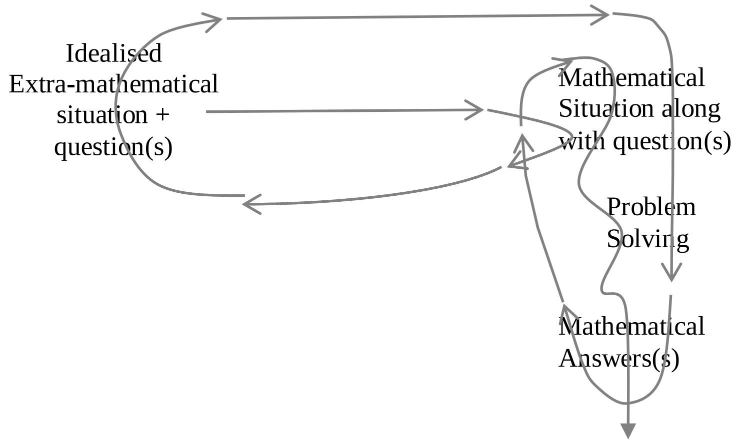

Fig. 2 Niss's model of ideal mathematisation (after Niss 2010, p. 57)

Niss (2010) identifies four enablers to successfully using implemented anticipation in mathematising a real or realistic situation. These are that modellers need to: (1) possess relevant mathematical knowledge, (2) be capable of using this when modelling, (3) believe a valid use of mathematics is modelling real phenomena, and (4) have perseverance and confidence in their mathematical capabilities (p. 57). It is reasonable to expect that new modellers, especially, would experience the challenge of ideal mathematising and have difficulties related to the three aspects of implemented anticipation. Furthermore, these difficulties could possibly be explained by lack of one or more of these enablers. As successful and unsuccessful attempts at applying mathematical knowledge to real situations provide opportunities for developing deeper "metaknowlege about modelling and mathematisation, in particular" (Schaap et al. 2011, p. 145), both should be the foci of any study of mathematisation.

Researchers (e.g., Schaap et al. 2011; Sol et al. 2011; Stillman et al. 2010) have found evidence of beginning modellers in secondary schools having difficulties with mathematising because of impeding formulations of the problem statement. However, Niss has confirmed that no one to date, other than ourselves in a shorter report (Stillman and Brown 2012), has published attempts to use his model of ideal mathematisation in analysing classroom or other data[2]. In this article the model is the basis for analysis of data from firstly groups of Year 10-11 students at an extra-curricular modelling event, then secondly, a year 9 class of beginning modellers who participated in a program of quite structured modelling over one year.

[2] Martin Niss (2012) alludes to a similar need for further research into mathematisation processes in the context of viewing real world problem solving as modelling in Physics.

To operationalise the mathematising construct for research purposes we take as its starting point the formulated statement of the problem situation. This may or may not be formulated as a question. The end point of mathematising is the mathematical model.

The research questions to be addressed are:

1. To what extent is there evidence for the existence of Niss's *implemented anticipation* in mathematisations by beginning modellers?
2. Do Niss's four enablers explain unsuccessful mathematisations by beginning modellers?

Data Sources

As our purpose is to provide "paradigmatic cases", to use Freudenthal's term (1981, p. 135), for the existence of Niss's construct in theorising mathematisation in modelling tasks, data examined to answer the questions come from two sources—one that allowed open modelling where students chose the task and followed their own solution pathways and a classroom context where students worked on a task chosen by the teacher. As tasks in the classrooms where we have had access as researchers have been formulated already as mathematical statements or questions by the task setter, a context where students were allowed to choose their own situations to investigate was needed in order to look for evidence of the first aspect of the Niss model, hence the need for two data sources. During the annual AB Paterson Modelling Challenge, on the Gold Coast, Queensland, Australia, Year 10-11 students are allowed such freedom. These students are expected to engage in problem finding (Getzels 1979) and problem posing (Stillman in press) as well as other aspects of modelling. Two groups were purposefully selected (Flick 2006) as being most relevant to the purpose of the first research question (Richards 2005, p. 41) which was to provide evidence of the existence of "implemented anticipation" if indeed it did exist. The modelling of two groups, one each from the 2009 and 2011 Modelling Challenge, is to be examined for evidence of implemented anticipation occurring in an open modelling context. The work of the two groups selected was considered by the researchers as likely "to yield the most information about the phenomenon of interest" (Merriam 2002, p. 20).

The second data source is a classroom from the RITEMATHS project. As part of this project, a series of three modelling tasks were used in a class of 21 Year 9 students. The data analysed are from the implementation of *Shot On Goal*. The main part of *Shot on Goal* offered structured scaffolding for student modelling so this part of the task would be seen by Niss (2010) as largely "(pre) mathematized by the presenter of the task" (p. 47). Niss's enablers 1, 3 and 4 are still needed to select relevant information and to subject this to mathematical problem solving. Two further questions at the end of these tasks included no details of how to mathematise or approach them mathematically and these are the focus of analysis for this article in addressing the research questions.

Evidence from Open Modelling Context – Year 10/11 in Extra-curricular Event

Students participating in the Modelling Challenge come from many different schools. The researchers are university academics with expertise in facilitating modelling activities who have participated in this event as mentors of Year 10-11 students for several years. Mentors facilitate students' pathways in modelling rather than impose particular approaches or choose situations for groups to model. Thus, mentors intervene as little as possible with the role of the mentor being to enable students to have opportunities to evaluate their own ideas in a productive manner (Doerr 2007). The Challenge commences for these students by an introduction to modelling by one mentor followed by working for approximately 2 hours in small groups on one common modelling task (e.g., optimum location of a high care facility for aged persons) chosen by the mentor to ensure a common understanding of what constitutes modelling. Next, each group freely chooses its own real situation to investigate then work on these for approximately 9 hours over two days.
At the 2009 Challenge, the first author mentored 16 students in 4 mixed groups of 3-5 students from different schools in south-east Queensland and/or Singapore. A group of 5 Australian students (three Year 11 and two Year 10) who were interested in use of alternative renewable energy sources was chosen as focus. At the 2011 Challenge, the same researcher mentored 18 students in 5 such mixed groups of 3-4 students. A group of 3, one Year 11 from Singapore and two Australian Year 10 students, was chosen. Data collected relevant to the focus of this article consisted of transcriptions of video-recordings and digital photographs. A Flip camera was used to video groups interacting with themselves and the mentor in the allocated classroom and the computer laboratory. Groups were photographed at various intervals throughout the two days as they explored possibilities for a situation to model, collected data once a problem was decided on, constructed posters to display their modelling solution and presented their modelling to the class and mentor. In addition students from both groups gave their finished posters and sheets of rough working to the researchers for further analysis.

Feasibility of Wave Power for an Island Resort

The group of 5 students were interested in use of alternative renewable energy sources other than solar, nuclear and wind power. They saw as an alternative the use of wave power through collecting "kinetic and potential energy from energy from the waves and converting it into electricity for general use." They decided to limit the scope of their investigation to providing power for Fraser Island or "similar sized islands". Fraser Island is a large sand island off the Queensland coast attracting tourists and recreational anglers. Electricity usage was estimated as 4,666 kWh per year per household based on the average consumption of all Australians (sourced from the internet). They initially decided the wave station would be located 2km from the shoreline. The question posed was: "*Will wave energy be sufficient to satisfy Fraser Island's power consumption needs?*" The variables deemed of relevance were "amplitude (A), period (T) and wavelength (λ) of waves (which is affected by where the wave station is placed in terms of distance from shore, weather, wind, [and] movement of tectonic plates).” Average values for these three parameters were sourced from the Australian Bureau of Meteorology for near shore waves 50m from shore and ocean waves 2km from shore. Assumptions identified included there was “little variation in the height and energy of each wave in the relevant location”, “the power station obtains the same amount of power at all times”, “no loss due to resistance” during transferring power to land, “weather conditions … have no impact on the strength and height of the waves” and “the wave power station runs consistently at 100% efficiency”. Awareness that some of these assumptions were over simplifications was indicated by both Gen and Tom[3]:

> Gen: …we had to make a few assumptions: that the waves in the area were consistent and that there was no variation in their height, speed … . I know that is unrealistic but in order to build models for this topic we had to make that assumption. [Video, Day 2]

> Tom: … we had to make the assumption that the waves would be continuous because if a wave was too small, or too big even, it would not cause the crank to turn properly and therefore the entire power supply would be interrupted and it wouldn’t work if it was the sole power supply for an island. [Video, Day 2]

A single “bob” model for the power station generator was used although it was noted in limitations of the modelling that “the frequency of the electricity output would be very low (typically 0.1 Hz) as the buoy follows the frequency of the wave whereas typical power output of household wall sockets is typically 50-60 Hz. Therefore for this model to be suitable for household consumption, the transportation of electricity must be in DC electricity.” As Mei explained pragmatic concerns also influenced their choice of power generator:

> Mei: There are other types of wave power stations that in reality can extract much more power from waves. It relies on more complex models. Due to time constraints only a simple bob model was used. [Video, Day 2]

All models and calculations were based on near shore values. Jon explained that considering off shore waves further could have been a future refinement.

> Jon: Also if we were going to do this again, umm, we would have a look at the effects of changing the shape of the wave that we simulated [see Fig. 3] because the one that we simulated was one that was close in to shore.

> Kit: It was averaged.

> Jon: Yeah. If we were to look at ones further out at sea those are, they’re bigger, longer, long waves which are…more regular than coastal, the shore waves; however they obviously don’t move as fast as the coastal ones either. [Video, Day 2]

A statement in the strengths and limitations section of the poster report gave loss of power in longer transmission wires from the station as a further consideration in deciding to model in-shore waves only. The wave function $y = A\sin\left(\frac{2\pi}{T}x\right)$ with no phase shift or vertical translation was chosen, then specified as $y = 1.2\sin\left(\frac{2\pi}{5}x\right)$ and used as a model for the displacement (y) of the bob with respect to time (x) in the wave energy generator device. This function was then graphed using technology. To find the energy output from the device, the displacement function was differentiated with respect to time to give velocity and the kinetic energy formula used to model kinetic energy (KE) with respect to time as: $KE = \frac{1}{2}m\left(1.2\,\frac{2\pi}{5}\right)^2\cos^2\left(\frac{2\pi}{5}x\right)$. Jon explained the thinking behind their model on the video:

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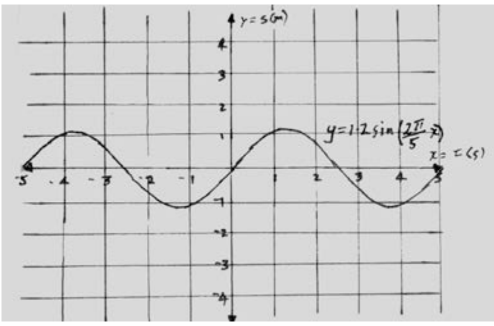
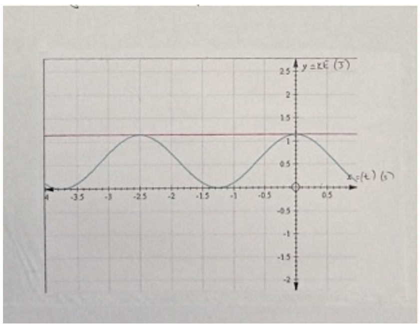
[3] Student names used throughout this article are pseudonyms.

> Jon: ... the way we went about modelling this [sic] data was simulating … a typical type of wave…. [pointing to Fig. 3a] here is a simulated wave [runs finger along sinusoidal graph] … the first thing that we did to get our model was to do the derivative of this equation [points to $y = 1.2\sin\left(\frac{2\pi}{5}x\right)$] that we made for our wave. The reason we needed the derivative was because if we look at the derivative of the periodic function [points to Fig. 3b] it is going to give us the velocity of the crest of the wave with respect to time [indicating up and down motion with hand] and we know the velocity of the crest of the wave and we have assumed that the bob follows the crest of the wave perfectly… we can work out the kinetic energy of … the bob [makes a fist] using the equation ½ m v^2. [Video, Day 2]

Graphs were also produced for velocity vs. time and KE vs. time where *m* was taken to be 1 so the shape of the graph could be inspected on a graphing calculator. "To work out the amount of power generated in 24 hours we can simply find the amount of KE generated in every period and then multiply this result by the number of periods in 24 hours."



(a)



(b)

Fig. 3 Graphs for algebraic models of (a) typical wave (b) velocity

To calculate the kinetic energy generated in one period they decided to "look at the ΔKE between each trough and each peak of the model". Argumentation was based on graphs of kinetic energy vs. time and maximum kinetic energy vs. time. The change in kinetic energy from peak to trough was said to be $m \times 1.13698$ and from this the total ΔKE in one period was twice this "because there are two peaks and because of the fact that, umm, the symmetrical nature of periodic functions" [Jon, Video Day 2]. This resulted in an amount of 39294*m* joules where *m* is the mass in kg of the bob riding on the front of the wave. Using the yearly electricity usage estimate, an average of 2.6 people per household (from internet source), and population of 300 they calculated 1470 kWh of energy were used on average daily. As Kit commented on the video, an extra 10% power usage was added to account for seasonal demand fluctuations and "public needs like street lamps". This increased average daily power usage to 1617 kWh.

Jon pointed out on the video that "in [their] working [they] did not put in the mass of the bob that was following the crest of the wave because [they] decided to solve for the mass of the, the bob that would be required to … meet Fraser Island's energy needs for one day." The mathematical question that was finally posed to mathematise the initiating question was: *What is the required mass of the bob for the wave generation device that will power the whole of Fraser Island?* The answer came after conversion of a previous result from Mega Joules to kWh and solving the model $1617 = \frac{39294m}{1000000 \times 3.6}$ giving m = 148145 kg. This output from their modelling had to be translated back to the real domain (demathematised) to answer their initial question about the sufficiency of wave energy. The group then concluded on their poster:

> … this result is ludicrous and unrealistic because building a bob or several bobs to make up 148.145 tonnes of floating mass [is not feasible]. Building enough power stations to make up this total mass would also be totally unrealistic because the costs of building would far outweigh the benefits of the wave power generation. Therefore, it can be concluded that wave power is an unfeasible way to generate electricity on Fraser Island.

There was some disagreement among group members on this point so what was recorded on the poster was more of a negotiated agreement.

> Kit: We got the mass needed to be 148 tons, so obviously that's really unrealistic because you know 148 tons would probably sink to the bottom of the ocean. Umm, so we probably suggest, umm, using multiple buoys with smaller masses to create the same amount of power [Jon's facial expression shows he disagrees].
>
> Jon: Even if we were to use [shrugging his shoulders] multiple buoys, the cost of that would be just unfeasible for the small needs of Fraser Island…
>
> Mei: The model is simple and can be considered a reasonable representation of waves in real life. This means that not a lot of time is required to figure out the calculations of the model. [Video, Day 2]

Mathematisations

We now take a closer look at what was involved in completing the mathematisation of the task. These interpretations have been cross checked through alignment of student dialogue, gestures and actions on the video record and photographs, student written work during the two days and the final poster. The initial structuring of the wave power situation requires anticipation of the potential involvement of mathematics and the nature and usefulness of this involvement with regard to the modelling purpose, namely, the sufficiency of the amount of power generated to meet consumption. As Jon pointed out, function and differentiation applied in an energy context were anticipated as being useful in representing the situation and the question posed about it. The sine function in particular was anticipated to be a satisfactory initial model for the periodic vertical displacement of the bob in the power station generator - a buoy generator following the frequency of the waves. The actual values to specify the parameters of this model for the location of the generator were anticipated to be available on the internet. Graphical representations generated with technological tools (graphing calculators) were also anticipated to be useful in analysing how the particular models involved for energy generation could produce output that would eventually be able to be used in answering their question. All students in the group were familiar with using these technological tools in mathematics. The modellers had to be able to envisage, at least in outline form, how they could progress from the model for displacement to a model for kinetic energy output and to decide whether or not they had the mathematical and technological skills within the group to carry through this modelling strategy. All but Mei were confident in this respect but she was clearly scaffolded by the others. Another complication in this example was that the mathematical question that was finally posed to mathematise the initiating question turned out to be about the mass of the bob required in order to generate the estimated amount of electricity consumed rather than the amount of energy the wave station potentially generated. Thus, in idealising the situation further specification had to be anticipated and conducted in order to model the energy consumption. The group members had to anticipate that this output from their modelling was what had to be translated back to the real domain (i.e., de-mathematised) to answer their initial question about sufficiency of wave energy to satisfy Fraser Island's power consumption needs.

Their modelling is not without flaws as the root mean square value should have been used in their energy calculation. This would have been outside their mathematical knowledge being Year 10 and 11 students. Alternatively, they could have sketched the graph of $(\text{velocity})^2$ and found its average value over a period and used that for the KE calculation. However, this involves an integral-based mean value calculation – again stretching their knowledge. Thus, in terms of Niss's enablers for successful implemented anticipation, relevant mathematical knowledge was outside the knowledge base of the group. Another area where their modelling could have been improved, and they probably had the skills to implement, was to do with energy lost. Gen indicated "we assumed that the process in which the power is transferred to the land there is no loss due to resistance because that has been known to happen, but in order to create all these models we had to assume that" [Video, Day 2]. They could have researched some typical values of the percentage of kinetic energy 'lost' in various transformation processes and used these, as they were obviously worried about some assumptions made. Time constraints could have precluded this.

Predicting the Longevity of Pop Stars and Bands

A group of one Year 11 and two Year 10 students in the 2011 Challenge chose to investigate the popularity of pop stars and bands. They were particularly interested in teen idol, Justin Bieber, the band Perry and the group Lady Antebellum. What intrigued them was that "all three Music makers have started to gain an increasing number of fans in the industry and the question remains, 'Will they stay popular till the end of time like Queen and the Bee Gees, or will they fade out of the music industry like many others?'" The group had chosen a real world situation to investigate that interested all members but they were not sure that it was particularly mathematical or what mathematics to apply. After much discussion they arrived at the idea of "Predicting the longevity of pop stars & bands" as the social phenomenon they would model mathematically. The group expected there would be relevant aggregated internet data available but they could not find any. Thus they needed to decide which variables were relevant to the situation themselves and collect their own data. Firstly, the group decided on some simplifying assumptions: 1. All pop stars are of similar appeal. 2. All music genres have the same appeal. 3. Bands and individual singers have the same longevity of appeal. They decided on career starting age, career length, number of hits and number of singles as the variables of interest. Although data were collected on albums, it was decided not to include albums as some singles and hits were in album collections as well. Data were collected for 20 artists or groups.

The group used scatterplots and various regressions using technology to look at trends in their data to see if they could use these for prediction (see Fig. 4). Firstly, they "compared the number of singles the artist made to the time they stayed in the music industry". They observed that there was "a very vague relationship but we could still conclude that 'Artists who made more singles enjoyed more longevity.'" They fitted a power function to these data. Secondly, they found "a similar but more distinct trend” when number of hits in the weekly top 100 chart (*y*) was compared to longevity of an artist’s career (*x*). They were able to conclude that “artists who have more hits in the top 100 generally stayed in the music industry for a longer period of time”. Thirdly, they attempted to model the data for career length and career starting age. A quadratic model was fitted and the group concluded that, “An artist had a higher chance of becoming successful if he/she started younger”. All models were selected by best fit by eye. Conclusions were based on the fitted model, rather naively at times. As evident on the video taken in the computer laboratory on Day 2, there was some discussion about the suitability of the models. Ben had produced three scatterplots and regression models that were displayed on his computer screen whilst Fleur and Carol discussed them and Fleur typed their interpretation on the computer next to Ben’s. She had already done this for Trends 1 and 2 but the scatterplot on display for the second pair of variables was Career length versus Hits in Top 100. The fitted model was clearly non-linear. Fleur indicated “That is the Log A data then you can analyse the power”. The variables were eventually reversed and the data were modelled by $y = 1.4\,x$.

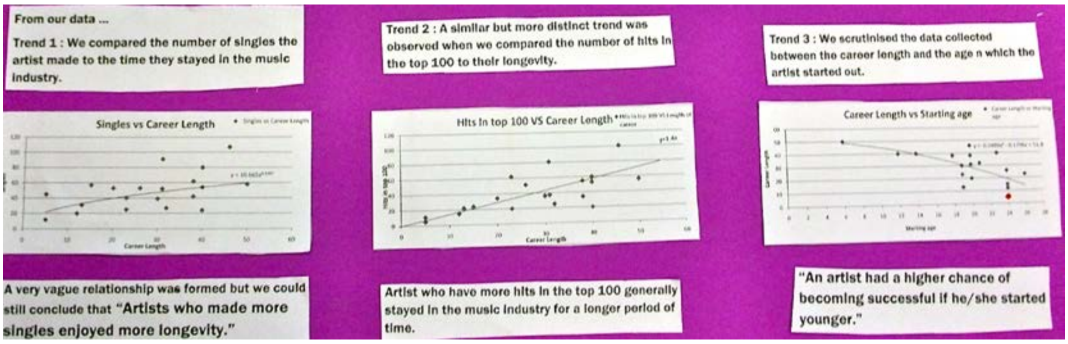


Fig. 4 Graphs and algebraic models of data from previous artists

These three models were then tested with the band, Powderfinger, to see if this combination of models allowed a reasonable prediction of “longevity in the music industry” through mathematical modelling. The band’s starting age was 20. From their modelling Powderfinger’s career should have spanned 28.6 years and produced 46.6 singles and 42.9 hits (an error as this should be 40 using the group’s linear model). These predictions were compared with actual data of 29 years, 25 singles and 23 hits, respectively. The group concluded on their poster that: “Our predictions for the longevity was quite accurate however we over predicted the number of singles and hits. Reasons for this is [sic] probably due to using more popular musicians such as the Beatles, the Rolling Stones and Michael Jackson for our model.” Clearly, they had forgotten to enact their first assumption when choosing artists for data collection.

They then used their models to make predictions of the longevity of the other artists. They were aware that “only time will tell” how accurate their predictions were realising their modelling had limitations. These were that it only applied to English speaking artists, statistics from only 20 artists were used and other factors such as the effects of “number of scandals, history of crime and drug abuse” needed to be used to refine the model. If they had more time they “would have collected more data and removed people with too many abnormity [sic] from our calculations” in order to increase the accuracy of the modelling. On the last scatterplot, the point corresponding to Jimi Hendrix’s career of only 5 years (cut short by his death from a drug overdose) was highlighted in red as was his row in their hand written data table. Clearly, the idea of what is an abnormal career would have stimulated quite a bit of discussion if they had attempted this refinement of the models. Fleur, for one, was aware some artists had had "shocking, ridiculous careers" [Video, Day 2].

Mathematisations

Let's now examine the group's mathematisation. These interpretations have been cross checked as before. To be successful the group needed to possess amongst its members knowledge of specific modelling instruments, namely graphical representations of scatterplots and various algebraic functions. Fleur, the Year 11 student, was well versed in polynomial, log and power functions. In addition at some point they needed to be able to anticipate ways in which these instruments could be employed to answer the question they had posed. As they had used such instruments for similar purposes in a three day Modelling Forum[4] that preceded the Modelling Challenge, all three were aware of this capability of digital tools. There needed to be knowledge within the group of how this strategy could be implemented and the requisite mathematical and technological skills (as technology was used) and the ability to convince others to follow this path. Ben was able to do this using a TI-Nspire calculator as well as upload data onto a computer and use an Excel spreadsheet for making charts and different regression equations. The crucial steps in this mathematisation are not obvious from the outset. The modellers had to look for mathematical objects and techniques familiar to them (or be able to be persuaded by another with this knowledge, namely Fleur, to agree to their use) that could serve as modelling instruments in this particular context that none of them had seen mathematised as was clear from their initial discussions. Once the models had been established, mathematising the question, answering it using their models and translating those figures back to the real situation involved calculation techniques that they were able to leave to technological tools. If no one within the group had been familiar with the use of scatterplots and regression beyond linear regression, the group could not have done this modelling. Also, without that person or persons being able to map out a sense of direction for the implementation of that modelling, the others within the group would not have been convinced to continue to investigate a phenomenon for which no group member could suggest a means of mathematising. Fleur was instrumental in taking on this role mainly as a driver rather than an implementer except when it came to de-mathematising.

Thus, the development of the models in question have had to involve an anticipation of which elements or features are essential to idealisation of the real situation from the extra-mathematical domain and the posing of their statement of the problem and its related question in light of anticipated usefulness in mathematising. Intuitively, they did expect from the outset that data on particular aspects of the real-situation would be accessible to them. When mathematising their particular formulation of the problem situation, the group clearly were anticipating mathematical

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[4]In the Forum, Singaporean students with their Australian hosts (including our focus group) visited tourist theme park, Movie World, to collect data using various technological tools about amusement rides such as the Batwing Spaceshot Ride. Students posed their own problems about the rides and tried to solve these using mathematical modelling [Video, Day 3 Forum].

representations that, from previous experience of at least one group member, had been beneficial when put to similar use, albeit in vastly different contexts. When anticipating these mathematical representations, the group had to be aware (or able to be convinced) of the utility of the selected mathematisation and the resulting model in future solution processes to provide mathematical answers to the mathematical questions posed by the mathematisation, thus anticipating mathematical techniques to be used in solving after mathematisation was complete. In this example, implemented anticipation projected back onto current actions at the three points identified by Niss is crucial for Fleur's persuading the group to continue with their choice of situation to investigate through mathematical modelling.

Evidence from Classroom Context - Year 9 Study

The second site to be examined for evidence of implemented anticipation by beginning modellers was a classroom in the RITEMATHS project. The data are from the last of a series of three modelling tasks used in a Year 9 class. The data analysed here relate to the implementation of *Shot On Goal*. The task has a soccer context (see Fig. 5) where the modelling involves optimising a position for a player to attempt a shot on goal whilst running parallel to the sideline (see Stillman and Brown 2007 for details). Students worked in 7 groups of 2 to 4 students. The teacher allocated each group a particular distance for their run line from the near goal post (e.g., 18m as in Fig. 5).

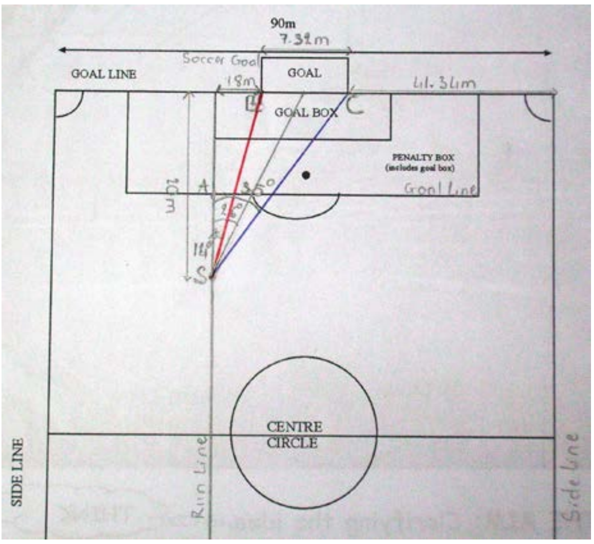

Fig. 5 Student diagram of *Shot on Goal*

The main part of *Shot on Goal* (Tasks 1 to 10) offered structured scaffolding for student modelling. See Appendix A for an outline of a solution to this part of the task. How student groups and individuals from this class tackled this part of the task is reported elsewhere (e.g., Stillman et al. 2010) and is not the focus here. Two further questions at the end of these tasks included no details of how to mathematise or approach them mathematically (see Fig. 6). Fifteen students attempted part or all of these. Tasks 11 and 12 are the focus here. The task was implemented over three lessons on consecutive days, and these final tasks were attempted during the third lesson. Data collected which are relevant to the focus of this article consisted of transcriptions of 3 video-recordings and a further audio-recording of groups working on the tasks, individual task scripts from 15 students and interview responses of nine of these students (see Appendix B for example questions).

TASK 11 – CHANGING THE RUN LINE

Investigate whether the position of the spot for the maximum shot on goal changes as you move closer or further away from the near post. [Collect data from other students' results to help you see if there are any patterns in the position of spots for the maximum angle.] What does the relationship between *position of the spot for the maximum shot on goal* and the *distance of the run line from the near post* reveal?

TASK 12 – CHANGING THE RULES

Soccer is often a low scoring game. Some have suggested that it would be a better game if the attackers had more chance of scoring, so the width of the goal mouth should be increased. Others claim it would be a more skilful game if the goal keeper was given more of a chance to stop goals by reducing the width of the goal mouth.

Investigate what effect changing the width of the goal mouth would have on the position of the maximum shot on goal for the run lines and give your recommendation.

Fig. 6 Tasks for mathematising at end of *Shot on Goal*

To analyse the data student responses to Tasks 11 and 12 were classified as (a) mathematisations showing (i) successful or (ii) unsuccessful implemented anticipation; (b) qualitative statements not supported by any mathematical objects or representations (i) identifying a relationship between relevant variables or (ii) identifying variables only and (c) incomplete as only raw data with no translation into mathematics or interpretations were recorded. As a(ii), (b) or (c) classifications indicated unsuccessful modelling attempts, post task interview data and video and audiotape data were scrutinised in detail for possible explanations of lack of success. These were then compared to Niss's four enablers of successful implemented anticipation.

Results and analysis

Group 5 and two students from Group 7 were behind the others because of difficulties with task formulation or using technology so did not attempt either Tasks 11 or 12. None of these students' work features in this analysis. Fifteen students from 6 different groups attempted Task 11. Eleven students from 5 different groups also attempted Task 12. The others ran out of time.

Changing the runline – Task 11

All groups collected data from other groups to begin Task 11. Len from Group 6 recorded their raw data systematically in two columns from which his partner Ned produced the partially ordered Table 1. Ned correctly identified the relevant variables as the distance from the run line to the goal post, the distance along the run line to the shot spot where the angle maximised and this angle. Ned and Len created an error in their collected data when they decided Group 4's angle for a 14m run line distance maximised at 18m as angle data were recorded to only one decimal place and thus were the same for 16 to 19m. They felt correcting this datum later was unnecessary as the other data supported their conclusion.

Table 1

Group 6's Tabulation of Data for Changing the Run Line

Run line to goal post	Dist along RL	Angle
12	15	13.51°
14	18	11.9°
10	13	15.54°
15	18	11.31°
16	19	10.73°
20	23	8.99°

After Ned had tabulated the data and scrutinised it briefly, the following exchange took place:

> Ned: God Len, I've got a clu-e-e
> Len: What?
> Ned: Check it out. The distance along the run line you have to be is, with the exception of-f-f 14, ah, is 3 more than the distance, what the distance is from the run line to the goal post. [Video, Lesson 3, Group 6]

In recording his interpretation of the data Len added: "It also proves our theory that the closer you come to goal, the closer you have to be on your run line to achieve maximum angle." Neither student recorded this symbolically. However, when interviewed in response to being asked if he could write an algebraic model for this (see Appendix B: Q10.1), Len wrote "$y = x + 3$ where maximum angle is = y [sic] and distance from near post = x". Even though he was unable to graph the relationship this was not pivotal for a successful solution. Ned, in contrast, was able to plot points in interview and name the relation as linear. This work was classified as successful implemented anticipation (ai).

Ozzie and Jaz of Group 2 used correctly ordered tables with Jaz also identifying and labelling the variables in the columns correctly on his script. Jaz concluded:

> The closer you were to the post, the bigger angle you got but you had to run in [along run line] further. [Script, Jaz, Group 2]

This typified the conclusions of all three group members. When asked in interview if he was able to write his answer algebraically, Ozzie responded: "I think you could but I would have to think about it." However, he recognised the response from Group 6 (see Appendix B: Q10.2) as being a linear function that produced a straight line graph. In interview, Jaz claimed to have seen from the start of the task that "there was a relationship, three metres". When asked to express this algebraically he said he "could have" but his written conclusion (as above) was trying to capture this. He elaborated:

> Jaz: Yeah, yeah. Like the closer you get, the closer your run line is to your goal post the bigger angle, you get a much bigger maximum angle but you have to run towards the goal further but I think it is every 3 metres you have to run in by figuring out all of this [seems to be describing: position of maximum angle on run line = run line distance from post – 3]. [Interview, Jaz, Group 2]

When asked to show Group 6's model algebraically, he used a diagram (see Fig. 7) and d, the distance from the near post to the run line, to show the maximum shot angle occurring on a run line at $d + 3$. He wrote $y = 2d + 3$ and labelled $d + 3$ on his diagram also as y. When asked what y equalled, he replied, "$y = 2d + 3$" but his explanation described $y = d + 3$. Ozzie and Jaz's work was classified as unsuccessful implemented anticipation (aii) as their ordered table did not help produce a mathematical answer although the representation was correctly anticipated as potentially allowing this. Sven (Group 7) also used an ordered table but no conclusion was drawn. The response was classified similarly as (aii). This student appeared to run out of time.

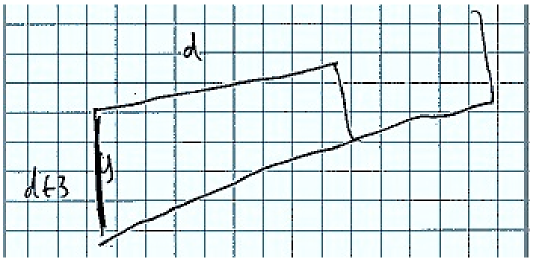

Fig. 7 Jaz's diagram of Group 6's solution to Task 11

Molly and Christine (Group 3) collected data from five groups systematically recording these in two-celled rows across the page. When they attempted to translate these, together with their own results for a run line of 14, into an ordered mapping of *run line distance from near post* → *shot distance down run lin*e → *maximum angle of shot* (Fig. 8), three mappings (shown in italics) reversed *run line distance*→*shot distance down run line*. This mapping was interpreted as showing: "In most cases the further away the run line from the goal post the smaller the angle [meaning maximum angle] and the larger the distance down the run line." This work was classified as unsuccessful implemented anticipation (aii) as the mapping does not support this conclusion even though the representation is correctly anticipated as a useful tool to do so.

10	14	*15*	*18*	*19*	20
↓	↓	↓	↓	↓	↓
13	17	*12*	*15*	*16m*	23m
↓	↓	↓	↓	↓	↓
15.54°	11.96°	13.51°	11.31°	10.37°	8.99°

Fig. 8 Group 3's attempted ordered mapping for changing the run line (error italicised)

All of Groups 1 and 4 and the third member of Group 2 produced only qualitative descriptions of the collected data. Discussions captured on video showed Group 4 were aware they were expected to construct a mathematical model for the data but Simon, Max and Lori (Group 4) only described a relationship between the run line distance from the goal post and the maximum angle (bi).

> It reveals that the closer the run line is to the near post, the bigger the angle gets. [Script, Simon, Group 4]

Jim (Group 1), Raza (Group 2), and Rose (Group 4) described two relationships between relevant variables (bi), for example:

> The closer your run line is to the goal, the better angle you will have, but you have to get closer to the goal line [along your run line]. [Script, Raza, Group 2]

Ahmed (Group 1) merely identified the variables, "the further from post" and "distance on the run line to a certain point", as playing "a crucial factor in getting the best angle" (bii). Rod's work (Group 7) was classified as incomplete (c) as data were merely listed. The video record showed he ran out of time as he was recording data at the end of the lesson.

Changing the rules – Task 12

The mathematisation of Task 12 required students realise that the representations and models used earlier for finding the angle of the shot for their particular run line distance from the near post and a standard soccer goal width of 7.32m could be used to capture the changed conditions created by varying the goal width. Work from four groups (1, 2, 6 and 3) was classified as showing successful implemented anticipation (ai).

Groups 1 and 2 used their original equations (e.g., $y = \tan^{-1}(17.32/x) - \tan^{-1}(10/x)$ for Group 1) changing the widths in the formula so they could graph the three functions displaying size of shot angle versus distance along run line on their graphing calculators (Fig. 9) to locate maximums and compare. They recorded examples of the functions they had used with the exception of Ahmed (Group 1) who merely recorded goal mouth width, run line distance for maximum shot angle and size of the maximum angle for three specific cases. In addition, Ozzie of Group 2 explained where the changes in the algebraic model came from in the real context.

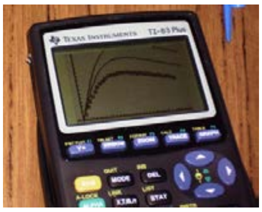

Fig. 9 Group 1's graphs for goal widths of 5, 7.32 and 10m for a run line from near goal post of 10m.

For example, as Group 1's distance of their run line from the near post was 10m and they widened then narrowed the goalmouth to 10m and 5m respectively, their function became:

$y = \tan^{-1}(20/x) - \tan^{-1}(10/x)$ and $y = \tan^{-1}(15/x) - \tan^{-1}(10/x)$ respectively [Script, Jim, Group 1]. Jim then noted: "The bigger the goal is the less you have to run on the goal and the angle will be bigger. The angle is generally bigger for a larger goal. The smaller the goal the more you have to run on the goal line to get a big angle and the angle will still be smaller." Neither group made a recommendation as to which goalmouth width was preferable.

Group 6 used their previous formula varying it to generate numerical data for two cases in their graphing calculator LISTs. They then compared with their original data set for their run line distance. Finally, they produced two tables of ordered data for goal widths of 8m and 6m. These showed distances of 1 to 5 metres from the goal line along the run line and corresponding angles and were labelled "examples". The video confirmed that distances much further down the run line were examined allowing them to see where the angle was optimising. Their observations and recommendation were:

> After some calculations, it seems that you would have to move further along your run line if the goals were widened to achieve the maximum angle. If you were to decrease the width of the goal, you would have to move closer to goal on your run line to achieve the best maximum angle for shooting at goal. I would recommend decreasing the goal width because it is much easier to obtain a good shooting chance at close range if it is decreased. [Script, Len, Group 6].

The two Group 3 students again used an ordered mapping (see Fig. 10) of *shot distance down run line* → *angle of shot* for several shot distances along their particular run line for a goal width of 9m and then 6m. The mapping was correctly recorded. Angle data were generated by changing formulae in the calculator for a larger then smaller goalmouth. Only Christine commented that the angle was reaching a maximum then decreasing but she mentioned this only for the smaller goalmouth case. The group recommendation was: “Make goals larger for more scoring shots”.

If you make the width of the goal 9m:					
10 down the runline	14	15	18	19	20
↓	↓	↓	↓	↓	↓
12.039°	13.671°	13.864°	14.078°	14.056°	13.999°

Fig. 10 Part of Group 3’s ordered mapping for changing the goalmouth width

Simon and Max from Group 4 made only a qualitative statement indicating “changing the width of the goal would change the angle” (bii). Max added that “the bigger the width of the goal the bigger the angle” and the reverse of this, indicating it was a direct relationship (bi). The former was classified as identifying variables and the latter as identifying a relationship but both were without mathematical support other than collected data.

Explanations of unsuccessful mathematisations

Niss’s four enablers (relevant mathematical knowledge, capability of using this when modelling, believing a valid use of mathematics is modelling real phenomena, and mathematical perseverance and confidence) were then investigated as possible explanations of unsuccessful mathematisations. Perseverance was not considered to be playing a part for the 15 students who attempted these tasks as all 21 students in the class whether they had reached Tasks 11 or 12 or not were persevering on the task as a whole at the end of the three lessons (i.e., 165 minutes). Students who were unable to complete because of time pressure or for whom there were no interview data of relevance were eliminated from the analysis leaving the work of nine. As seen in Table 2, Task 12 did not prove to be discriminatory either as all student work was rated as (ai) except for that of the two members of Group 4 who the video revealed were still discussing and recording their interpretation of collected data for Task 11 as the teacher began to wrap up the lesson. Task 12 was therefore eliminated from further analysis.

Table 2

Possible Reasons for Unsuccessful Mathematisations

Student	Classification		Mathematical Knowledge	Can use in Modelling		MathsView
	Task 11	Task 12	Interview Responses	Task 11	Task 12	
Group 1						
Ahmed	bii	ai	$y = x + 3$ names as algebraic straight line then turns into Angle vs Dist graph	no	yes	RW +
Jim	bi	ai	$y = x + 3$ straight line	no	yes	RW +
Group 2						
Ozzie	aii	ai	Linear - straight line	no	yes	RW +

			graph			
Raza	bi	ai	*X* + 3 refuses to graph	no	yes	RW -
Jaz	aii	ai	*Y* = 2*d* +3 correct graph for this	no	yes	RW +
Group 4						
Rose	bi	NA	Local point (15, 18); Plots points straight line graph names as linear	no	NA	More abstractly oriented
Lori	bii	NA	*X* + 3 straight line	no	NA	More abstractly oriented
Group 5						
Ned	ai	ai	linear relation plots points – says should be straight	yes	yes	ambivalent
Len	ai	ai	*y* = *x* +3 unable to graph	yes	yes	RW +

Note. NA = no attempt; RW +/- = positive/negative attitude to modelling real world phenomena

To see if students whose responses to Task 11 received classifications other than (ai) possessed the relevant mathematical knowledge of linear relationships, their responses in interview (see column 3, Table 2) to being asked to describe (Appendix B: Q10.2) and graph (Appendix B: Q10.2.2) such a relationship were examined. This revealed that all but Jaz were able to describe the relationship as the correct linear relation or expression. Jaz's response, described previously, was, however, a linear relation. Rose used the relation correctly in terms of mappings of coordinate points so understood how to use it mathematically to demathematise a qualitative description. Clearly, being able to use this mathematical knowledge in the modelling context (column 4, Table 2), rather than knowing the mathematics in the first place, was the difficulty for those students who were unsuccessful in mathematising Task 11.

Evidence of students' appreciation of modelling real world phenomena and references to reality in mathematics (see column 5, Table 2) was gleaned from the video and audio interactions in their group and through responses to interview questions (see Appendix B: Q11.1-Q11.3). These were then classified as displaying a preference and positive attitude to modelling real world phenomena (RW+) or negative attitude to modelling real world phenomena (RW-) or a preference for more abstractly oriented tasks (this did not imply that the person was necessarily negatively oriented to real world tasks), or displaying ambivalence towards the real world or abstract nature of mathematical tasks.

Several students were highly positive to the real world modelling tasks saying they preferred such tasks. Len's response was typical.

> Len: … it's like because it is in a real life situation and, you know, you can relate to it as in thinking about it … Yeah, I do, I do prefer that to a normal Maths one, yeah.

Only Raza was negative towards the utility of the mathematics they were using in class, trigonometry and algebra in particular, in the real world.

> Raza: You don’t need to do advanced trigonometry or, you know, advanced algebra. Like my dad has used algebra for the first time since he left school, he is like 40, last year he used it once! And it wasn’t even complicated so I really can’t see the point.

He did, however, concede that there could be some application of this task for a soccer coach:

> Raza: But it wasn’t particularly, applicable. Like you couldn’t really do it in soccer, unless you were a coach, because there are so many things you can do. Like that, things that can change in soccer.

Rose and Lori preferred doing mathematical exercises.

> Rose: I like maths straight-forward, and doing exercises, that’s how I find it easiest. It’s okay. I don’t mind it [a real world task] but I prefer to just do normal tasks, like just normal exercises.
> Interviewer: But you said you really didn’t find this difficult?
> Rose: I didn’t find it difficult but I find it easier just to do normal exercises.

Although both were more abstractly oriented in their mathematical preference they were not opposed to doing real world tasks nor did they view real world connections negatively, although Lori did not like *Shot on Goal,* in particular, despite being a soccer player herself. The other tasks were set in the contexts of orienteering and bungee jumping.

> Q10.3 Do you like the fact these tasks are set in a real world context?
> Lori: Umm, yeah. But I didn’t like the task [*SOG*] at all, so. …I found this one harder. I liked the other ones better. I didn’t really like, this one. I had trouble with it. The other ones were okay.

Ned remained ambivalent towards the need for a task context to be real or not although he appreciated mathematics being in some sort of context.

> Ned: Exercises from the book I tend to avoid as much as possible, yeah. I just do the first couple to make sure I understand it … . They don’t really interest me. I just see them as sort of revising over and over for something you already know. [This task] actually made me think rather than doing mindless work. I prefer the fact they [real world tasks] are worded tasks rather than just blank equations on a sheet but I don’t think wherever they were set would matter to me.

Thus, appreciating the modelling of real phenomena was not seen as discriminating between successful or unsuccessful mathematisation for the students in the study.

Discussion

Open modelling contexts allowing student input to formulation are required to demonstrate the first aspect of Niss’s model of mathematisation, namely, the basing of idealisation and specification of the extra-mathematical situation on anticipation of their potential in mathematising (2010, p. 56). In our experience as researchers, open modelling occurs rarely within classrooms to which we have had access. This is not saying it does not occur in a schooling context as, for example, in other projects (e.g., Stillman and Galbraith 2011) students have been given quite open modelling assignments to complete but often the formulation is done outside class. We have therefore been compelled to use data from outside the classroom in a context where students are allowed to find their own real situations and pose their own problems for modelling to confirm existence of this aspect. According to PISA 2012 draft mathematics framework (OECD 2010, p. 37), the “mathematisation demand” in such a context is at a high level as modellers need to identify and even define many assumptions, variables, and relationships and set the constraints of what they will investigate themselves. Nevertheless, in the two examples we have investigated from the AB Paterson Modelling Challenge, both groups have mathematised their chosen situation by first structuring the real world situation by anticipating elements, features and even questions that have potential for the coming mathematisation. In both examples this entailed the employment of knowledge-based anticipation of what is required to progress the solution of the problem in order to resolve the problem posed in the chosen extra-mathematical domain. In other words, there is evidence of current actions having been informed by the foreshadowing of future mathematical decisions being projected back onto themselves. This is what Niss has proposed as "implemented anticipation" (Niss 2010, p. 55). In the first example the mathematisation was flawed because of a lack of relevant mathematical knowledge which was outside the knowledge base of the participants although they were clearly unaware of this. In the second example it was successful.

Vygotsky points to a dilemma that arises in our work on finding evidence for anticipated implementation, namely that "the method [of enquiry] is simultaneously pre-requisite and product, the tool and the result of the study" (1978, p. 65). As beginning modellers engage in mathematisation of a messy situation implemented anticipation is manifested simultaneously with the tool by which that implemented anticipated can be evidenced. This tool is dialogue. However, what this means is that it is only possible to tell that it is implemented anticipation, that is in the sense of being successful as Niss requires for ideal mathematisation, after the fact.

As the Year 9 tasks were already formulated as mathematical statements or questions by the task setter, all the successful mathematisations could be said to have involved students identifying relevant variables, foreshadowing representations (i.e., ordered tables, ordered mappings, graphs) that would be useful in identifying relationships between variables and producing mathematical answers. Furthermore, these required the realisation that the representations and models used earlier could be used to capture changed conditions involving a mathematisation demand at a higher level than the lowest conceivable but lower than that expected in the Year 10/11 examples above (OECD 2010). They have provided, nevertheless, empirical evidence of the existence of two aspects of Niss's implemented anticipation: (1) anticipating "relevant mathematical representations which are capturing the situation" and (2) anticipating "the use that may be made of the mathematization chosen and the resulting model. …[thus anticipating] the problem solving tools that can be selected after the mathematization is completed" (2010, pp. 56-57).

In both open modelling where students choose their own situation, its formulation and methods of proceeding mathematically and structured modelling situations, in situations outside the classrooms (i.e., the modelling challenge) and inside the classroom, "'anticipation' allows planning and continuous feed back" (Boero 2001, p. 99) at an intuitive level from the foreshadowed goal of each aspect of activity. Thus, if students are going to develop the capability of mathematising as a fundamental mathematical activity as suggested in the PISA framework (OECD, 2010) then there needs to be a focus in schooling on developing anticipating of mathematical activity and its output as a characteristic form of thinking when doing mathematics particularly when mathematising. Not only is this emphasis needed but also a focus on how to implement this anticipation in decision making and the carrying out of actions for subsequent steps in tasks.

With respect to Niss's four enablers (2010, p. 57) for successful implemented anticipation in mathematisation, namely, (1) possessing relevant mathematical knowledge, (2) being capable of using this when modelling, (3) believing a valid use of mathematics is modelling real phenomena, and (4) displaying mathematical perseverance and confidence, these were able to be used by the authors to explain unsuccessful mathematisations. They also were present in the data for successful mathematising but this is not claimed to be unequivocal evidence for their necessity as enablers. For example, a reason for students engaging with a modelling task might not be a belief that a valid use of mathematics is modelling real phenomena but rather deference to the norms of the classroom as engagement with such a task is clearly expected by the teacher. This could be what is being displayed by Raza, for example, who displayed a strong negative attitude to using mathematics in real world situations. We thus see this as an area for further research into Niss's model.

Conclusion

We have found empirical evidence for all three aspects of Niss's implemented anticipation (Niss 2010) in open modelling situations and for the last two in the classroom data we analysed. Thus, what we believe we have presented is a set of exemplars from practice that form the basis for "paradigmatic cases" (Freudenthal, 1981, p. 135) for the existence of Niss's construct in theorising mathematisation in modelling or realistic tasks.

Unsuccessful attempts at mathematisation were able to be explained using Niss's enablers. In the open modelling situation the flaw in mathematisation in one exemplar projects was due to lack of relevant mathematical knowledge as what was needed was beyond the students' previous mathematical content knowledge. Where evidence was available, unsuccessful attempts at mathematisations in the classroom context were, in this instance, related to inability to use relevant mathematical knowledge in the modelling context rather than lack of the relevant mathematical knowledge per se, or an application oriented view of mathematics or perseverance on the task. As this was only the third in a series of modelling tasks the students had attempted as their first experience of modelling, it is not surprising that this was the most discriminating of the enablers for successful implemented anticipation in mathematisation. In both contexts, the examples of successful implemented anticipation were congruent with all four enablers being present; however, as we have pointed out this is subject to alternative interpretation in the case of the third enabler and further research is needed regarding the necessity for the enablers in the model.

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Appendix A: Solution Elements to Shot on Goal

An outline of essential steps in the solution for a run line 18m from the near goal post follows. Table 3 shows calculations obtained using the LIST facility of a TI-83 Plus graphing calculator. Calculations are shown for positions of the goal shooter at distances from the goal line of between 14 and 28m along the run line (see Fig. 5 and 9). Width of goalmouth is 7.32m. (The students carried out calculations from 1- 30m.) The maximum angle is highlighted in the table, which was generated by the LIST facility of the calculator, following hand calculations to establish a method.

Table 3.

Sample calculations from a typical solution to *Shot on Goal*

Distance (m) L1	∠ASB (°) L2	∠ASB (°) L3	Shot Angle (α°) L4
14.00	52.13	61.06	8.94
15.00	50.19	59.36	9.16
16.00	48.37	57.71	9.34
17.00	46.64	56.12	9.49
18.00	45.00	54.59	9.59
19.00	43.45	53.12	9.66
20.00	41.99	51.70	9.71
21.00	***40.60***	***50.33***	***9.73***

22.00	39.29	49.01	9.72
23.00	38.05	47.75	9.70
24.00	36.87	46.53	9.66
25.00	35.75	45.36	9.61
26.00	34.70	44.24	9.55
27.00	33.69	43.16	9.47
28.00	32.75	42.15	9.39

Note. Calculator LIST formulae used were L2 = "$\tan^{-1}$(18/L1)", L3 = "$\tan^{-1}$((18 + 7.32)/L1)" , L4 = "L3 – L2"

A graph (Fig. 10), showing angle against distance along the run line is drawn, using the graph plotting facility of the calculator. Additional points near the maximum can then be calculated, to provide a numerical approach to the optimum position (9.73° at 21.35m from the goal line) or an algebraic model, $y = \tan^{-1}(25.32/x) - \tan^{-1}(18/x)$, can be constructed and the maximum found using graphing calculator operations.

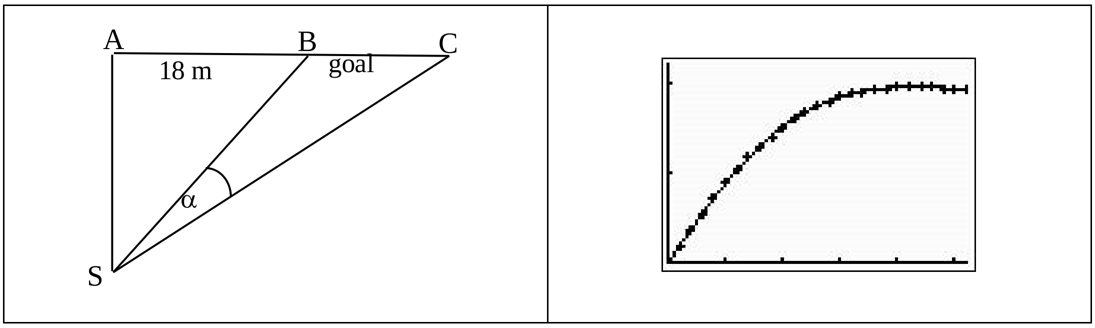

Fig. 10 Angle (α) to be maximised

Appendix B: Selected Interview Questions

Q10.1 Could you have written an algebraic model for your answer to Task 11?

Q10.2 One group [Group 6] said that the position of the spot for the maximum shot on goal was 3 metres more than the distance of the run line from the goal post. What type of mathematical relationship is this?

Q10.2.2 Draw me a graph to show it.

Q10.2.3 How could a coach of a defending player use this information?

Q11.1 Do you like doing challenging tasks like this in maths? Can you elaborate on that? [Prompt: What makes it challenging/not challenging?]

Q11.1.1 What types of maths tasks do you prefer?

Q11.2 What is the purpose of tasks such as *Cunning Running* and *Shot on Goal*?

Q11.3 Do you like the fact these tasks are set in a real world context?