A FRAMEWORK OF GROWTH POINTS IN STUDENTS' DEVELOPING UNDERSTANDING OF FUNCTION

Submitted by

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DECLARATION

This thesis contains no material published elsewhere or extracted in whole or in part from a thesis by which I have qualified for or been awarded another degree or diploma. No other person's work has been used without due acknowledgment in the main text of the thesis. This thesis has not been submitted for the award of any degree or diploma in any other tertiary institution. All research procedures reported in the thesis received the approval of the relevant Ethics committee.

Erlina R. Ronda

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ABSTRACT

This research developed a framework describing students' developing understanding of function. The research started with the problem: *How might typical learning paths of secondary school students' developing understanding of function be described and assessed?* The following principles and research questions guided the development of the framework.

Principle 1. The framework should be research-based.

Principle 2. The framework should include key aspects of the function concept.

Principle 3. The framework should be in a form that would enable teachers to assess and monitor students' developing understanding of this concept.

Principle 4. The framework should reflect students' *big ideas* or *growth points* which describe students' key cognitive strategies, knowledge and skills in working with function tasks.

Principle 5. The framework should reflect typical learning trajectories or a general trend of the growth points in students' developing understanding of function.

The following questions guided the development of the framework of growth points:

1. What are the growth points in students' developing understanding of function?
2. What information on students' understanding of function is revealed in the course of developing the framework of growth points that would be potentially useful for teachers?

The framework considered four key domains of the function concept: Graphs, Equations, Linking Representations and Equivalent Functions. Students' understanding of function in each of these domains was described in terms of *growth points*. Growth points are descriptions of students' "big ideas". The description of each growth point highlights students' developing conceptual understanding rather than merely procedural understanding of a mathematical concept. For example, growth points in students' understanding of function under Equations were:

1) interpretations based on individual points;

2) interpretations based on holistic analysis of relationships;

3) interpretations based on local properties; and,

4) manipulations and transformations of functions (in equation form) as objects.

The growth points in each domain are more or less ordered according to the likelihood that these “big ideas” would emerge.

To identify and describe these growth points, Year 8, 9 and 10 students in Australia and the Philippines were given tasks involving function that would highlight thinking in terms of the process-object conception and the property-oriented conception of function. Students’ performance on these tasks and their strategies served as bases for the identification and description of the growth points.

The research approach was interpretive and exploratory during the initial stages of analysis. The research then moved to a quantitative approach to identify typical patterns across the growth points, before returning to an interpretive phase in refining the growth points in the light of these data. The main data were collected from students in the Philippines largely through two written tests. Interviews with a sample of students also provided insights into students’ strategies and interpretations of tasks.

The research outputs, the research-based framework and the assessment tasks, have the potential to provide teachers with a structure through which they can assess and develop students’ growth in the understanding of function, and their own understanding of the function concept.

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CHAPTER 1

INTRODUCTION

Context

Discussing his view regarding the construction of knowledge, von Glasersfeld (1987) argued:

> If, then, we come to see knowledge and competence as products of the individual's conceptual organization of the individual's experience, the teacher's role will no longer be to dispense 'truth' but rather to help and guide the student in the conceptual organization of certain areas of experience. Two things are required for the teacher to do this: on the one hand, an adequate idea of where the student is and, on the other, an adequate idea of the destination (p. 16).

Von Glasersfeld then emphasised the need to develop "a conceptual model of the formation of the structures and the operations that constitute mathematical competence … because it alone, would indicate the direction in which the student is to be guided" (p. 16). As a mathematics teacher and educator, I believe that this conceptual model should reflect as well the students' process of understanding of the concept as this provides a basis for pedagogical decisions. This conceptual model may be in a form of a framework of typical *learning paths* or *landscapes* in students' developing understanding of mathematical concepts.

The need for a model that describes students' developing understanding of mathematical concepts is reflected in the National Council of Teachers of Mathematics (NCTM) document *Principles and Standards for School Mathematics* (NCTM, 2000) which reported that the changes made in the teaching of mathematics in the name of the *Curriculum and Evaluation Standards,* (NCTM, 1989)] "have been superficial and incomplete [because] … some of the pedagogical ideas from the NCTM *Standards*—such as the emphases on discourse, worthwhile mathematical tasks, or learning through problem solving—have been enacted without sufficient attention to students' understanding of mathematics content" (NCTM, 2000, pp. 5-6).

In the same vein, Hiebert and Wearne (1991) argued that while features of learning which are domain-independent, such as the idea that students are active constructors of knowledge, are important, for these to impact on instruction, these features of learning should be interpreted in terms of specific mathematical domains. They proposed that there be more studies, which identify and describe students' key cognitive processes in understanding specific mathematical domains in order to inform teaching.

That teachers' knowledge of students' thinking in acquiring concepts and procedures in a specific mathematical domain can be a powerful tool in informing instruction has been demonstrated by the results of studies such as the Cognitively Guided Instruction project (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996) and the Early Numeracy Research Project (Clarke, 2001). These studies developed research-based models of learners' key cognitive processes in understanding specific domains of mathematics in the early years that teachers could use to assess and monitor students' understanding.

Recent developments in classroom assessment such as the domain-based approach to assessment (see, Shafer & Romberg, 1999) also put emphasis on the levels of reasoning or mental activities and strategies students are applying in the domain.

While there is already a considerable wealth of knowledge regarding the general stages learners pass through in acquiring concepts and procedures in mathematics in the early years, very few studies have been undertaken which describe students' developing understanding in specific mathematical domains in high school mathematics.

The potential of a research-based model of the students' process of understanding of a specific domain of mathematics in informing teaching and in assessing students' understanding and the lack of this model for mathematics studied in high school, inspired me to work towards developing a framework showing key aspects of students' growth in understanding in one of the major component of mathematics studied in high school – function.

Focus of the Study

Function is a core concept in mathematics and students' understanding of this concept has been the subject of many studies (see, e.g., Dubinsky & Harel, 1992; Leinhardt, Zaslavsky & Stein, 1992). However, many of these studies highlighted what students *do not know* about functions and why this might be the case rather than giving more emphasis on students' developing conceptualisation of function (Thompson, 1994). Although there are proposed frameworks that describe students' developing understanding of this concept, they are all theoretical frameworks (see, e.g., DeMarois & Tall, 1996; O'Callagan, 1998; Slavit, 1994). Hence, the way these frameworks are structured may be too general to be useful to classroom teachers especially in monitoring and assessing students' understanding of this concept. These frameworks need to be reinterpreted and articulated, taking into consideration secondary students' experiences with function, which is limited to linear and quadratic relationships and some basic knowledge of other functions. I believe that it would be more useful for teachers if students' developing understanding of function is described in terms of their reasoning about representations of relationships and properties of relationships (Thompson, 1994), as reflected in the strategies, knowledge, and procedures they apply in working with function tasks, since these are reflective of developing abstraction in students' thinking. Describing students' developing understanding of function this way would give secondary teachers a guide in terms of recognising landmarks in students' growing understanding of the concept. It would also encourage teachers to focus on the level of mathematical thinking students used, which could be more informative in designing classroom experiences, compared to referring to expected outcomes outlined in curriculum documents. The premise of the study is that unless teachers can recognise the differing level of abstraction in students' thinking and reasoning it would be unlikely that they are well equipped to design appropriate pedagogy to lead students towards a deeper understanding of function.

It is true that the competencies provided in school curricula and other documents, for example the *Curriculum and Evaluation Standards* (NCTM, 1989) already provide teachers with a structure and direction in which to guide students. The *Standards* and its revised version, *Principles and Standards for School Mathematics* (NCTM, 2000) provide a comprehensive list of competencies students should learn in the study of function and its representations in high school. However, most of these competencies are stated in terms of outcomes. While these may be useful for teachers, these competencies do not describe the strategies and thinking that students use (Horne & Lindberg, 2001). For example, one common competency is *to translate among the tabular, symbolic and graphical representations of function* (NCTM, 1989, p. 154). Students can do this in many ways. They can use a *point-by-point* analysis, use *patterns/trends* or some *global properties*, or use their knowledge of the *invariant properties* of the function. These strategies certainly represent different levels of understanding of the concept but the differences are not made explicit in the ways competencies are stated. There is therefore a need to describe typical learning paths in students' developing understanding of function in terms of these kinds of strategies and reasoning to inform teaching. This, in fact was the aim of the study.

I started with the research problem:

> How might typical learning paths of secondary students' developing understanding of function be described and assessed?

After an initial review of related literature, I decided to develop a framework describing understanding in terms of *growth points* or *big ideas* in students' developing understanding of function which would highlight differing level of abstraction in students' thinking and reasoning. The theoretical underpinnings of the framework are described below, and discussed in detail in Chapter 2.

Theoretical Underpinnings of the Framework

The aim for the framework was to provide a structure that describes secondary students' developing understanding of function, based on the consideration of understanding as a growing network of conceptual nodes that is continuously being constructed and reorganised (Hiebert & Carpenter, 1992; von Glasersfeld, 1987), and of the understanding of mathematics as a dynamic, multilevel process as articulated in the process-object and property-oriented perspectives, described below.

Process-Object Route in the Understanding of Function

Generalising, formalising and abstracting are intrinsic to mathematics, and mathematical activities in the classrooms are aiming towards these, implicitly or explicitly. The understanding of mathematical concepts in terms of the process-object theory highlights this nature of mathematics. In this theory, which some authors traced back to Piaget's theory of *reflective abstraction*, an individual starts by engaging in computational processes that then lead them to a process conception, which later is encapsulated as a mental object (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Selden & Selden, 1992). Freudenthal also articulated this process: "My analysis of the mathematical learning process has unveiled levels in the learning process where mathematics acted out on one level becomes mathematics observed on the next" (1978, p. 33). Sfard (1991), using historical examples and in the light of schema theory, also argued that for most people, concepts are conceived as a process first before they are conceived as a mathematical, mental object.

In the process-object theory, an object conception is generally attained after experiences in performing *actions* on the concept. Sfard and other authors (see, e.g., Breidenbach, Dubinsky, Hawks & Nichols, 1992; DeMarois & Tall, 1996) have identified stages leading to a structural, object conception of a mathematical concept. Although there are differences in the number of stages identified and names used for the stages, the sets of stages are parallel to each other. Sfard's stages are briefly described below.

Sfard identified three stages learners go through to accomplish the transition from process to object conceptions: *interiorization*, *condensation* and *reification*. The stage of *interiorization* is the stage when "a learner gets acquainted with the processes which will eventually give rise to a new concept" (p. 18). When the learner compresses in chunks the lengthy and detailed sequences of operations he or she has been doing during the stage of interiorization, then the learner is said to be in the *condensation* phase of concept development. At this stage, the learner becomes "more and more capable of thinking of a process as a whole without an urge to go into details" (p. 19). This condensation phase lasts as long as the learner associates the "new" concept with a certain process. The concept is said to be *reified* when this "new" concept is conceived as a full-fledged object.

Property-oriented Route towards Understanding of Function

Slavit (1997) proposed an alternate route toward an understanding of function as an object or as a permanent construct. He argued that through experiences with various function exemplars and noting their properties, students could conceive function as objects either possessing or not possessing these properties. Students can think of "functions as entities possessing various growth properties of a local and global nature" (p. 260). Slavit also identified stages toward an object conception of function along this property-oriented route. The first stage involved understanding of equivalence procedures across representations, the second involved understanding of equivalence of procedures across function classes and the third, understanding of procedural networks as permanent constructs (Slavit, 1994). There is little chance of course that the majority of high school students would attain these stages. There is a need therefore to reinterpret these stages in terms of the limited experience with function of students in secondary schools.

The Principles and Questions Guiding the Development of the Framework

As stated earlier, the objective of this research was to develop a framework describing secondary students' developing understanding of function. The following principles were formulated to inform the development of the framework:

Principle 1. The framework should be research-based.

Principle 2. The framework should include key aspects of the function concept.

Principle 3. The framework should be in a form that would enable teachers to assess and monitor students' developing understanding of this concept.

Principle 4. The framework should reflect students' *big ideas* or *growth points* which describe students' key cognitive strategies, knowledge and skills in working with function tasks.

Principle 5. The framework should reflect typical learning trajectories or a general trend of the growth points in students' developing understanding of function.

The collection of the data and development of the framework were guided by the following questions:

1. What are the growth points in secondary school students' developing understanding of function?
2. What information on the students' understanding of function is revealed in the course of developing the framework of growth points that would be potentially useful for teachers?

Understanding of function as used in the present study does not refer to the students' understanding of the *concept's definition* but of the *concept's image* (Vinner, 1992). That is, it refers to students' understanding of the representations, relationships and properties associated with the function concept.

Students' developing understanding was described in terms of growth points. Growth points as they are referred to in this research are *cognitive structures* or a *meaningful chunk* of information which students construct in the process of understanding a concept. It is related to the notion of schema (Marshall, 1990), theorems-in-action (Vergnaud, 1997) and key cognitive processes (Hiebert & Wearne, 1991). Growth points could be thought of as major conceptual nodes in the network of students' understanding of a mathematical concept.

In this study, each growth point describes students' "big ideas" in terms of the strategies, knowledge and procedures they apply in working with tasks and problem situations. The growth points are more or less ordered according to the likelihood that these "big ideas" would emerge. The description of each growth point highlights students' developing conceptual understanding rather than merely procedural understanding of a mathematical concept.

The present research used the phrase *growth point* to describe students' big ideas in the process of understanding rather than schema or theorem-in-action. The phrase growth point vividly reflects the essence of understanding as something that is growing and developing. The phrase is simple, less technical and can easily become a part of teachers' everyday language (Clarke, Sullivan, Cheeseman, & Clarke, 2000).

Process-object and property-oriented perspectives in the understanding of function underpinned the identification and descriptions of the growth points. For example, the growth points identified under the domain *Equations* were: (1) Equations as procedure for generating values; (2) Interpretations based on holistic analysis of relationships; (3) Interpretations based on local properties; and, (4) Manipulation and transformation of equations seen as objects. The first two growth points are reflective of an understanding of equation as a process while the fourth growth point is reflective of an understanding of equation as a mathematical object. The third growth point is reflective of a property-oriented perspective of understanding of function in equation form.

The Research Procedure

The development of the framework of growth points started with an initial framework based on the literature reviewed. The aim of the research was not to confirm the initial framework of growth points but to expand and enrich it, based on students' performance on tasks and the strategies and reasoning used.

The research approach was initially more exploratory and interpretive in order to select and develop the range of tasks as well as identify a range of students' strategies that would more or less assess the initial list of growth points or show other possible growth points. Then, having identified and developed the tasks and categorised the strategies that would assess the identified growth points, the study moved to a more empirical approach to identify trends in the growth points, before going back to a qualitative approach to further refine the framework. The data were collected in natural settings. There was no treatment introduced and no experiment was involved.

Because the study focussed on describing typical learning trajectories, the study required data from a large number of students to determine students' distribution across the growth points. Initial studies in Melbourne, Australia and in the Philippines were conducted before the main data collections. The main data collection involved 149 Year 8 students, 152 Year 9 students and 143 Year 10 students, from three Regional Public Science High Schools in the Philippines. The students from science high schools in the Philippines could be considered representative of high performing students, because they are specially selected through their performance in Science, Mathematics and English. Data collection focussed on three year levels, Years 8, 9 and 10, since function is formally introduced in Year 8 in the Philippines and students in Year 10 more or less have considerable experience with other families of functions and their representations. The data were collected twice from the same students, the first at the beginning of the school year, and the second five months later, to gain insights into students' movement, if any, in the growth points. The main data analysed for this purpose came from written responses, solutions and explanations of the students to the tasks, which were administered, in a test-like environment. There were also interviews conducted with a sample of students using the tasks to gain further insight into students' strategies and interpretations of the tasks.

The development of the instrument (set of assessment tasks) and the development of the framework of growth points went hand in hand, one informing the other. Chapter 3 explains the details of the process in the development of the framework and the instrument used.

Importance of the Study

Schools' and teachers' enactments of pedagogical ideas are incomplete if there is insufficient attention to students' understanding of mathematics content (NCTM, 2000). The framework of growth points in students' understanding of function contributes to focussing on this major factor in the teaching-learning process: the students' developing understanding of mathematical content, particularly in the area of mathematical function.

The outputs of the study could contribute to bridging the gap between research and practice, an area where "scholarship in education is most often found to be lacking" (Boaler, 2002, p. 8). The research-based framework and the assessment tasks have the potential to provide teachers with a structure through which they can assess and develop students' growth in the understanding of function and their own understanding of the function concept.

The framework could serve as an organising structure so that functions studied in a particular year level are linked into a powerful whole. It could also be used as a tool for investigating how knowledge of students' understanding of function informs teaching and provides a basis for designing the secondary school mathematics syllabus in the area of functions, and in writing textbooks and teaching materials. In addition, it may initiate research of its kind not only in the domain of function but also in other specific domains of school mathematics.

Summary

I have argued the importance of developing a framework that describes growth points in students' developing understanding of a particular mathematical domain or concept, that of function. The key domains considered in the final framework include Graphs, Equations, Linking Representations and Equivalent Relationships. The development of this framework started with an initial framework based on other studies of students' understanding of this concept, and recent frameworks used in analysing students' understanding of function. The descriptions of the growth points were based on students' understanding as described in the process-object and property-oriented perspectives of students' understanding of a mathematical concept. The framework has the potential to inform the work of teachers and researchers, with benefits to all students studying this important mathematical topic.

Outline of the Thesis

This chapter gave an overview of the thesis. Chapter 2 outlines the literature reviewed, focussing on the theoretical underpinnings of the framework and the literature from which the initial framework of growth points was based. Chapter 3 describes the methodology of the research. The next four chapters are devoted to the four key domains of function considered in the Framework: Chapters 4 focuses on Graphs, Chapter 5 on Equations, Chapter 6 on Linking Representations, and Chapter 7 on Equivalent Relationships. Each of these four chapters describes the correspondence between the growth points and the tasks used to assess the growth points. Each chapter presents the empirical data supporting the typical learning trajectory of function in terms of growth points. Each chapter also discusses the results and their implications specific to the domain considered. Chapter 8 consolidates the results of the study by presenting the Framework of Growth Points. The last chapter, Chapter 9, presents important implications and recommendations.

CHAPTER 2

LITERATURE REVIEW

This chapter summarises the literature, which informed the development of the framework of growth points. The review is divided into two parts: the first part discusses the literature about understanding function; and the second part discusses the nature of growth points. The chapter concludes with the initial framework of growth points that the study enhanced in the course of the research.

The Concept of Function

Mathematics is not just about the study of number and shape but also about the study of patterns and relationships (Steen, 1990). Function, which can represent some of these relationships, is an indispensable tool in its study. It is the central underlying concept in calculus, which is usually the final goal of all science students not majoring in mathematics (Vinner, 1992). Function is also one of the key concepts of mathematics, which can easily be applied to real life situations.

The following discussion summarises relevant literature on understanding function which informed the development of the initial framework of growth points: the definitions of function, literature on how the concept of function is learned, recent frameworks developed for analysing students' understanding of function, aspects of functions included in secondary schools, and the kind of tasks and skills associated with the concept. Results of some studies on students' understanding of function are also discussed.

Definitions of Function

The study of function is a much later addition to the fundamental concepts included in algebra. The concept of function "was born as a result of a long search after a mathematical model for physical phenomena involving variable quantities" (Sfard, 1991, p. 14). In 1755, Euler (1707-1783) elaborated on this conception of function as a dependence relation. He proposed that, "a quantity should be called a function only if it depends on another quantity in such a way that if the latter is changed the former undergoes change itself" (p. 15). Seventy-five years later, Dirichlet (1805-1859) introduced the notion of function as an arbitrary correspondence between real numbers. About a hundred years later in 1932, with the rise of abstract algebra, the Bourbaki generalised Dirichlet's definition. Thus, function came to be defined as a correspondence between two sets (Kieran, 1992). This formal set-theoretic definition is very different from its original definition. Function is no longer associated with numbers only and the notion of dependence between two varying quantities is now only implied (Markovits, Eylon, & Bruckheimer, 1986). The Direchlet-Bourbaki definition allows function to be conceived as a mathematical object, which is the weakness of the early definition. However, the set-theoretic definition is too abstract for an initial introduction to students and is inconsistent with their experiences in the real world (Freudenthal, 1973; Leinhardt, Zaslavsky, & Stein, 1990; Sfard, 1992).

Textbooks, which often define function as a set of ordered pairs usually start the discussion with relation and introduce function as a special kind of relation. But relation is more abstract than function. Thus the supposed pedagogical value of having to learn relation first before one understands function is, in the opinion of Thorpe (1989), wrong. Freudenthal (1973) also expressed strongly that "to introduce function, relations can be dismissed" (p. 392). Thorpe went on to say that the use of the set-theoretic definition which defines function as a set of ordered pairs "was certainly one of the errors of the sixties and it is time that it were laid to rest" (p. 13). For this reason, the present study did not consider the relationship of function to relation in describing students' developing understanding of function.

Concept Definition versus Concept Image

Understanding the definition does not imply understanding the concept. In order to understand a concept one must have a concept image for it. One's concept image includes all the non-verbal entities, visual representations, impressions and experiences that are created in our mind by a mention of a concept name (Vinner, 1992). Vinner stressed that the concept definition is not the first thing that is learned in understanding a concept but the experiences associated with it, which becomes part of one's concept image. Vinner believes that in carrying out cognitive tasks, the mind consults the concept image rather than the concept definition. For this reason, the present study did not consider understanding of the definitions of function in describing students' developing understanding of the concept of function. The study focussed on the concept image of function.

Vergnaud (1997) also noted, "it is misleading, even in mathematics [despite its precision in defining], to consider that the properties of a concept are self-contained in its definition" (p. 5). To study and understand how mathematical concepts develop in students' minds through their experience both in and outside school, Vergnaud proposed that one needs to consider a concept C as a three-tuple of three sets (see Figure 1):

C = (***S***, ***I***, ***R***)
S: the set of situations that make the concept useful and meaningful.
I: the set of operational invariants that can be used by individuals to deal with these situations.
R: the set of symbolic representations, linguistic, graphic or gestural that can be used to represent invariants, situations and procedures (p. 6)

Figure 1. Key aspects of a concept.

For the concept of function studied in the secondary school, the set ***S***, in Vergnaud's equation, would include the set of tasks involving function like modelling, representing relationships, and linking representations. The set ***I*** would include the strategies used to work out the tasks like point-plotting techniques and composition. The set ***R*** would include the different representations of function. These would be the graphs, tables, ordered pairs and, equations.

One of the key elements, in Vergnaud's equation is the set of representations for the concept. The representations of function commonly used in junior high school are described in the next section.

Representations of Function

Kaput (1989) argued that "there are no absolute meanings for the mathematical word *function*, but rather a whole web of meanings woven out of the many physical and mental representations of functions and correspondences among representations" (p. 168). Understanding of function therefore may be done in terms of understanding of its representations. There are at least three representational systems used to study function in secondary schools. Kaput described the strengths and limitations of each of these representational systems. This is summarised below:

> Table: displays discrete, finite samples; displays information in more specific quantitative terms; changes in the values of variables are relatively explicitly available by reading horizontally or vertically when terms are arranged in order (this is not easily inferred from graph and formula).
>
> Graph: can display both discrete, finite samples as well as continuous infinite samples; quantities involved are *automatically* ordered compared to tables; condenses pairs of numbers into single points; consolidates a functional

relationship into a single visual entity (while the formula also expresses the relationship into a single set of symbols, individual pair of values are not easily available for considerations unlike in the graph).

Formula: a shorthand rule, which can generate pairs of values (this is not easily inferred from tables and graphs); has a feature (the coefficient of x) that conveys conceptual knowledge about the constancy of the relationship across allowable values of x and y — a constancy inferable from table only if the terms are ordered and includes a full interval of integers in the x column; parameters in equation aid the modelling process since it provides explicit conceptual entities to reason with (e.g. in $y = mx$, m represents rate).

It is obvious that the strength of one representation is the limitation of another. A sound understanding of function therefore should include the ability to work with the different representations confidently. Furthermore, because these representations can signify the same concept, understanding of function requires being able to see the connections between the different representations since "the cognitive linking of representations creates a whole that is more than the sum of its parts" (Kaput, 1989, p. 179). Thus, the study gave emphasis to students' understanding of the links between the different representations in the framework.

The next section discusses understanding of function from a psychological point of view.

Process – Object Conceptions

Using historical examples and in the light of schema theory, Sfard (1991) conjectured that in the acquisition of a new mathematical concept, most people have an operational conception first before having a structural conception. That is, concepts are conceived as a process first before they are conceived as object. She characterised a structural (object) conceptions as "static, instantaneous and integrative" while the operational (process) conception is "dynamic, sequential and detailed" (p. 4).

Table 1 shows some of the examples Sfard provided to explain and distinguish structural and operational conceptions.

Table 1
Structural and Operational Descriptions of Mathematical Notions

	Structural	Operational
Function	Set of ordered pairs (Bourbaki, 1934)	Computational process or Well-defined method of getting from one system to another (Skemp, 1971)
Symmetry	Property of a geometrical shape	Transformation of a geometrical shape
Circle	The locus of all points equidistant from a given point	[a curve obtained by] rotating a compass around a fixed point

Sfard identified three stages learners go through to accomplish the transition from process to object conceptions: *interiorization*, *condensation* and *reification*. The stage of *interiorization* is the stage when "a learner gets acquainted with the processes which will eventually give rise to a new concept" (p. 18). In the case of function for example, she classified a learner in this stage when the learner has the idea of a variable and has acquired the ability to use a formula to find values of the dependent variable.

When the learner compresses in chunks the lengthy and detailed sequences of operation he or she has been doing during the stage of interiorization, then the learner is said to be in the *condensation* phase of concept development. At this stage, the learner becomes "more and more capable of thinking a process as a whole without an urge to go into details" (p. 19). This chunk of detailed procedures (an example Sfard provided is a recurrent part of a computer program) is eventually treated as a whole and might be given a name, thus becoming a new entity. In the case of function, Sfard considered learners to be in this phase when they can investigate the function, draw their graphs or combine a couple of functions. This condensation phase lasts as long as the learner associates the "new" concept with a certain process. When this "new" concept is conceived as a full-fledged object, then the concept is said to be *reified*.

Sfard defined reification as "an ontological shift — a sudden ability to see something familiar in a totally new light" (p. 19). Hence, while interiorization and condensation are "gradual, quantitative rather than qualitative changes", reification is "an instantaneous quantum leap: a process solidifies into object, into a static structure" (p.19). "The stage of reification is the point where interiorization of higher level concepts (those which originate in processes performed on the object in question) begins" (p. 20). Sfard considered proficiency in solving equations in which the "unknowns" are functions, ability to talk about general properties of different procedures performed on functions and the recognition that computability is not a necessary characteristic of the set of ordered pairs which are to be regarded as function as evidences that a student is in this highest stage of concept development.

Sfard's stages of acquiring an object conception of a mathematical concept are similar to Breidenbach, Dubinsky, Hawks and Nichols' (1992) characterisation of the different levels of conception of function. These levels of conceptions are *prefunction*, *action* (or pre-process), *process* and *object*.

> For *prefunction* we consider that the subject really does not display very much of a function concept. Whatever the term means to such a subject, this meaning is not very useful in performing the tasks that are called for in mathematical activities related to function. An *action* is a repeatable mental or physical manipulation of objects. Such a conception of function would involve, for example, the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time (e.g., one evaluation of an expression). A *process* conception involves a dynamic transformation of objects according to some repeatable means that, given the same original object, will always produce the same transformed object. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. When a process is transformed by some action, then we say that it has been *encapsulated* to become an *object* (Breidenbach et al., 1992, p. 250, 251).

Sfard (1991) and Breidenbach et al's (1992) stages can be thought of as stepping-stones in understanding the concept. The present study investigated the existence of these conceptions and articulated them through describing students' strategies and reasoning as they work through tasks about functions included in mathematics for secondary schools.

Some authors (see, e.g., Dubinsky & Harel, 1992) traced the process-object theory back to Piaget who "developed the concept of *reflective abstraction*, which is, at its most elementary, a process of interiorizing physical operations on objects" (Selden & Selden, 1992, p. 9).

From this point on, this thesis uses the phrase *process-object theory* or *process-object perspectives or conceptions* or *process-object route* to refer to this path towards the understanding of a mathematical concept. Process conception in this context corresponds to what Breidenbach et al calls "action conception" and "process conception" or Sfard's "interiorization" and "condensation" stages. The object conception corresponds to the other end of the process-object continuum, which is the reification stage or object level.

Property-oriented View

Slavit (1997) presented an alternate route to objectification of the process conception of function. He argued that through experiences with various function exemplars and noting their properties, students could conceive function as objects either possessing or not possessing these properties. Students can think of "function as entities possessing various growth properties of a local and global nature" (p. 260). Examples of global properties of function are periodicity, symmetry, and asymptotes. Local properties are slope, intercepts and points of inflection.

Slavit (1994) hypothesised that the property-oriented view develops in three stages, which correspond to the instructional sequencing of a typical elementary function course:

> *Stage 1: Equivalence of procedures across representations*. The first stage of the functional property view involves an ability to realize the equivalence of procedures, which exist in different representations. Prerequisite to this ability are translation skills (Janvier, 1987). Noting that the processes of symbolically solving $f(x) = 0$ and graphically finding the x-intercepts are structurally equivalent demonstrates this awareness. It is here that the student first begins to deeply understand the underlying mathematical purposes of the procedures by making connections between analogous procedures across different representations.
>
> *Stage 2. Equivalence of procedures across function classes.* A view of function as a related set of properties also involves the ability to generalise procedures across classes of functions. Students at this second stage can translate procedures across representations (Stage 1), but are also beginning to realise that some of these procedures have analogues in other function classes. For example, the symbolic algorithm used to solve linear and quadratic equations are different, but they are equivalent when discussed in terms of finding zeroes of functions. Moreover, these procedures are identical in graphic setting, and they are structurally the same across symbolic and graphic representations.... Hence, the ability to translate procedures across representations combined with knowledge of various function classes yields a more general understanding of the conceptual result of a procedure and defines the stage of structural thought.
>
> *Stage 3. Procedural networks as permanent constructs.* The final stage of this process extends the student's ability to identify functional properties. Once the properties are identified, the student can "see" a function as an object either with or without these functional properties. For example, a quadratic function could be viewed as a continuous function with exactly one extrema that is symmetric about a vertical line (with, of course, second degree growth). Seeing a function as possessing these properties would allow the student to view the function as a well-defined object. (pp. 7-8)

Slavit cautioned that to see a function as a well-defined object possessing these properties requires an enormous amount of prior knowledge and familiarity with specific examples of functions and their properties. This therefore may not be expected of junior high school students whose experiences with the different kinds of functions are limited to basic ideas and properties of the specific functions such as linear, quadratic and exponential. They may not fully attain the stages described above but it is still important for teachers to assess students' understanding in terms of the properties of specific function included in the syllabus.

The property-oriented view expands the notion of function as an object that can be transformed by performing action on it (as described in Sfard's and Breidenbach et. al.'s work) to the notion of a function a permanent construct, an object that either possesses or does not possess properties of the concept. The present study incorporated this view of function as an object. That is, students that use the invariant properties of function in working with a particular task on function were considered as having an object conception of function.

Correspondence/Relational View and Covariance View

"Understandings based on causal and dependency relationship between input-output pairs comprise the essence of a relational view" (Slavit, 1997, p. 262). The relational view is emphasised in most textbooks and teaching approaches. In this approach, one builds a rule that will determine a unique y-value from any given x-value. Hence, a correspondence between x and y is built (Confrey & Smith, 1994). The covariance view involves an understanding of the way in which the dependent and independent variable changes. It involves analysing and comprehending the relationship between the changing quantities (Slavit, 1997). That is, it "entails being able to move operationally from y_m to y_{m+1}, coordinating movement from x_m to x_{m+1}" (Confrey & Smith, 1994, p. 137).

The correspondence/relational view is reflective of the definitions of function as a correspondence or dependence relation. The covariance view on the other hand goes beyond noting relationship between the two quantities to noting the relationship between the changes in each quantity. This is an important step in understanding because it paves the way to an understanding of rate of change. It is not clear from the literature which of these views, correspondence or covariance, students commonly use or is acquired first. It seems that in terms of level of thinking involved, the correspondence view will be much easier to acquire since it involves a causal and dependency relationship of input-output pairs. It involves point-by-point interpretations, which is reflective of conceiving function as a procedure or a process. The covariance view on the other hand supports interpretations in terms of important properties of function and holistic thinking of relationships between the varying quantities.

Summary of the Different Perspectives in the Understanding of Function

The preceding discussions described the different conceptions of function, namely the *process* and the *object* conceptions. Sfard, Breidenbach et al. and Slavit recognise that the object conception comes later. Breidenbach et al. identified three stages before one can achieve an object conception. These are *prefunction*, *action*, and *process* conceptions, which also *roughly* correspond to the *interiorization* and *condensation* and *reification* stages, which Sfard described. Slavit proposed an alternate route to conceiving function as an object and that is through identification and recognition of equivalent procedures performed on different representations, on different function classes and finally the identification of properties of function in general or properties unique to a specific family of functions. The two other views, *relational* and *covariance*, both emphasise on the relationship between the quantities involved in the function. The former relates the quantities between *x*'s and *y*'s while the latter relates the quantities between *change in x's* and *change in y's*.

In the next section, recent frameworks used in studies on students' understanding of function will be described.

Some Theoretical Frameworks for Analysing Understanding of Function

This section describes some recent frameworks for understanding of function. Early works on function focused on various modes of representation and the translation between representations (see Janvier, 1987; Kaput, 1989). More recent frameworks combine the process-object perspectives of function and the different representations of function. The frameworks developed by Moschkovich, Schoenfeld and Arcavi (1993), DeMarois and Tall (1996), and O'Callaghan (1998) were of this type.

Moschkovich, Schoenfeld and Arcavi's Framework

To work competently with function, Moschkovich, Schoenfeld and Arcavi (1993) proposed that one should think along at least two dimensions. The first dimension involves the different representations of function. They considered the three most basic: tabular, algebraic and graphical. The second dimension involves the process and object perspective of function. The framework is shown in Figure 2.

Representations

		Tabular	Algebraic	Graphical
Perspectives	Process			
	Object			

Figure 2. A schematic characterisation of the framework, indicating alternate perspectives of functions in typical representations.

Moschkovich et al. considered a student competent in working with function when he or she "knows which representations and perspectives are likely to be useful in a particular problem context and is able to switch flexibly among representations and perspectives as seems appropriate" (p. 74).

The framework in Figure 2 is simple and shows what needs to be considered in analysing understanding of the concept. The framework does not indicate an order between the process or object perspective. Moschkovich et al. argued that the ability to switch between perspectives depending on what is needed in a problem should be the aim of instruction.

DeMarois and Tall's Framework

DeMarois and Tall's (1996) framework (see Figure 3) is an elaboration of the framework of Moschkovich, Schoenfeld and Arcavi. Each facet of the diagram corresponds to the different representations of function: the *function notation* (including the meaning of $f(x)$); *colloquial* (use of function machine as input-output box); *symbolic* (algebraic formulae); *numeric* (table); *geometric* (graph); and, *written* and *verbal* representations. The reification (or encapsulation) of function from process to object conception was subdivided into five levels or layers as they are represented in the framework. These five levels of understanding are *pre-procedure*, *procedure*, *process*, *object* and *procept*.

In the framework, a student is assigned in the *pre-procedure* layer when he or she is at the lowest level, with respect to a concept. It denotes that the student has not attained the *procedure layer*, which is indicated by a need for a specific algorithm. The *process layer* is not dependent on individual steps of the procedure but rather on the result produced from the original input. An example provided by DeMarois (1997) to distinguish between the procedure and process layers involves the recognition of whether $f(x) = 2x + 6$ and $f(x) = 2(x+3)$ are the same function or not. Students who say they are different functions were considered in the procedure layer while those who say they represent the same function were considered to be in the process layer. Those students classified in the *object* layer were able to treat function as a manipulable mental object to which a procedure can be applied. The students' understanding reaches the *procept* layer when they have the ability to move between the process and mental object in a flexible way. Moschkovich et al. emphasise that this should be the aim of instruction. Except for the procept layer, the layers described in this framework correspond to Breidenbach et al.'s levels of conception of function discussed earlier.

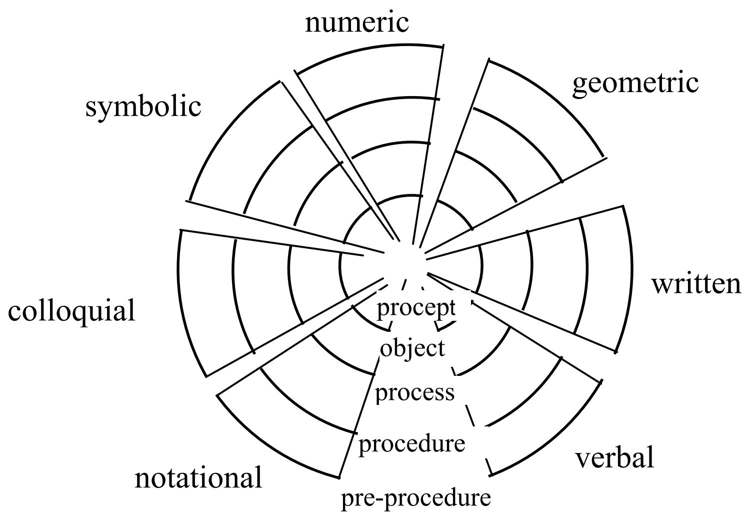


Figure 3. DeMarois and Tall's framework.

O' Callaghan's Framework

The function model proposed by O'Callaghan (1998) consists of four component competencies: *modelling*, *interpreting*, *translating*, and *reifying*. The model is firmly rooted in problem solving.

Modelling according to O'Callaghan refers to a student's ability to represent a problem situation using functions. This component can be divided into a number of subcomponents depending on the representation system used – equations, tables or graphs. The second component is the *interpretation* of function in their different representations in terms of real-life situations. As with modelling, this component can also be partitioned into subcomponents depending again on the representation system used. The students may be asked to make different types of interpretations or to focus on different aspects of a graph, for example, individual points versus more global features.

Translating refers to students' ability to move from one representation of a function to another. *Reifying*, the final component in the framework as had been already elaborated in previous discussions is taken to mean conceiving function as a mathematical object and having the ability to perform operations with it.

Associated with the four competencies are the procedural skills, which consist of transformations and other procedures that allow students to operate within a mathematical representational system.

O' Callaghan's framework added a third dimension to the frameworks in Figure 2 and Figure 3. It included competence on applying function in real-life situations. Although rooted in problem solving, the framework also recognises the importance of acquiring procedural skills as part of understanding function.

The three frameworks just described are theoretical frameworks hence they are too general to be useful for teachers. For example, O'Callaghan and Moschkovich et al.'s frameworks are more of an account of the important aspects of function. Their frameworks do not describe what students do as they come to understand function. DeMarois and Tall's framework gives a clear idea as to where teachers should lead their students in developing understanding of function. However, it does not reflect in a more specific form, information about students' strategies and reasoning as they come to understand function. Studies have shown that teachers' knowledge of students' strategies and reasoning as they come to understand a mathematical concept can be a powerful tool in informing instruction (see e.g., Clarke, 2001; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996).

The Study of Function in the Secondary Schools

Syllabi and curriculum framework usually outline some operational invariants situations and that make a particular concept meaningful especially to the learners. Because the aim of the study was to describe developing understanding of function of students in the secondary school, the study considered the components of function studied in this level. The study of function is usually part of the study of algebra. This section presents function as included in the curriculum in United States of America, Australia and the Philippines.

The Study of Function in the U.S.A.

In the United States of America, typical topics in algebra include: properties of real and complex numbers; forming and solving first and second degree equations in one unknown; simplifying polynomial and rational expressions; representing linear, quadratic, exponential, logarithmic, and trigonometric functions graphically and symbolically; and, sequences and series (Kieran, 1992). In the National Council of Teachers of Mathematics (NCTM) document *Curriculum and Evaluation Standards for School Mathematics* (1989), there is a separate section for function to emphasise its importance as one of the central ideas in the study of mathematics.

> In grades 5-8, the mathematics curriculum should include explorations of patterns and functions so that students can –
>
> - describe, extend, analyse, and create a wide variety of patterns;
> - describe and represent relationships with tables, graphs and rules;
> - analyze functional relationships to explain how a change in one quantity results in a change in another;
> - use patterns and functions to represent and solve problems (p. 98).
>
> In grades 9-12, the mathematics curriculum should include the continued study of functions so that all students can –
>
> - model real-world phenomena with a variety of functions;
> - represent and analyse relationships using tables, verbal rules, equations and graphs;
> - translate among tabular, symbolic, and graphical representations of functions;
> - recognize that a variety of problem situations can be modelled by the same type of function;
> - analyze the effects of parameter changes on the graphs of function;

so that, in addition, college-intending students can –

- understand operations on, and the general properties and behaviour of, classes of function (p.154).

The Study of Function in Australia

In the *National Statement on Mathematics for Australian Schools* of the Australian Education Council (1990), function is also included in the algebra strand, which is subdivided into three subheadings: expressing generality, function and equation.

The document recommends that students should be provided with experiences with functions that would enable students to

- represent (verbally, graphically, in writing and physically) and interpret relationships between quantities;
- identify variation in situations and use the idea of variable;
- draw freehand sketches of and interpret graphs which model real phenomena qualitatively;
- use graphs to model real life situations and make predictions including those based on interpolation, extrapolation, slope and turning points;
- recognise algebraic expressions of linear, reciprocal, quadratic and exponential functions and the graphs which represent them; and,
- use algebraic expressions (formulae) to model situations and make predictions based on the general characteristics of the formulae. (pp. 193 -201)

These experiences are considered in the document as necessary for typical citizens. For those who want to pursue higher learning, the document recommends that in addition to the above list, experiences with functions should be provided which enable these students to

- identify and express recursion and periodicity in various contexts
- recognise and determine important features of families of function
- recognise different situations which can be modelled by the same function and fit curves to data sets. (pp. 207-208)

The Study of Function in the Philippines

Almost half of the topics included in the algebra syllabus in the Philippines are devoted to the study of function (see Bureau of Secondary Education, 1998). The concept of function is formally introduced in Year 8 (aged 14) and each family of function is studied almost exclusively. That is, linear functions are studied in Year 8, quadratic functions in Year 9 and exponential, polynomial, and functions related to circles in Year 10. The structure for studying each family of function is almost the same. It includes identifying properties of function in different representations, translating between representations with emphasis on finding the algebraic representations using different techniques especially for linear and quadratic functions, investigating the effects of parameters on the graph of the function, and applying function in real life situations.

In the Regional Science High School (RSHS), from which the present study collected the data, the teachers are only provided with the topics to cover in each year level. The topics covered in the study of function are linear relationship for Year 8, quadratic equations and functions, variation, polynomial function and circular function for Year 9 and review of functions and introduction to analysis for Year 10 (Department of Education, Culture and Sports, 1994).

Although there are differences in the emphases given, common competencies in each country include the following:

- translating between representations
- interpreting the different representations
- recognising/identifying the different properties or features of the function
- modelling/applying functions

The competencies are all stated in terms of outcomes and the level of reasoning or the process to obtain the outcome is not explicitly stated in the curriculum framework. There is therefore a need to supplement the curriculum with a framework that would focus on the level of abstraction involved in the competencies. This was the aim of the study.

The next section outlines some of the skills associated with function understanding.

Skills/Tasks Involved in Learning the Concept of Function

Concept acquisition is intimately linked to the action or operations performed on it (see Schwarz & Dreyfus, 1995; Vergnaud, 1997). This section presents a taxonomy of skills involved in learning function and the tasks related to function, which have been used in most research on students' understanding of this concept. Students' difficulties and misconceptions were also looked into to identify big ideas in students' developing understanding of functions.

Tasks and Skills

Schwarz and Dreyfus analysed a large set of problems involving function. From these, they formulated a taxonomy of skills involved in learning the concept. They used the term *representative* to refer to the different representations of function, arguing that in solving problems about function, "one is in fact dealing with (acting on, operating on, transforming) one or several representatives of that function" (p. 125). The skills are classified into three classes, S_1, S_2, and S_3. Listed under each skill are specific skills associated with it.

S_1: Be able to cope with the fact that representational information is partial

S_{1a}: PARTIAL DATA: To be able to recognize discrete numerical and/or graphical information about pre-image-image pairs (graphed, tabulated or listed) as belonging to a continuum of data points. Being able to infer properties such as increase of the function from this discrete information. Such recognition is closely linked to S_{1b}

S_{1b}: INTERPOLATION, particularly interpolation between points in a graph; often, interpolation needs to be sufficiently smooth to take concavity into account.

S_{1c}: PARTIAL GRAPH: To be able to recognize and use the fact that any representative from the graphical setting has properties that derive from the abstract mathematical graph

S_{1d}: LINKING PARTS OF A GRAPH: To be able to integrate into a single graph of a function several partial graphs of that function from different, possibly partially overlapping domains and with possibly different scales

S_{1e}: ARBITRARINESS: To be conscious of the fact that a function is "arbitrary" and to use this in order to think in a flexible manner about several/many/all possible interpolations (see skill S_{1a}), even if the given data suggest a very specific interpolation

S_2: Be able to link between representatives belonging to different settings

S_{2a}: LINKING BETWEEN GRAPHICAL AND NUMERICAL INFORMATION: This includes for example, the ability to move points from a table or list into a graph. But it also includes the ability to induce numerical information using qualitative properties of graphs.

S_{2b}: LINKING BETWEEN NUMERICAL AND ALGEBRAIC INFORMATION: realize that any algebraic rule is the representative of a set of (numerical) pre-image-image pairs and be able to reason within this framework; in other words: be able to handle functional properties as relationships between these ordered pairs. A more advanced aspect of this skill is symbolization from numerical information; e.g. find a quadratic function with vertex (0,5) which goes through (5,0).

S_{2c}: LINKING BETWEEN ALGEBRAIC AND GRAPHICAL INFORMATION: This includes graphing on the basis of algebraic information, including many levels from point plotting to calculus-based discussion.

S_3: Be able to carry out transformations between representatives within the same setting

S_{3a}: REORDERING TABLE: Understanding that order in a table is irrelevant and be able to reorder a table for example, according to increasing values of the independent variable.

S_{3b}: SCALING: This is the prototype link within the graphical setting. It is the ability to recognize and carry out, at least intuitively, a stretch (or shrink) transformation on one or both axes of a graph; in other words, to construct the transform of a given graph under a different linear setting. This is a visual, analytic rather than an algebraic skill.

S_{3c}: LINKING PARTS OF A GRAPH: The ability to integrate into a single graph of a function several partial graphs of that function from different domains and with possibly different scales.

S_{3d}: TRANSFORMING FUNCTIONS: The ability to create new functions from given ones by specified rules such as shifts and reflections (pp. 267-270).

Leinhardt, Zaslavsky, and Stein (1990) also analysed tasks on functions, graphs and graphing, which have been the focus of the studies on function. They identified *interpretation* and *construction* as the two actions that learners do with functions and graphs. The four tasks which involve these actions are:

prediction tasks, which rely mostly on construction and address the issue of pattern;

classification tasks, which require interpretation and address the definition and special properties of functions;

translation tasks, which can be either interpretation or construction and address the issue of representations; and

scaling tasks, which also can be either interpretation or construction and involve decisions regarding scale and unit that are characteristic in particular to the domain of graphing (p. 4).

The taxonomy of skills by Schwarz and Dreyfus and the different kinds of tasks on function identified by Leinhardt, Zaslavsky, and Stein provided the present study with an initial bases for the development of tasks on function and the skills on which to focus.

Students' Difficulties and Misconceptions with the Function Concept

Sierpinska (1990) argued that the act of understanding and the act of overcoming epistemological obstacles are two sides of the same coin. It is therefore possible that overcoming difficulties and misconceptions, whether they are *didactically related* or *epistemologically related* (Artigue, 1992) would reveal some big ideas in students' understanding of function. Students' difficulties and misconceptions were therefore considered in the study especially in designing the assessment tasks that would have the potential to draw out the most sophisticated strategy and depth of understanding of function.

Two of the most reported persistent difficulties in students' understanding of function involve understanding of the links between the different representations and compartmentalisation of students' understanding of function especially in the different representations (see e.g., Leinhardt, Zaslavsky & Stein, 1990; Schwarz & Dreyfus, 1995; Sierpinska, 1992).

Junior high school students' difficulties with function were also identified in several studies. Markovits, Eylon and Bruckheimer (1986) reported that whatever the particular nature of the question, three types of function caused difficulty: the constant function, a function defined piecewise, a function represented by a discrete set of points. There was also general neglect of domain and range, whether attention to them was explicitly required by the question or only implicitly. "Complexity" of technical manipulations inhibited success and when examples of functions were required, there was an excessive adherence to linearity.

Bell, Brekke and Swan (1987) reported a more specific study on students' difficulties in understanding graphs. In developing the teaching and examination module, *The language of function and graphs*, a diagnostic test before and after using the materials was given to eight third-year mathematics classes in a comprehensive school in England. From the results of the test, students' responses were grouped into general types of difficulties involved. Some of the difficulties and misconceptions as reported by Bell et al. included: the misconception that a graph is a picture of the situation; it is a big step for students to realise that graphs can also show the relationship between two variables; and, students find difficulty in coordinating information relating to two variables and the two axes. They also noted faulty conclusions often stem from attention to one variable only.

Tasks developed to assess the growth points address many of the identified difficulties. It is expected that students' strategies in overcoming these difficulties will reveal some "big ideas" in their making sense of the concept of function.

Some Studies on Stages in Students' Understanding of Function

One of the earliest study describing levels of understanding of function was conducted by Thomas (1971). Thomas studied American high school students' understanding of function. A group test on function was given to 201 seventh- and eighth-grade pupils (about 13 year olds) and 20 students for individual interview. All the students were of above average ability. From the students' responses, Thomas suggested the following stages in the growth of the idea of function.

> Stage 1. Concept identification: ability to discriminate instances and non-instances of function.
>
> Stage 2. Process: ability to work with various representations and names of function in finding images, preimages, domain, range, and set of images.
>
> Stage 3. Operations: ability to carry out operations on functions with an indication that the result of the operation is understood again to be a function.

Kerslake's (1981) study, which is a part of a large-scale study in England, called the Concepts in Secondary Mathematics and Science (CSMS), investigated the underlying ideas, which are necessary components of the understanding of graphs. After a series of interviews and preliminary class tests, test items were constructed and given to a sample of 459 second year pupils (13 years old), 755 third year pupils (14 years old) and 584 fourth year pupils (15 years old) from different secondary schools. The items were grouped into three. Each group is called a level, and a student is assigned to the highest level in which he or she is successful on about two-thirds of the items. The three levels were described according to the tasks that were involved.

> Level 1: included plotting points, interpreting block graphs, recognition that a straight line represents a constant rate, and simple interpretation of scattergrams.
>
> Level 2: included simple interpolation from a graph, recognition of the connection between rate of growth and gradient, use of scales shown on a graph, interpretation of simple travel graphs and awareness of the effect of changing the scale of the graph.
>
> Level 3: consist of items that require understanding of the relation between a graph and its algebraic expression (p. 134).

Both Thomas' and Kerlake's stages and levels do not include the strategies the students used. Moreover, Thomas' study focused on the understanding of the definition of function and associated subconcepts such as image and pre-images while Kerslake's focused on levels of tasks. The present study described students' understanding in terms of strategies and "big ideas". Students' strategies are reflective of the degree of abstraction and sophistication the student is working on rather than competencies.

Garcia-Cruz and Martinòn (1998) investigated students' processes of generalisation. The main goal of the research was to identify some hierarchical levels of generalisation that can reflect students' performance when dealing with problems involving generalisation. The first phase of the study consisted of video-recorded interviews with eleven students (15-16 year olds). The second phase was an interactionist teaching experiment with a group of 18 students. The researchers summarised their findings as follows:

> *Level 1 (Procedural Activity)*
>
> At this level, the student recognises the iterative and recursive character of the linear pattern…These strategies are not generalisable but are important in highlighting the constant difference of the linear patterns….
>
> *Level 2 (Procedural understanding. Local Generalization)*
>
> At this level, the student has established a local generalization. This means that he or she has been able to establish an invariant from an action performed on the picture or numerical sequence, within any new problem given, although this invariant could be different from problem to problem….
>
> *Level 3 (Conceptual understanding. Global Generalization)*
>
> At this level, the student has generalized a *strategy*. That means that he or she has performed the same action and established the same invariant in a new but similar problem. The *rule* developed and used in an early problem is now an object, which serves as a stimulus for an action…At this level, what is achieved as a generalization is the students' overall performance when dealing with these situations, and this is what we call a strategy. … The students' cognitive behaviour could now be considered as conceptual understanding (p. 334).

Garcia-Cruz and Martinòn's study was not directly about function but it does tell how students generalise from a linear numerical pattern, which might eventually lead them to express the generalization in symbolic form. This learning trajectory is reflective of a process to object understanding of algebraic formulae.

Components of Function Concept

The literatures reviewed show that function is a complex concept. The definition of function has evolved from the more intuitive dependence relationship between two quantities to the more abstract definition as a correspondence not only between numerical quantities but also between any two sets in general.

There are at least three representational systems used to represent the concept of function in the secondary school: the tables (including ordered pairs), graphs, and formulae or equations. Each representational system has its own strengths and limitations in representing the concept so that a full understanding of the concept of function necessitates not only an understanding and facility in working with each of these representations but also the flexibility to think of function in terms of the other representations. That is, learners must be able to recognize the link and connection between these settings. They should be able to recognize that the properties of the function remain invariant in different settings.

Furthermore, there are at least two conceptions of function with which one has to be familiar: the conception of function as a *process* and the conception of function as an *object*. Recent frameworks for looking at students' understanding of function combine the process-object conceptions and the different representational systems. Acquiring an object conception is a long and difficult process. More experiences in working with function and with various families of function are needed in order to have a fully-fledged object conception.

Various authors used different terms to describe the conceptual path towards having a formal and abstract conception of function. For example, Sfard used the terms interiorization, condensation and reification. Breidenbach et al used prefunction, action, process and object and DeMarois and Tall used pre-procedure, procedure, process, object, and procept. This path generally indicates that at first, students conceive of function as a computational process, for example, using the functional equation to find the "*y*" given a specific value of "*x*". Then students start to think of function as a whole "chunk" or a single entity but still as a set of procedures. Then later they start to think of function no longer as a set of procedures but as a mathematical object in itself that they can manipulate or transform.

Being a concept, function has properties like intercepts, growth, etc., and some familiarity with these properties is needed to appreciate and use it. Thus, a different route to acquiring an object conception of function was proposed by Slavit. This is through recognising the equivalent procedures performed on the different representations, recognising equivalent procedures performed on the different families of function, and recognition of the invariant properties of functions.

As mentioned earlier, context or situations that make the concept meaningful are important factors in acquiring understanding of a concept. The curricula examined in the literature are explicit on this aspect as well as the framework developed by O'Callaghan.

The image of understanding the function concept that emerges based on the literature considered is three-dimensional. On one dimension are the different conceptions of function; on another dimension are the different representations of function students have to work with and on the third dimension are the situations in which functions are used or applied. This is illustrated in Figure 4.

Tasks related to function would more or less include a significant amount of each of the components in Figure 4. Some of the tasks would be nearer to one of the planes than in the other planes. If one would think of the taxonomy of skills provided by Schwarz and Dreyfus (1995) as points in space, most of those will be located in the plane containing the conceptions of function and representations of function in Figure 4. A good number of the competencies in school syllabi would be located near the plane containing the situations and representations of function. The aim of the present framework is to increase teacher awareness of the vertical dimension (the conceptions of function) as this path points towards having a more abstract notion of the concept.

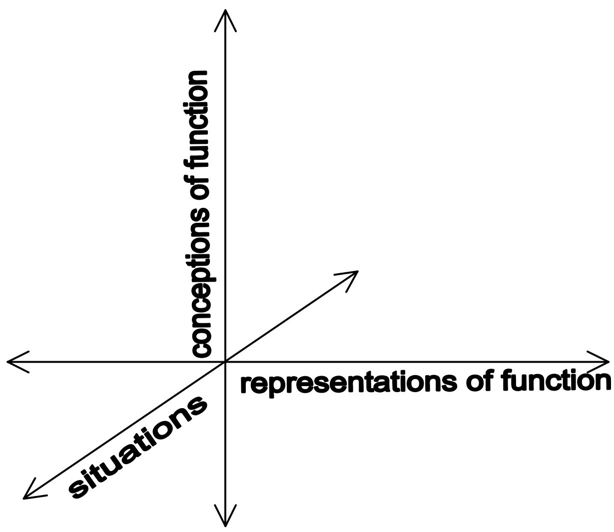


Figure 4. Key components in understanding function.

The second part of the literature review discusses some of the theoretical underpinnings of the notion of growth points in understanding, as they are understood in the present study. This chapter concludes with an initial outline of the framework of growth points.

Notion of Growth Points and Understanding

The following sections describe the notion of growth points and understanding.

Cognitive Structure, Chunking and Growth Points

The notion of growth point is based on the view of "knowing as having *structures of information and processes* that recognize and construct patterns of symbols in order to understand concepts and exhibit general cognitive abilities, such as reasoning, solving problems, and using and understanding language" (Greeno, Collins & Resnick, 1996, p. 18) [italics, mine]. Symbolic information processing, constructivism and the tradition of Gestalt psychology were identified by Greeno et al. to espouse this view. Gestalt psychology "emphasized the structural nature of knowledge and the importance of insight in learning" (p. 16). Symbolic information processing, which "focused on characterizing processes of language understanding, reasoning and problem solving", also gave "emphasis on the organization of information in cognitive structures" (p. 16). Likewise, constructivism within which "most recent research on students' conceptual learning in subject matter domains has been organized" (p. 22), also assumes the existence of "cognitive structures that are activated in the process of construction" and that these "cognitive structures are under continual development" (Noddings, 1990, p. 10).

The importance of the acquisition of cognitive structures has been shown in studies of people who have developed expertise in areas such as mathematics, physics, chess, etc. Bransford, Brown & Cocking (1999) summarised the key principles of experts' knowledge. Some of these are:

1. Experts notice features and meaningful patterns of information that are not noticed by novices.
2. Experts have acquired a great deal of content knowledge that is organised in ways that reflect a deep understanding of their subject matter.
3. Experts' knowledge cannot be reduced to sets of isolated facts or propositions but, instead, reflect contexts of applicability: that is, the knowledge is "conditionalized" on a set of circumstances.

4. Experts are able to flexibly retrieve important aspects of their knowledge with little attentional effort (p. 19).

The superior ability of experts to retrieve important information from memory has been explained in terms of how they "chunk" aspects of information that are related by an underlying function or strategy (Bransford, Brown and Cocking, 1999). The idea of chunking is a key concept in information processing theories. Biggs and Moore (1993) explained that chunking could be done either by repeating it over and over (rehearsal), or by linking it to something, one already knows (coding). For Biggs and Moore, rehearsal is useful in acquiring physical skills or verbal tasks where there is no intrinsic structure, or when the individual is unable to use the structure if there is. On the other hand, coding is used when the material has some kind of structure (e.g., mathematical concepts) and when the individual is able to make use of that structure. They stressed that coding is "almost always preferable as a means of chunking" (p. 215).

Growth points as they are referred to in this research are some form of a *cognitive structure* or a *meaningful chunk* of information which students construct in the process of understanding a concept.

Other Concepts Related to Growth Points

The notion of growth point is also related to the notion of schema, theorems-in-action (Vergnaud, 1997) and key cognitive processes (Hiebert & Wearne, 1991). A schema is a network of well-connected ideas, skills or strategies an individual uses in working with a particular task (Marshall, 1990). Vergnaud proposed the notion of theorems-in-action and described this as propositions students hold to be true "even though they may be totally implicit, partially true or even false" (p. 14). Hiebert and Wearne described the characteristics of key cognitive processes: they can be owned by students, they can be linked into students' growing web of conceptual knowledge, they possess features of higher order thinking and should involve the construction of meaning, and they transfer to novel contexts.

Conceptual and Procedural Knowledge

Conceptual knowledge in mathematics is "knowledge of those facts and properties of mathematics that are recognized as being related in some way" (Hiebert & Wearne, 1991, p. 200). It is "knowledge that is understood, … a knowledge that is rich in relationships. … A *unit* of conceptual knowledge is not stored as an isolated piece of information; it is conceptual knowledge only if it is a part of a network" (Hiebert & Carpenter, 1992, p. 78) [italics, mine]. This implies that the quality of conceptual knowledge is a function of the strength of the connection or relationships between the concepts involved.

Hiebert and Carpenter defined procedural knowledge as a sequence of actions and as such, the connection between concepts involved is minimal. An example of procedural knowledge is knowledge of standard computation algorithms, which consist of a step-by-step sequence of procedures of symbol manipulation. Hiebert and Carpenter argued that procedural knowledge could contribute to mathematical expertise *only if* it is related to conceptual knowledge: "From the expert's point of view, procedures in mathematics always depend upon principles represented conceptually" (p. 78).

The growth points in understanding function as they are referred in the research include ideas, skills and strategies students use in working with tasks about function. They therefore include both procedural and conceptual knowledge. However, there is more emphasis on conceptual knowledge.

Understanding and Growth Points

Skemp (1986) identified two kinds of understanding: relational understanding and instrumental understanding. He described relational understanding as "knowing both what to do and why" and instrumental understanding as "rules without reason" (p. 20). A close link between one's conceptual knowledge and procedural knowledge promotes relational understanding.

Another notion of understanding is the idea of understanding as making connections. In mathematics, this idea implies that one understands when he or she can make connections between ideas, facts or procedures (Hiebert & Wearne, 1991). In making connections, one not only links new mathematical knowledge to prior knowledge but also creates and integrates knowledge structures (Carpenter & Lehrer, 1999). Thus, the process of understanding is like building a network. Networks are built as new information is linked to existing networks or as new relationships are constructed (Hiebert & Carpenter, 1992). If one imagines a weblike structure, the mental representations constructed in the process of understanding can be thought of as nodes. These nodes are themselves "networks". These smaller networks resemble what is called a *schema* in cognitive psychology which as described earlier is a network of well-connected ideas, skills and strategies an individual uses in working with a particular task (Marshall, 1990).

Von Glasersfeld (1987) described understanding as a "never-ending process of consistent organization" (p. 5). It is not an all or none phenomenon hence "it is more appropriate to think of understanding as emerging or developing rather than presuming that someone either does or does not understand a given topic, idea or process" (Carpenter & Lehrer, 1999, p. 20). Each growth point in the framework therefore can be considered as students' "reorganised knowledge" and it is continuously being reorganised.

The Early Numeracy Research Project (ENRP) in Victoria, Australia which developed a framework of growth points in early mathematics learning had come to describe growth points as conceptual landmarks or key stepping stones along paths to mathematical understanding (Clarke, 2001). The growth points are understood as pupils' big mathematical ideas and hence it is possible that there are other "interim" growth points. It is also acknowledged that pupils might not necessarily pass through all the growth points along the way.

Summary and Conclusion

The preceding sections discussed the notion of growth points and understanding. Understanding in mathematics means making connections and it is a continuous process. The process of understanding is like building a network. The network grows as a new network is linked to existing networks or as it is reorganised as a new relationship is built. Some of the networks that are constructed are very powerful, in the sense that it allows one to perform a particular mathematical task. These structures are referred to as schemas, theorems-in-action, key cognitive processes or growth points.

The phrase growth point is used in the present research. The phrase captures the essence of understanding as something that is growing and developing. Growth points, as they are used in ENRP, describe learners' strategies and conceptual growth in specific mathematical domains. It describes learners' "big ideas" so that they are referred to as conceptual landmarks along the path to mathematical understanding.

The last part of the chapter describes the nature of the framework of growth points developed in the study. The first section summarises the theoretical considerations from which the framework of growth points was built. The second section presents the initial framework and the last section describes briefly the tasks associated with function that were used to identify strategies and thinking of the students.

The Initial Framework

As indicated in previous discussions one should think along three dimensions in considering understanding of function especially in the secondary schools: the representations of function, the situations which make the concept useful and meaningful and the conception of function as a process and/or object. The growth points identified and described in the framework developed in the present study can be thought of as major nodes or points in this space.

The conception of function as process and/ or object is the least emphasised in the curriculum. The framework of growth points developed in the study put importance on this dimension of understanding along with the other dimensions by describing the growth points in terms of the process-object theory in the understanding of mathematical concept. As previously discussed in the first part of this chapter, there are two routes toward the object conception of a concept. One is the *property-oriented path* proposed by Slavit and the other is process-object path proposed by Sfard and others. In the latter path, an object conception is generally attained after experiences in performing *actions* on the concept.

Proponents of each of these conceptual paths have identified stages or levels in understanding of a mathematical concept. Their descriptions however may not be suitable for the ideas of function studied in Year 8 to 10. In fact, they acknowledged that it requires an enormous amount of knowledge and experiences with specific kinds of functions in order to have full property-oriented and process-object perspective of function. It does not follow however that "shades" of this kind of understanding is not present in high school students' mathematical understanding. It was the objective of the present study to develop a framework that describes students' understanding of function by reinterpreting these stages in terms of the experiences with function provided in secondary schools. The study seeks to identify and describe manifestations of these stages in secondary school students' understanding of function

Table 2 shows the initial framework of growth points for understanding function. The description of the growth points are reinterpretations of the different conceptions of function based on the function studied in junior high school (Years 8 to 10).

Table 2
The Initial Framework of Growth Points

Conceptions of function	Tables	Equations	Graphs	Modelling
Procedure	Point by point interpretation to infer information	Interpretation of equation as a procedure for generating values	Point by point interpretation to infer information	Shows awareness of varying quantities
Property	Use of trends or patterns to describe related quantities and predict values	Interpretations based on properties	Interpretations based on properties	Uses properties/ trend to sketch or represent situations
Process	Uses point by point interpretation to link representations			
	Interpretations based on trend/patterns			
Object	Uses knowledge of the invariant properties of function to link representations and create new function			

Procedure, property, process, and object more or less correspond to the kind of interpretation the students used in working our function tasks: point-by-point; global (use of trends and patterns); use of their knowledge of the properties of the function; or, interpretation of function as an object. The growth points are more or less ordered according to level of abstraction. It is one of the objectives of the research to establish a general trend for the growth points.

To identify and describe the growth points, the following tasks related to function were developed. The range of tasks is described below.

Tasks Assessing the Growth Points

It is not claimed that the list of tasks includes all the tasks related to function but it does incorporate the key components. The tasks more or less correspond to O' Callaghan's component competencies in function understanding: interpreting, modelling, translating and reifying. The range of tasks should not be understood as distinct from one another. They simply indicate the focus of the tasks.

Interpreting includes tasks on predicting and generating pairs of values, interpreting the representation in relation to the given context or situations and describing function or the relationship between quantities in natural language. Interpretation may be in quantitative terms or qualitative terms.

Modelling functional situations is more on *constructing* representations of functional situations. Interpreting representations in relation to the given context is included in the first set of tasks (interpreting). The situations that will be used involve visual patterns as well as situations familiar to students.

Linking representations includes recognising equivalent representations as well as translating between representations. This therefore involves both *interpreting and constructing.*

Creating new functions involves performing operations on a given representation thereby creating a new function in the same representational system or in another.

The four sets of tasks are more or less arranged according to the degree of sophistication in knowledge and skills required.

The literature reviewed in this section informed the development of the initial framework and the analysis of the data gathered. In the course of the study, other studies and ideas from other investigators informed the analysis of the data and discussion of the results. These are incorporated where they were used.

The next chapter describes the methodology of the research to further develop the initial framework of growth points.

CHAPTER 3

METHODOLOGY

In the previous chapters, an overview of the research and a detailed discussion of relevant literature were presented. This chapter describes the methodology of the research. The discussion starts with the objectives, assumptions and limitations of the study. The second part presents an overview of the conduct of the study. The third part situates the research approach of the study within common methodologies. This is followed by the discussion of issues regarding validity and reliability of the research. The last part describes how validity and reliability issues were addressed through discussion on choice of respondents and data collection methods, choice of participants, instrumentation, and data analysis procedures used in the study.

Objectives, Assumptions and Limitations of the Study

The main objective of the research was to develop a framework describing growth points in secondary students' developing understanding of the concept of function. The purpose of the framework was to provide teachers with a structure and a tool for assessing and monitoring their students' understanding of the concept, as well as contribute to the theoretical perspectives of students' developing understanding of this mathematical domain. Hence the major question addressed in the research was *How might typical learning paths of secondary school students' developing understanding of function be described and assessed?* The specific questions that guided the collection and analysis of data were:

1. What are the growth points in students developing understanding of function?
2. What information on the students' understanding of function is revealed in the course of developing the framework of growth points that would be potentially useful for teachers?

Inherent in the objective of the research were the following principles, which also guided the formulation and development of the framework.

Principle 1. The framework should be research-based.

Principle 2. The framework should include key aspects of the function concept.

Principle 3. The framework should be in a form that would enable teachers to assess and monitor students' developing understanding of this concept.

Principle 4. The framework should reflect students' *big ideas* or growth points which describes students' key cognitive strategies, knowledge and skills in working with function tasks; and,

Principle 5. The framework should reflect typical learning trajectories or a general trend of the growth points in students' developing understanding of function.

Principle 1 ensured that the framework was empirically and theoretically supported.

Principle 2 acknowledged that the result of the research was informed by external factors other than data from the students. That is, the framework was partly determined by the structure of mathematics, particularly of the function concept. The framework was also influenced by what the researcher and persons consulted about the study thought the focus should be, and in general by what the community of those involved in mathematics education deemed important to learn about the concept. These influences were apparent in the literature from which the initial framework described in Chapter 2 was based, and the decisions made in the subsequent development of the framework of growth points presented in the succeeding chapters.

Principle 3 identified the intended users of the framework. Thus, the form of the framework was informed by what the researcher and mathematics education colleagues thought about how the framework might be of support to teachers. For example, a major feature that emerged from this study was the development of the set of tasks and the corresponding rubric that teachers could use to assess students' understanding against the framework of growth points. The descriptions of the growth points incorporated both procedural and conceptual understanding by describing the thinking processes, strategies and knowledge used by students to encourage the teachers to focus on these rather than just on the outcome and completion of the tasks. In addition, the description of the growth points makes use of language commonly used by teachers in teaching mathematics or function, in particular. Most importantly, the study identified the understanding demonstrated by Year 8, 9 and 10 students. The ultimate aim was to provide teachers with a guide that would develop their awareness of the ways students reveal their understanding.

One limitation of the present study was not to have had the framework and the assessment tasks used by teachers themselves as this would further refine or validate the framework. This is the plan for further study.

Principle 4 identified one of the research questions that guided the collection of data and analysis, which was *What are the growth points in students' developing understanding of function?* These growth points, as described in Chapter 2, were "big ideas" or meaningful chunks of information students use in working out mathematical tasks. They were described in terms of students' knowledge, skills and strategies. For example, growth points in students' understanding of function in Equations in the final form of the framework include: *interpretations based on individual points; interpretations based on analysis of relationships; interpretations based on local properties;* and *manipulation and transformations of functions (in equation form) as objects*. To identify and describe these growth points, students were given tasks involving function that would highlight thinking in terms of process-object conception and property-oriented conception of function. Students' performance on these tasks and their strategies informed the identification and description of the growth points.

Principle 5 also addressed the main aim of the study, which was to describe a typical *learning trajectory* (Cobb & McClain, 1999) of students' understanding of function. To do this, the study identified "big ideas" in students' developing understanding of the concept.

A general map of students' developing understanding of a specific mathematical domain would be useful to teachers since they would have a sense of students' typical learning paths (Von Glasersfeld, 1997; Carpenter & Lehrer, 1999). Thus, while the present study acknowledged that individuals have their own way of making sense, of giving meaning to their own experiences, have their own learning trajectory, the study also assumed that in the process of understanding a particular concept most learners also follow some sort of a general path. The van Hiele hierarchy in geometry (see, e.g., Clements & Battista, 1992) and the framework of growth points in numeracy of K-2 developed by the ENRP (Clarke, 2001) are examples of these.

Having clarified the objectives, assumptions and limitations of the study, I will now briefly describe the conduct of the study. This is followed by the discussion of the research approach adopted and the reasons for choosing these methods.

Conduct of the Study

The conduct of the research was in two stages. The purpose of Stage 1 (see Figure 5) was to develop the assessment instrument, collect a range of tasks and strategies students used in working on these tasks and identify some big ideas that reflected students' developing understanding of function. Stage 1 started with the development of an initial framework of growth points. Assessment tasks were then developed that would assess evidence of achievement of the growth points identified or would draw out strategies reflective of students' understanding of function related to the growth points identified in the initial framework. A record sheet of students' possible responses and strategies was also developed.

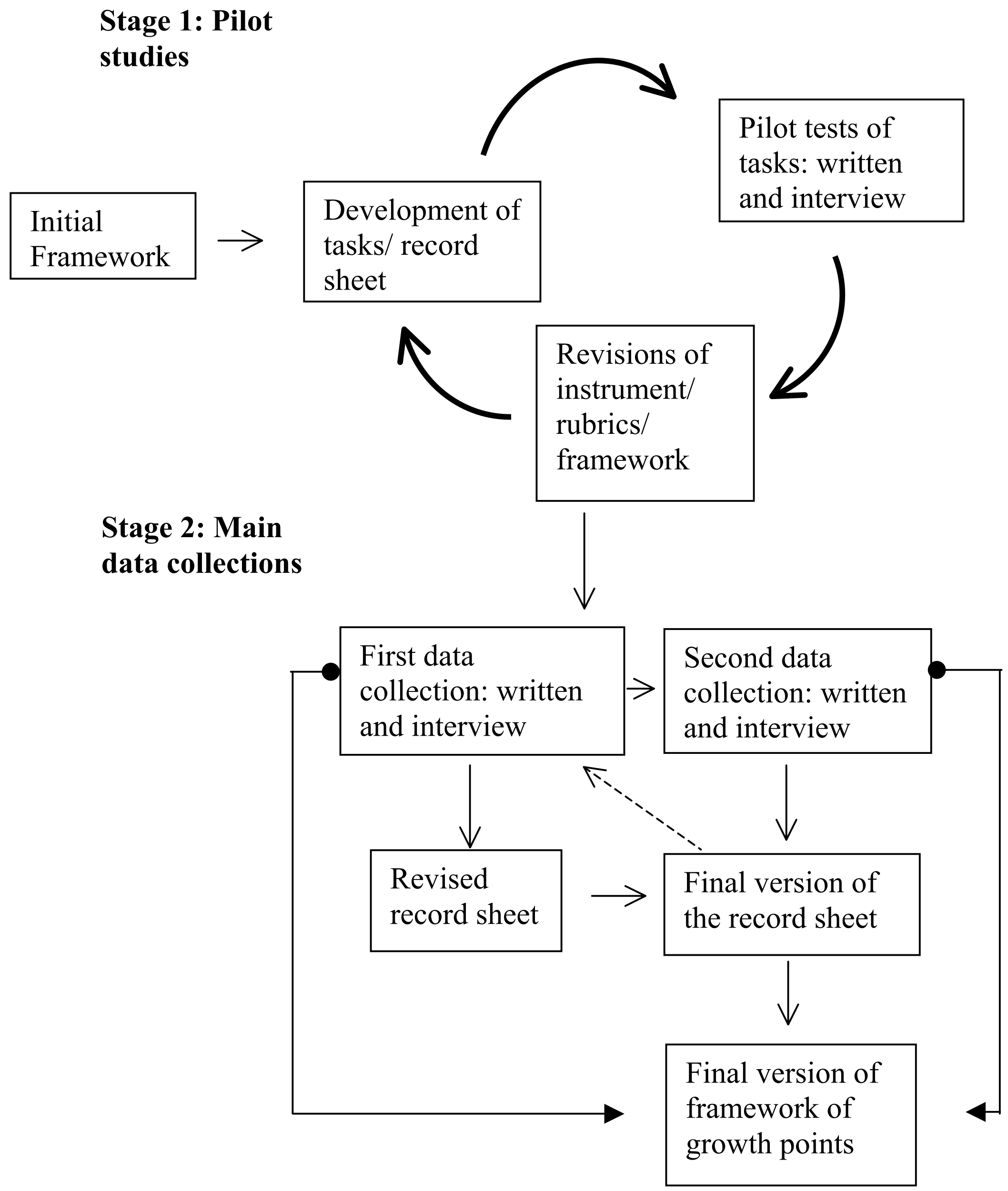


Figure 5. The research procedure.

The cyclic bold arrows in Stage 1 connecting the development of the instrument, piloting of tasks and revisions made on the instrument, record sheet and framework illustrates that this process occurred more than once. There were in fact four pilot testings of the assessment tasks conducted. Revisions made on the tasks, record sheet and the framework of growth points were informed by each other. They were continually revised after each pilot testing. Thus, the results of pilot studies were very much a part of the final framework. It was not conducted just for the development and validation of the instrument.

The purpose of Stage 2 was to further enhance and refine the framework and identify typical learning paths in students' developing understanding of function through two large-scale collections of data. The first data collection period was about three weeks after the beginning of the school year and the second collection was five months later. The same instrument was used to collect the data from the same students. This will be discussed in more detail later in the chapter.

The following section describes the research methods adopted by the present study.

Describing the Research Methodology

The discussion on the guiding principles that informed the development of the Framework of Growth Points in the first part of this chapter shows that the Framework should be both descriptive and normative. The research approach adopted in order to develop this Framework is described below.

The research approach of the present study is found within a qualitative-quantitative research methodology continuum. I will give a brief description of these two methodologies before characterising the research approach of the present study. The summary was drawn from the works of Brown and Dowling (1988), Cohen, Manion and Morrison (2000), Flick (2002) and Weirsma (1995).

Qualitative methodologies (also described as naturalistic or interpretive) such as ethnography, case studies, grounded theory and the like assume multiple realities for its subjects. In these methodologies, "objects under study are not reduced to single variables but are studied in their complexity and entirety in their everyday context" (Flick, 2002, p. 5). These studies are concerned with the production of meaning, which is carried out inductively through understanding events from the viewpoints of the subjects. They use elaborated descriptions to present an argument and to establish the validity of their interpretations.

Results are usually in a form of detailed explanation. Generalising of findings, if done, is approached with caution.

Quantitative methodologies (also generally described as positivistic and normative) such as survey, correlational and experimental studies on the other hand, assume one reality for their subjects. The world is construed as ultimately describable in terms of equally likely events. These assumptions allow the investigator to generalise the local findings to be true to a bigger if not entire population. The research procedures are carried out deductively and rigorously through discrete and distinct steps. They operate in a highly controlled setting. They are concerned with the search for facts, relationships, and cause and effect between observable, well-defined and quantifiable constructs.

The research approach of the present study will be briefly described in terms of the characteristics of quantitative and qualitative research listed in Figure 6. The figure was adapted from Weirsma's (1995) summary of the characteristics of the qualitative-quantitative continuum in educational research. These characteristics are not distinct from each other and are in fact related. Each of these will now be discussed in relation to the present study.

Inductive Inquiry	◄----------------------►	Deductive Inquiry
Theory-generation	◄----------------------►	Theory-based
Holistic Inquiry	◄----------------------►	Focused on Individual Variables
Context-Specific	◄----------------------►	Context-Free (Generalisation)
Production of Meaning	◄----------------------►	Relationship, Effects, Causes
Text	◄----------------------►	Numbers

◄--►

Qualitative Approaches | Quantitative Approaches

Figure 6. The research approach within the qualitative-quantitative research continuum.

Inductive-Deductive Inquiry

An inquiry is "deductive insofar as some orientating constructs - informed by the prior knowledge, the experience and the values of the investigator - have been put forward and operationalized and matched to a body of field data" (Sowden & Keeves, 1988, p. 514). The use of this approach has the advantage of focussing and reducing the data that could be collected. This indeed was one of the reasons why an initial framework was developed at the onset. However, the aim of the research was not to confirm the initial framework but to develop and enhance it. To do this, the present study collected a range of students' strategies in working on function tasks and used these data to enhance the framework. In this sense, it is using an inductive approach. “Induction is employed in so far as the gathered data are used to modify and rebuild the original constructs” (p. 514).

Theory-generation and Theory-based

The present study was theory-based in the sense that the process-object theory and the property-oriented conception of function, which I described in Chapter 2, underpinned the identification and description of the growth points. However the present study had also generated a “theory”, which was the Framework of Growth Points in students’ developing understanding of function.

Focussed on Individual Variables Rather than Holistic Inquiry

Holistic inquiry would involve consideration of most if not all aspects of the setting that would have an effect on the object of investigation. The present study only focussed on students’ performance and strategies in working out function tasks to inform the development of the framework of growth points. However, the research acknowledged that whatever understanding or learning students reflected in the instrument, these were products of several factors including the prescribed curriculum, various teaching styles, and students’ previous experiences. The list may be endless and the research did not tease out the effects of these factors. However, this would be an interesting further study.

In order to minimise the effects on the data of some of the factors just mentioned, the present study used non-standard textbook tasks and constructed the tasks so that they could be solved generally by natural or intuitive strategies. This is explained in more detail later in the chapter when I describe the instrument used for this research.

Context-free rather than Context-based

This heading is related to the characteristic of the research described in the previous paragraph. The effect of context on the growth points in students’ developing understanding of function was not the focus of this study. The study’s objectives were to identify and describe the growth points and identify a typical map of students’ understanding of this concept. This objective necessitates consideration of a large number of students. Therefore, while it is ideal to know the effect of context on the growth points, the present study could not focus on this as well.

Production of Meaning

The present study sought to identify big ideas in students' understanding of function and then organise them into a framework that would show a typical map of students' developing understanding of the concept. The identification and description of the growth points were based on students' performance and strategies in working on function tasks. Most of these data were qualitative or in text form and were the study's bases for analysis and interpretation. In this context, the aim of the present study was to *produce meaning* from the data gathered that would describe "big ideas" or conceptual landmarks in secondary students' developing understanding of function.

Text and Numbers

As already mentioned in the preceding discussion, the present study started with qualitative data, which were then categorised in terms of big ideas. Categorisation of the data allowed the research to deal with them in quantitative form.

To summarise, the research fits what Creswell (2002) calls *exploratory, mixed-method design*. The research was partly exploratory, qualitative, and quantitative and used inductive and deductive reasoning approaches in forming generalisations. In general, the research techniques adopted were more interpretive and exploratory during the initial stages of analysis. Then, the research moved to a quantitative approach to identify typical trends and distribution across the growth points before returning to an interpretive phase in refining the growth points in the light of the data. It could be said that in identifying and describing the growth points, the present research was more aligned with qualitative research but in establishing trend and patterns and in data collection procedures adopted, it was more aligned with the assumptions and techniques of quantitative research.

In the field of education, findings of studies must be believable, trustworthy and present insights and conclusions that ring true to teachers, educators and other researchers in order to have an effect on its theory or practice. Thus, all scientific ways of knowing, whatever is the discipline or method used in collecting data and in doing the analysis, strive for reliable and valid results (Merriam, 1988; Shulman, 1988). The following discussion describes the measures undertaken to establish confidence in the research through addressing reliability and validity issues.

Validity and Reliability Issues

Issues regarding validity and reliability are construed and addressed depending on whether the research is quantitative or qualitative (Borg & Gall, 1989; Cohen, Manion & Morrison, 2000; LeCompte & Goetz, 1982). The present study has both qualitative and quantitative components. I now discuss issues regarding validity and reliability of the research as deemed applicable.

Addressing Validity Issues

Validity necessitates demonstration that the investigators "actually observe or measure what they think they are observing or measuring" and that the result can be generalised to some extent (Le Compte & Goetz, 1982, p. 43).

Kinds of validity relevant to the present study included *interpretive validity* and *theoretical validity*, *internal and external validity*, and *content* and *construct* validity. These types of validity are discussed below in relation to the study.

Interpretive and theoretical validity.

Interpretive and theoretical validity are associated with qualitative studies (Johnson & Christensen, 2000). Interpretive validity refers to "accurately portraying the meaning given by the participants to what is being studied" (p. 209). Since the present study investigated students' understanding of function through the strategies, knowledge and skills they use in working with function tasks, it was important to establish the accuracy of the interpretation of students' performance.

Theoretical validity refers to "the degree to which a theoretical explanation developed from a research fits the data and is therefore credible and defensible" (p. 210). In the present study, these "theoretical explanations" were the growth points. Students' performance with the assessment tasks informed the identification and description of the growth points. Thus, it was important to establish the correspondence between the growth points, the assessment tasks and students' performance.

Proposed strategies that would enhance interpretive and theoretical validity include *low-inference descriptors*, *data triangulation*, *peer review*, *participant feedback* and *pattern matching* strategy (Johnson & Christensen, 2000).

Low-inference descriptors are descriptions phrased very close to that of the participants' account or researcher's notes during the data collection period. Since the study's main data were students' solutions, strategies and explanation, low-inference descriptors used to identify and assess the growth points took this form. Low-inference descriptors were used in the research as evidence of the correspondence between the growth points and the assessment tasks and students' strategies.

Data triangulation is a strategy, which involves the use of several sources to help understand the phenomenon under investigation. This was used in the present study through collecting data from three year levels and in two different times using the same instrument.

Participant feedback involves the discussion of the interpretation and conclusions of the researcher with the participants. In the present study, this strategy was implemented through conducting an interview with selected students who had taken the test regarding the strategies they used and their interpretation of the tasks.

Peer review involves the discussion of the researcher's interpretations and conclusions with other people. Discussion with research supervisors regarding the correspondence between the assessment tasks and the growth points occurred regularly. Comments from mathematics education colleagues regarding the growth points and the assessment tasks were also sought.

Pattern matching involves predicting a pattern of results and determining if the actual results fit the predicted pattern. This was done in the present study during the series of pilot studies. Predictions were based on the use of the initial framework. Content analysis of assessment tasks also enabled the researcher to predict possible growth points.

Internal and external validity.

One notion of internal validity addresses "the problem of whether conceptual categories understood to have mutual meanings between the participants and the observer actually are shared" (LeCompte & Goetz, 1982, p. 44). In this sense, it is related to interpretive validity, which also refers to accurate portrayal of the meanings given by the participants.

Another notion of internal validity refers to the degree that a researcher is justified in concluding that an observed relationship is causal (Johnson & Christensen, 2000). The study acknowledges that there are many factors affecting the students' understanding or growth point, but the study was not about identifying what "causes" the growth point so in this context, it need not address internal validity. However, in analysing the growth points, in explaining the trend in the growth point and students' movements in the growth points within a period of five months or within year levels, speculations on what "causes" them were made. Because the study was conducted in a natural setting and no treatment or experimenting on the participants was introduced, natural maturation and history such as curriculum and learning experiences were assumed to be likely factors that would affect students' performance.

External validity refers to "the extent to which research results can be generalised to populations, situations and/or conditions" (Wiersma, 1995, p. 8). It necessities demonstration that "the abstract constructs and postulates generated, refined or tested … [are] applicable across groups" (Le Compte & Goetz, 1982, p. 43). Survey studies use random sampling in order to be able to generalise the results to the population. Qualitative studies, which are more likely to use purposive sampling of participants in their investigation, address external validity by establishing the typicality of the phenomena under investigation which could then be made a basis for comparison with other groups or settings (Le Compte & Goetz, 1982).

The present study collected data from a large number of participants because it was investigating trends in the order of the growth points and aiming for the generalisability of its results. However, the selection of participants was partly purposive because the study needed students who could provide a wide range of strategies in working with the assessment tasks. This is discussed in more detail in the section on the selection of participants for the study.

The next discussion addresses validity issues related to the instrument used to collect the data: content and construct validity.

Content and construct validity.

Content validity refers to the degree to which the instrument fairly and comprehensively covers the construct, domain or concept it is investigating. It is not based on measure but a matter of judgment (Cohen, Manion & Morrison, 2000; Kerlinger, 1986). Content validity was achieved in the study through peer review. During the development of the assessment tasks and throughout the pilot studies, regular consultations with supervisors regarding the appropriateness of the tasks were made. Content analysis of the tasks in relation to the initial framework and insights gained during the pilot studies also contributed to making informed judgment regarding the content validity of the assessment tasks. My eight-year experience as a high school mathematics teacher and my five-year experience in my present work with high school mathematics teachers in professional development training programs also provided me with the necessary experience to judge the suitability and comprehensiveness of the tasks used to assess students' understanding of mathematics studied in high school.

Construct validity refers to the extent to which one can infer a theoretical construct from the result of the data collection (Johnson & Christensen, 2000). Like content validity, this is also a matter of judgement rather than measure (Cohen, Manion & Morrison, 2000; Kerlinger, 1986). "Evidence of construct validity involves formulating hypotheses about the expected behaviours that should occur from individuals who score high or low on a test and a tentative theory on why high and low scorers should behave differently" (Johnson & Christensen, 2000, p. 110). That construct validity was achieved in the present study is evidenced by the students' performance in the assessment tasks where students with more experience with mathematics performed well and were more likely to use strategies involving generality and abstraction while the opposite was true of students with lesser mathematical experience. The order of the growth points was also consistent with the process-object theory. Details of these are found Chapters 4 to 7, which present and analyse the research results in students' understanding of key domains of the function concept in Year 8, 9 and 10.

Another interpretation of construct validity concerns the extent to which meanings of terms and constructs are shared. This could be achieved through data triangulation and peer review (Le Compte & Goetz, 1982). These strategies have been used in the present study as described earlier.

The following discussion will address the issue on reliability of research findings.

Addressing Reliability Issues

Reliability of research "requires that a researcher using the same methods can obtain the same result as those of a prior study" (Le Compte & Goetz, 1982, p.35). It is concerned with the "replicability and consistency of the methods, conditions and result" (Weirsma, 1995, p. 9). Measures undertaken to achieve reliability already contribute to attaining accuracy of findings (Guba & Lincoln, 1981). Other strategies used in the present study are described below.

Strategies that would enhance the replicability of the research include: (1) outlining the theoretical premises and defining constructs that inform and shape the research; (2) explicit identification of constructs and premises that underlie the choice of terminology and method of analysis; (3) clarification of the researcher assumptions at the outset of the study; (4) adequate description of the conditions and procedures of the research; (LeCompte & Goetz, 1982; Merriam, 1991, Weirsma, 1995).

Chapter 2 of the study addressed strategies 1 and 2. It described in detail the theoretical as well as conceptual bases of students' understanding of function that the present study investigated. The preceding discussion on the objectives, assumptions and limitations of the study, the characterisation of the research approach adopted, and the conduct of the study described at the beginning of this chapter addressed strategies 3 and 4. The last section of this chapter, which describes in more detail the research procedure adopted by the present study together with the decision process involved in the choice of research procedure, participants and instrumentation were also meant to facilitate replication of the study.

Crucial to the reliability of the present study is the reliability of the instrument used to collect data. The assessment questionnaire used in the study was more like a domain-referenced assessment test and hence correlational methods that are usually used to measure reliability of criterion-referenced test are not applicable (Borg & Gall, 1989). As with domain-referenced tests, the selection of the assessment task therefore was judged on the basis of its fitness into the domain. Thus, items that everyone answers or items that only very few could answer were not eliminated. This is discussed in more detail in the section, Designing the assessment tasks.

The reliability of the instrument was also established through peer-review. That is, mathematics education colleagues and research supervisors were consulted regarding the potential of the task to assess students' understanding in the identified domain. Results of the pilot studies and the two main data collections also showed consistency in the ranking of the tasks in terms of success rates in both data collection period.

The succeeding discussion will describe in more detail the research approach of the study.

Choice of Data Collection Methods

As discussed earlier, the main objective of the present study was to develop a framework of growth points that would describe typical trends in students' developing understanding of function. This implies that the framework of growth points should be both descriptive and normative. That is, the framework of growth points should be descriptive of students' "big ideas" in key domains of the function concept, and the framework should reflect some form of order of these big ideas or growth points. The following discussion describes the research approach adopted by the present study to collect the data necessary for the development of the framework.

Use of Initial Framework

To identify and describe students' understanding of function, the present study developed an initial framework based on the process-object and the property-oriented theoretical perspectives in students' developing understanding of function. These perspectives, described in more detail in the previous chapter, provided a theoretical base for identifying an initial list of growth points in key domains of the function concept. Hence, the initial framework of growth points served as a "conceptual framework" for the study (Eisenhart, 1991). The initial framework was also made the basis for the development of assessment tasks, which were used to collect the data for the study.

Use of Assessment Tasks

Growth points are students' "big ideas" or conceptual landmarks in understanding a particular mathematical domain. They are descriptions of students' use of concepts, skills, knowledge and strategies as resources in thinking about and solving problem situations. Thus, to identify and describe the growth points, the study needed data from students' performance and strategies in solving problems. Hence, the main instrument used for collection of data was a set of assessment tasks that would highlight and draw-out different levels of abstraction in students' strategies and understanding of function and its representations. The assessment tasks and students' solutions and justifications were then analysed and classified into *meaningful chunks* of information, which led to the identification of new growth points and refinement of the descriptions of existing growth points in the initial framework.

The present study collected the data from students using the assessment tasks given in a test-like environment. Students' responses in the tasks were documented and analysed noting the mathematical models, concepts, knowledge, skills or the strategies they used and the explanations they gave. Interviews were also conducted with a limited number of students to gain more insight of their strategies and thinking. Justification for collecting the data using written assessment, supplemented by interviews, is explained below.

Use of Written Assessment

The main data of the study were collected using the assessment tasks given in a test-like environment because the use of written assessment far outweighed the advantages afforded by use of interviews as far as the objectives of the present study were concerned. Written assessment, like questionnaires, could be administered to a large number of participants and required less time in administration. The study aimed to describe a typical map of students' developing understanding of function, so this would need more than a few participants for study, not only for establishing "typicality" but for providing a range of students' performance and strategies. Time needed to work out the tasks and the nature of the tasks was also important considerations. The study needed to assess conceptual understanding in terms of students' strategies and thinking processes used in working with function tasks and not simply skills and knowledge kinds of tasks require time to think through. In addition, to gather a range of students' performance and strategies also meant more assessment tasks would be needed. It was also desirable that students be given the chance to try all the tasks, as this would enhance the validity of the research result. Administering the tasks in the form of a written test would also enable the students to select which tasks they wanted to do first. This is difficult to do in an interview.

The study recognises the limitations of written assessment. Misinterpretations and non-response are some of the serious threats to the reliability and validity of the written assessment. Language and readability, clarity of language and instruction, cultural bias, layout and length of time required to complete the assessment are just some of the threats to the accuracy of the research result (see, e.g., Cohen, Manion & Morrison, 2000).

To minimise these threats, pilots of the tasks were done and tasks were revised to maximise clarity and appropriate interpretations. Most of the tasks were pilot tested three times in each year level. Other measures undertaken to minimise this threat are discussed in a later section in this chapter.

The study also acknowledged that the data, which were the written solutions and explanations of students, might not completely reflect how students' solved the problem (Clements & Ellerton, 1995). For example, the students might not show their solutions. To minimise this concern, I explained to them the intent of the study. That is, it was made explicit to the students before the test was given that since the research was investigating *how* they solved problems, answers without explanation or solution would not help in understanding how their minds worked in solving the problem. Other means such as physical layout of the booklet and the formulation of the assessment tasks are described in the section, Designing the Assessment Tasks.

Another example concerning students' responses not reflecting how they actually solved the problem involves students writing a logical solution when they had actually obtained their answers through guess and check. How the present study handled these cases is explained in the data analysis section in the latter part of this chapter.

Measures undertaken to address threats posed by the time element and the layout of assessment task booklet used during the test are described later in the chapter.

Use of Interview

The purpose of the interview during the pilot studies in Stage 1 was more on determining the clarity of the tasks and whether they were interpreted the way the researcher hoped they would be interpreted. During the main data collections (Stage 2), the interviews were focussed on gaining further insight into students' thinking that might help the researcher make sense of their solutions and explanation in the written form. Interviewing students using the tasks gave the researcher more confidence in interpreting the written responses and describing the growth points.

Use of Task Analysis

Task analyses in terms of knowledge, skills, and strategies reflective of the process-object and property-oriented perspectives were also carried out to inform the identification and description of the growth points. The analysis of the tasks was ongoing from the time of their development, during pilot studies and the main data collection as results from students' performance revealed other ways of interpreting the data. This is further explained in the Method of Data Analysis section in the last part of this chapter and in Chapters 4 to 7 where the tasks used to assess the growth points are presented and analysed.

Use of Three Year Levels and Two Data Collection Periods in Stage 2

The data were collected twice from the same students in order to determine students' movements in the growth points and the trend in the order of the growth points. The assessment tasks were also given to students from Years 8, 9 and 10 to cover a range of students' performance and strategies. This method in effect contributed as a means to check the validity of the research findings through what Denzin (1988) called *data triangulation*. This kind of triangulation is similar to achieving *predictive validity* (Cohen, Manion & Morrison, 2000) that "is achieved if the data acquired at the first round of research correlate highly with data acquired at a future date" (p. 111). That this was achieved in the present study was shown by near consistency of the ranking of the success rates in each task and the order of growth points for both data collection periods and in all the year levels. The statistics are presented in Chapters 4, 5, 6 and 7.

Designing the Assessment Tasks

This section describes the development of the assessment tasks used to develop the framework of growth points. The first part describes the principles informing the development of the tasks. The second part describes the criteria for selecting the tasks included in the instrument, the third part briefly describes the assessment booklets, and the last part present a brief description of each of the assessment tasks.

Designing the Tasks

The initial framework of growth points provided the basis for the development of the tasks, which would highlight students' strategies, and reasoning described in the growth points. The following principles informed the design and selection of the tasks to ensure that they would generate the kind of data needed to identify and describe the growth points in students' understanding of function. The principles were based on Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier and Human's (1997) descriptions of assessment tasks that build understanding.

Principle 1. Contexts familiar to the students should be used.

Principle 2. Tasks should be formulated in a way that what made tasks problematic would be the mathematics rather than the aspects of the situation.

Principle 3. The tasks should encourage students to use their natural strategies and use skills and knowledge they already possess.

Principle 4. The task could be solved in different ways and would encourage students to use the strategy that would highlight the depth of their understanding of the concept involved.

Principle 1 is about the use of context. As much as possible, in line with the purposes of this study, context was used in such a way that it would not be the determining factor in working out the task successfully or unsuccessfully. For example, the first six tasks in Booklet 1 involve height versus age (see T1 – T5 in Appendix A). Context was only used in these tasks to make them interesting. Height and age are familiar concepts to students and would therefore not get in the way of interpreting the tasks correctly. The other three tasks (see Tasks 7a, 7b, and 7c in Appendix A) used water flowing from a pipe as the context to assess students' concept of rate and interpretation of intercept. These tasks involved interpretation of a piece-wise function. Context was used in these tasks to make it appear less threatening and unfamiliar.

Principle 2 was about ensuring that the problem was the mathematics, not the setting. Using familiar and conventional mathematical symbols also ensured that what would be problematic would be the mathematics. For example, simple equations were used so students would not be caught up with complicated computations but rather spend more time on the mathematical thinking involved. The symbols used and the way the tables, graphs and equations were set was in the way they are usually set in textbooks. See for example, Task 11 in Figure 7.

In the Philippines, English is used in the teaching of mathematics, but because it is only a second language, the English language used in describing the tasks was that familiar to Filipino students. During the pilot studies, comments were sought from teachers regarding the readability and clarity of the tasks and whether or not the mathematical content covered in the instrument were normally taught in their classes. The final copy of the assessment tasks was also shown to mathematics educators who were experienced teachers in secondary schools, inviting comments.

Principle 3 was about ensuring that the tasks required the use of natural strategies and skills students already possessed. To do this, most of the tasks used were non-standard textbook problems. This was done to discourage students from using learned solutions or algorithms mechanically. For example, in Task 11 (see Figure 7), which involves the concept of inverse function, the term "inverse" was not used so as not to provide a hint to students who have been taught the algorithm of finding the inverse of function.

11. The relationship between x and y in Table 1 is $y = 2x + 1$. In Table 2, the values of x and y in Table 1 were swapped or interchanged. Please write the equation which shows the relationship between x and y in Table 2. Show how you obtained your answer.

Table 1

x	y
0	1
1	3
2	5
3	7
4	9

$y = 2x + 1$

Table 2

x	y
1	0
3	1
5	2
7	3
9	4

____

Solution or Explanation

Figure 7. Task 11.

Principle 4 was about ensuring that the assessment tasks would highlight students' depth of understanding. For example, most of the tasks involving graphs were not on grids, in order to encourage students to deal with the graph holistically (see Tasks 6, 12, 13, 13.1, 14 and 15 in Appendix A). But these tasks and the rest of the assessment could also be solved using a variety of strategies requiring minimum knowledge and skills like point-plotting techniques or evaluating equations using individual values.

Criteria for Selecting the Tasks

As previously explained, the initial framework of growth points informed the development of the set of tasks. Some of these tasks were adapted from previous studies about function, but the majority of the tasks were developed specifically for this study.

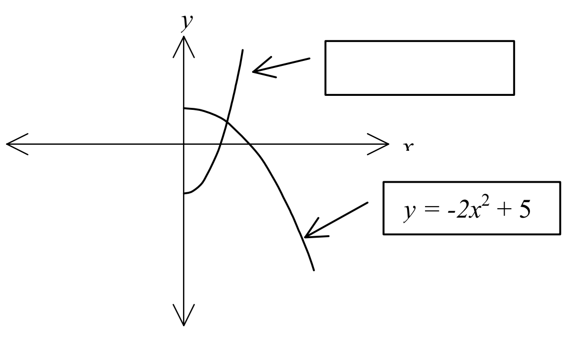
The same principles described in the previous section were used in selecting the tasks for inclusion in the instrument for the two main data collections. In selecting the tasks, frequency of correct answers on the task was never made a basis for adding or deleting a task, as is often done with quantitative research that uses tests to measure a particular variable.

The present study involved generating data reflecting a wide range of students' strategies and thinking. Hence, there were tasks included in the instrument even though very few students could answer them in the pilot studies, so long as results showed that these questions were clear, but a deeper understanding of the concept was required than most students possessed. There were also tasks included that almost everyone could answer. This latter set of tasks was considered assessing the entry level in students' understanding in a particular domain while the former set was considered assessing higher-level understanding.

During the development of the instrument an easy version of a difficult task was added to assess if students could do the same task involving a less difficult analysis than that of the present task. Likewise, tasks demanding a higher level of analysis were added when many students could answer the present task easily and there was reason to believe that some of the students were capable of thinking one level higher. This was done in order to capture the range of strategies of which students were capable. For example, Task 9.1 was designed when most of the Year 8 students in the pilot study had difficulty with Task 9 (see Figure 9).

Task 13 (see Figure 8) was one of the last additions to the set of assessment tasks, when results show that there was a possibility that students could still work in this level.

13. Part of the graph of $y = -2x^2 + 5$ is labelled in the figure below. What may be the equation of the other graph in the box? Please explain how you worked out your answer.



Explanation:

Figure 8. Task 13.

Overall, the main basis for inclusion of the tasks in the final instrument for the main data collection were: (1) the extent to which it drew out students' natural strategies, knowledge and skills; (2) its capability in drawing out the most "sophisticated" strategies from the students; and, (3) its contribution to providing the instrument with a wide range of students' understanding. The researcher largely made this judgment in constant consultations with the supervisors based on the results of the pilot studies. As Kerlinger (1986) emphasised, content validation is more a matter of judgment rather than measurement.

Important revisions to the tasks during the pilot studies are included in the discussion of the tasks in later chapters.

The Assessment Booklets

One of the issues that threaten the validity of test and research results involves the layout of the questions and overall appearance of the booklet, length of time and instrumentation. Measures taken to minimise the effects of these on the validity of the research are described below.

The set of tasks were divided into two booklets with the second booklet given the next day. For related questions, the more difficult question was in Booklet 1 and the easier question was in Booklet 2. For example, Task 9 was in Booklet 1 and Task 9.1 was in Booklet 2 (see Figure 9). This was done because the easier questions, which involved thinking in more specific terms, might provide hints on how to work out the more difficult questions that usually involved thinking in general terms. Having divided the administration of the test into two separate days also provided more time for students to work on the tasks.

9. The relation of s with p is shown in the equation $s = 5p + 3$. The relation of p with n is shown in the equation $2p = 6n$. From this information, please write the equation that will show the relation of s with n. Please show your solution.

9.1 The relation of s with p is shown in the equation $s = 5p + 3$. The relation of p with n is shown in the equation $2p = 6n$. If n = 5, what is s? Please show your solution.

Figure 9. Tasks 9 and 9.1.

The Assessment Tasks

Tables 3 and 4 provide a brief description of the tasks in Booklet 1 and Booklet 2 respectively. The assessment tasks are presented and analysed in more detail in the succeeding chapters. Appendix A also contains the two Booklets.

The first column of Tables 3 and 4 shows the title of the task and the domain in the Framework to which the growth point the task was assessing belongs. The second column gives a brief description of the task. The third column further describes the tasks in terms of the point of analysis or part of the representation involved in the analysis. That is, whether it involves individual points, an interval, the whole representation, or interpretation of properties. Where applicable, strategies used to work out the tasks are also briefly described.

The strategies may include point-by-point analysis, use of trend/pattern or properties, use of invariant properties, reasoning in terms of relationship, and working with the representation as objects that could be manipulated or transformed. Also included in the third column is a description of the task as involving interpretation or construction. The task is labelled *construction* if construction of representation – graph, equation, or tables – is required; otherwise, it is labelled *interpretation*.

Table 3
Assessment Tasks in Booklet 1

Tasks and Domain	Descriptions	Point of analysis involved/Strategies
T1: Reading off values (Graphs)	A graph of age vs. height on a numbered grid is given. The task is to determine the corresponding height to a given age.	Point Interpretation
T2: Amount of change (Graphs)	The task involves calculating the amount of change in height for a given particular interval using the same graph in T1.	Individual points Interpretation
T3a: Fastest growth (Graphs)	The task involves determining the interval showing fastest growth from a set of four intervals. The same graph in T1 was used here.	Interval Growth property Interpretation
T3b: Slowest change (Graphs)	The task involves determining the interval showing slowest growth from a set of five intervals. The same graph in T1 was used here.	Interval Growth property Interpretation
T4: Intersection (Graphs)	A second graph was added to the graph in T1 intersecting it at two points. The task was to interpret the points of intersection.	Individual points Interpretation
T5: Interval and continuity (Graphs)	The task involves writing the interval representing the age at which the second person was taller than the first. It requires interpreting interval and continuous property of the graph	Interval Property Interpretation
T6: Relating graphs (Graphs)	Two linear graphs are given. The first graph shows a direct linear relationship between x and y and the second shows an indirect linear relationship between y and z. The task was to construct the graph showing the relationship between x and z. The graphs were not on grid to encourage holistic analysis of the graph rather than point-by-point.	Whole representation Use of relationship or point-by -point Construction
T7a: Evaluating equation (Equations)	This task involves evaluating an equation by a single value. The given equation was part of a piece-wise function.	Point Interpretation
T7b: Rate (Equations)	This task involves interpreting rate from a given piece-wise function.	Interval Growth property Interpretation
T7c: Intercepts (Equations)	This task involves interpreting the intercept given a piece-wise function.	Point Intercept property Interpretation

T8: Rate (Equations)	The task involves determining the equation showing the fastest change in *y* when *x* takes values from 1 to 10. The choices given were all linear equations.	Interval Whole representation Growth property Interpretation
T9: Relating Equation/ Composition (Equations)	Two linear equations were given. The first relates *s* and *p* and the second relates *p* and *n*. The task was to determine the equation relating *s* and *n*.	Whole equations Use of relationship Construction
T10: Making equations (Equations and Linking Representations)	A table of values and its corresponding quadratic equation are shown. A second table showing the same *x* values as the first but with *y*-values two more than the *y*-values of the first table are also shown. The task was to construct the corresponding equation of the second table.	Whole representations Use of relationship or point-by-point Construction
T11: Inverse (Equations and Linking Representations)	A table of values and its corresponding linear equation are shown. The values in the given table were swapped and shown in a second table. The task was to write the corresponding equation of this second table.	Whole representations Use of relationship or point-by-point Construction
T12: Tables and parabola (Linking Representations)	A parabola and four sets of table of values are shown. The task was to identify the table of values corresponding to the parabola.	Whole representations Use of trend, local and invariant properties or point-by-point Interpretation
T13: Parabola and equation (Linking Representations)	A part of the graph of a parabola and its equation are shown. On the same axes, another part of a different parabola is shown. This second parabola is steeper than the first and with opening and intercept opposite that of the first parabola. The task was to construct the equation corresponding to the second parabola. The graphs were not on grids.	Whole representations Working with equation as object or reason in terms of invariant properties Construction
T14: Equation and lines (Linking Representations)	The task was to identify the graph that would match the given linear equation. The graphs were not on grid to discourage students from doing point-by-point analysis. The given linear equation was in slope-intercept form already.	Whole representations Use of trend, local and invariant properties or point-by-point Interpretation

Table 4
Assessment Tasks in Booklet 2

Tasks and Domain	Descriptions	Point of analysis involved/Strategies
T6.1: Relating graphs (Graphs)	This is the same as T6 but this time the given graphs were on a numbered grid.	Whole representation Point-by –point, use of property Construction
T9.1: Relating equations (Equations)	This is the same as T9. The only difference was instead of determining the equation relating the variables, the students were asked to find the value of *s* given a value for *n*.	Specific value Construction
T10.1: Generating values (Equations and Linking Representations)	This is similar to T10. A table of values and its corresponding quadratic equation are shown. A second equation with a different constant term is also given. The task was to complete the table of values for the second equation. The given *x*-values for this table are the same as those in the first table.	Whole representations Individual values or in terms of relationship Construction
T13.1: Parabola and equation (Linking Representations)	This is similar to T13. But unlike T13 where the students were asked to write the corresponding equation, in T13.1 the students were asked to select the graph that would correspond to the given equation. This task involved quadratic relationship.	Whole representations Use of trend, local and invariant properties or point-by-point Interpretation
T15: Lines and tables (Linking Representations)	This task involved determining the graph that would match a given table of values. This task involved linear relationship.	Whole representations Use of trend, local and invariant properties or point-by-point Interpretation
T16: Graphs (Equivalent Relationships)	This task involved selecting the linear graphs that represent the same function or relationship. The given four graphs were on separate numbered grids	Whole representations Use of trend, local and invariant properties or point-by-point Interpretation
T17: Tables (Equivalent Relationships)	This task involved determining the tables that represent the same function or relationship. The given four tables all represent linear relations.	Whole representations Use of trend, local and invariant properties or point-by-point Interpretation
T18: Equations (Equivalent Relationships)	This task involved selecting the equations that represent the same function or relationship. The given four equations all represent linear relations.	Whole representations Use of structure, invariance of letter symbol, point-by-point Interpretation

Choice of Participants

Pre-school children already have the notion of function or relationship between varying quantities (see Lovell, 1971), and elementary pupils through their experiences in handling data in and outside the school already have experiences with the notion of function. However, the more rigorous treatment of this concept in most countries usually starts in Year Year 8 or Year 9 (see, e.g., Howson, 1991; Sfard & Linchevski, 1994). In the 1998 mathematics curriculum in the Philippines, functions are introduced in Year 8 with the introduction of the two-dimensional coordinate system in their study of algebra. Thus, the study collected data from Year 8, 9 and 10 students. This section describes the participants in the pilot studies and in the two main data collection periods.

Participants in the Pilot Studies

During the pilot studies, data were collected from students with varied ability range, in order to cover a wider range of students' strategies and performance in function tasks, as well as assess the appropriateness and clarity of the tasks. The first two pilot tests were conducted in Melbourne. Two more pilot tests were conducted in the Philippines before the main data collections. The participants in the pilot studies comprised three levels, Years 8, 9 and 10, and represented low, average and high performing students.

In the pilot test of the assessment tasks, at least one student in each year level who took the test was interviewed regarding his/her answers, solutions and his/her interpretations of the questions.

Table 5 shows the details of the classes where the pilot studies were conducted. In the fourth and last pilot tests, the final version of the instrument was given in the form of a written test to four second year college students to check for strategies among students with more experience with mathematics.

In the Philippines, Year 8 classes have an average age of approximately 14 years, Year 9, 15 years and Year 10, 16 years. The second year college students who were given the test were 18 to 19 years old.

Table 5
Respondents in the Pilot Tests of Assessment Tasks

Pilot tests	School	Respondents	Purpose
Pilot test 1 (July and August, 2001)	Girls' School in Melbourne	Year 10: 23 Year 9: 20 (Above-average performing students) Year 10: 20 (Low-performing group)	These students were of above average ability and thus would provide rich data of students' strategies The tasks were also given to a low-performing group of Year 10 to get a sense of the kind of strategies this group would use.
Pilot test 2 (October, 2001)	Coeducational school in Melbourne	Year 10: 24 Year 9: 25 Year 8: 19 (Mixed-ability groups)	These were selected to check clarity of the revised version of the assessment tasks and get a sense of the distribution of students' strategies and performance on each task
Pilot test 3 (January, 2002)	Public general high school (Philippines)	Year 10: 51 (Low-performing group) Year 9: 43 (Average performing group)	To check the clarity of the assessment tasks (which now include a Filipino-language translation underneath each task) and get a range of students' strategies and performance.
	Public science high school (Philippines)	Year 10: 46 Year 9: 27 Year 8: 32 (High-performing group)	
Pilot test 4 (March, 2002)	Public general high school (Philippines)	Year 10: 40-50 students (Mixed-ability)	The researcher piloted the revised tasks to the Year 10 group in a class-discussion format to check clarity of the questions.
	Public College (Philippines)	2nd year college: 4	The final form of the instrument was given to 2nd year college students to get a sense on how this group who have more experience with mathematics would work out the tasks.

Participants for the Main Data Collections

Originally, the plan of the research was to consider a sample of the general population of Year 8, 9 and 10 students. This is why the pilot studies were with low, average and high performing groups of students from public general high schools and special science high school. The implementation of the new curriculum for mathematics in the Philippines, which was only finalised two months before the main data collections of the present study and after all pilot testing had been completed, led to the decision to consider only those schools not affected by this change. In the new curriculum, which was implemented in public general high schools, the studies of functions would be in Year 10. The Year 9 mathematics is about geometry. In the old curriculum, from which the plan of the present study was based, each year level from Year 8 to Year 10, study at least one family of functions and algebra topics are taught in all these year levels.

Among the schools not affected by the change were the Regional Science High Schools (RSHS). This change of respondents to a specific group of students however did not affect the validity of the research since the instrument had also been piloted with students from a RSHS.

It could be argued that because the study would not tease out the effect of such factors as curriculum in identifying and describing students' growth points in their developing understanding of function, then it would not matter if there was a change in the curriculum. However, the decision to consider only the RSHS was also influenced by the results of the pilot studies. Pilot studies in the Philippines showed that the majority of students from the RSHS were more likely to work out the tasks correctly. These students were also more likely to explain their answers or show their solutions. That students should be able to do these was important to the study, since the development of the framework depends on students' solutions, explanations, and some level of success. So, while the decision to consider the RSHS for the main data collection may have limited the scope by which the findings of the study could be generalised in terms of percentage of students at particular growth points, it also significantly increased the validity of the results because of the richness of data gathered.

Subjects for the main data collections came from three RSHS. The Philippines is divided into 16 regions and there is one RSHS in each region. The RSHS is a government high school. It is not always the only science high school in the area. There are also private high schools that are science-oriented high schools. RSHSs were chosen because there is at least uniformity in terms of composition, curriculum and school structure in all RSHSs. The same could not be said of private science high schools

Entrance to RSHS is competitive. An entrance test is given with strong emphasis on Science, Mathematics and English. Classes in these high schools are usually smaller with an average of 25 to 35 students in each class. General public high schools in the Philippines would have an average of 60 students in a class. According to the teachers in the RSHSs, most of their students come from middle-income families compared to those in the general high schools

Table 6 shows the number of students in each year level from the three schools selected in the study.

Table 6
Number of Respondents from Each School

	School O	School B	School M	Total
Year 8	28 (1 class)	70 (2 classes)	51 (2 classes)	149
Year 9	58 (2 classes)	53 (2 classes)	41 (2 classes)	152
Year 10	53 (2 classes)	50 (2 classes)	40 (2 classes)	143
Total	139	173	132	444

School O had a total of two Year 8 classes, two Year 9 classes and four Year 10 classes, but not all were chosen to participate. In this school, classes were organised according to the average grade of the students with the top students in section 1 class. The assessment tasks were administered to the first section Year 8 class, both classes of Year 9 and the Year 10's first and second section classes.

School B had two classes of mixed-ability students for each year level. The questionnaires were given to all classes except Year 7.

School M had four classes in each year level. In this school, one class of high performing students was organised for each year level and the other three classes consisted of mixed ability students. The assessment tasks were given to the top group and to one of the mixed-ability classes.

All students in each class were given the test, but only those who were able to take both parts of the test in the first and second data collections were considered for analysis. Aside from the 444 students who took the test in written form, three pairs of students from School O were asked to take the test together, one pair from each year level during the first data collection period. Two students from each year level from Schools B and M, who took the test, were also selected randomly for interview regarding their answers in the test. In the second data collection, one student from School O and School B in each year level were given the instrument in interview form.

The mathematics syllabus followed by RSHSs was not as detailed as the one followed by the general high school. In RSHS, the syllabus consisted of a general list of topics. The teaching order of the topics, the teaching strategies and learning experiences, the textbooks and reference materials were generally left for mathematics teachers to decide.

In general, students from science high schools have more experience with mathematics because they have an enriched curriculum compared to the general high school. The topics covered in each year level and the number of minutes allotted for teaching is shown below.

> Year 7 – Mathematics 1 (40 minutes daily)
> Working with fractions, decimals, using percent and ratio; Plane figures; Measurement; Graphs and tables; Integers; Algebraic expressions; Solving mathematical sentences; Enrichment activities related to the above topics.
>
> Year 8 – Mathematics II (80 minutes daily)
> Geometric relations, triangle congruence, quadrilaterals and similarity; Algebraic expressions and operations; mathematical sentences; linear relationships, systems of linear equations and inequalities; exponents; radicals; special products and factors; polynomial expressions and measures of statistics.
>
> Year 9 – Mathematics III (60 minutes daily)
> Quadratic equations and functions, variations, sequences and series; exponential functions; logarithmic functions; polynomial functions; permutations and combinations; complex numbers; linear correlations and probability; circles and circular functions.
>
> Year 10 – Mathematics IV (60 minutes daily)
> Topics in advanced algebra (functions, inequalities, mathematical induction and matrices); analytic geometry; differential calculus and introductory integral calculus.
>
> (Source: DECS Order No. 55, s. 1994. Curriculum in the Regional Science High School. July 1, 1994.)

In all three schools, Year 8 students were just starting their work with coordinate systems and doing some point-plotting activities when the first data collection occurred but all had covered the topic on linear functions before the second data collection was undertaken. In Year 9, Schools O and B had completed quadratic equations and function and School M was about to start with quadratics when the second data collection was undertaken. This means that revisions involving linear relations had been done. The Year 10 students had theoretically studied all the content covered in the instrument (linear and quadratic relationships) in Year 9. Because they study other families of function in Year 10, it could be assumed that they worked with linear and quadratic relationships as well during this time.

Overall, it could be claimed that the participants of the study were representative of the RSHS students in the country, and of other science high schools.

Administration of the Instrument

The main data collections were conducted twice with the same students in the Philippines. In the first data collection, three weeks after the start of the school year, *I* administered the test to make sure that teachers would not see the tasks given. In the second data collection, five months later, the teachers helped in the administration of the test. After explaining the purpose of the research and the test, there were occasions when I requested the teacher to stay in the classroom while I conducted interviews with a few of the students.

The second data collection was conducted five months after the first to provide enough time for students to have more experience with mathematics and thus enable them to show some growth in understanding. Five months was also thought to be long enough for them not to recall their solutions to the tasks or to recall discussions with classmates following the first test.

The test was given to the whole class each time but in the data analysis only the work of those students who were able to take the test on both occasions was considered. Table 7 shows the date of the main data collection periods.

Table 7
Dates for the Two Main Data Collections

Schools	Data collection 1	Data Collection 2
School O	June 24-27, 2002	November 26-28, 2002
School B	July 1- 4, 2002	November 19-21, 2002
School M	July 8 - 11 2002	December 9-11, 2002

As mentioned earlier, interviews were also conducted using the assessment tasks. Students were interviewed regarding their interpretations of the tasks. They were asked to explain how they worked out their answers for selected tasks. They were also asked if they could think of other ways of solving the problem. Because I knew students would be not be comfortable if they could not answer right away, I explained to the students that because my research was about understanding how students solve problems, I had formulated tasks that would make them think. Therefore, it was all right that they would take time to think. I switched to Filipino language when students showed difficulty expressing themselves to get them talking and to increase their level of comfort, but there were only a few instances when this happened.

Data Analysis

This section describes briefly how the data were analysed to develop the framework of growth points. The first part describes the initial analyses made to identify and describe the growth points, the second part describes the coding of students' responses and the third part describes how the typical learning paths were determined.

Identifying and Describing the Growth Points

The aim of the research was to identify "big ideas" in students' understanding of function in various aspects of the function concept as well as to develop tasks that would draw out and highlight these ideas from the students. As earlier mentioned, the assessment tasks and the identification and descriptions of the growth points informed each other.

Task analyses were conducted before the tasks were piloted to determine possible responses and strategies. After each pilot study, tasks were analysed in terms of skills and knowledge involved, in terms of the representation involved, that is, whether only point or single values were involved, an interval or a part of the representation, or the whole representation. These resulted to the development of the Record Sheet and the Framework for identifying and refining the description of the growth points. These are described below.

The record sheet.

Because the focus was on both answers and strategies, a record sheet of students' responses and strategies was developed and used. Data from the pilot tests and main data collections expanded the rubric for each task. Figure 10 shows the rubric for Task 10 and Task 14.

T10
0: incorrect ____; NA IR NE
1: $y = 2x^2 + 1$, guess and check
1.5: $y = 2x^2 + 1$, used pattern, or the intercept
2: $y = 2x^2 + 1$, subtracted 2 from the given equation

T14
0: incorrect: ____; NA IR NE
1: d, plotted more than 2 points or generate values
1.5: d, used patterns/trends or properties such as slope and intercept
2: d, used 2 points + slope or intercept and another point
3: d, used invariant properties (slope and intercept, x and y intercepts)

Figure 10. Rubrics for Tasks 10 and 14.

In the rubric, NA means no answer, IR means incorrect reasoning and NE means students selected the correct answer but did not show or explain how they worked it out. The incorrect response is written in the blank provided.

In the final coding for the growth points, some of the categories were combined. For example in Task 14, Strategies 1.5 and 2 were combined. The succeeding chapters, which describe the tasks and the growth points explains the strategies further. The final form of the rubric is in Appendix B.

Guide for tasks analysis.

Aside from the initial framework of growth points, the study produced another framework when the tasks were content-analysed and strategies classified for each task. The analysis of the tasks and students' strategies and therefore the identification and description of the growth points were also guided by the diagram in Figure 11.

The diagram shows that the tasks were classified according to the points of analysis involved: individual points versus the whole representation. Working with individual points is a manifestation of a process or procedure conception of function, while the latter points toward conceiving function as an object or permanent construct.

	Process	⟶	**Object**
Points of analysis:	• Individual points	• Set of points, interval	• Whole representation, Relationship
Strategies: (action performed)	• Perform series of same procedure (point-by-point)	• Combination of points and whole representation	• Perform general operation on the representation
Interpretation of properties:	• Local properties	• Trends and patterns and local properties	• Invariant properties

Figure 11. General guide used to analyse assessment tasks and identify and describe the growth points.

Students' strategies used in working out the tasks were also classified according to the procedure performed on the representation: series of the same procedure versus performing a general operation. The former is a manifestation of conceiving function as a process and the latter shows understanding of function more as an object than a process. Strategies in between these conceptions included the use of trend and patterns; use of properties and individual points; and interpretations based on local properties.

Tasks involving properties were also classified according to the kind of property: local properties versus invariant properties of the function and how knowledge of the properties of the function was used in the tasks. Use of invariant properties in working with function tasks was considered as evidence that students conceived function as a permanent construct.

Coding Students' Responses and Strategies

Students' responses were coded against the rubrics shown in the record sheet. In previous discussions, I mentioned the possibility that students may not reflect their actual solution or explanation or show an incomplete solution or explanation or write a logical solution, but have actually obtained their answers through guess and check. These are possible threats to the validity of the research findings. Although it was made explicit to the students before the test was given that since the research was investigating *how* they solved problem, answers without explanations or solutions would not help in understanding their reasoning, there were still a few who did not show their solutions. While it could not be completely claimed that students did not provide explanations because they did not know how to explain it or could not show how they worked it out, it could be claimed with confidence that these students had no problem explaining their thinking. Students who were interviewed were articulate enough to explain their thinking. Secondly, language could not be the problem since they could explain in Filipino language. Thirdly, there were students who did not provide an explanation in one task but showed very detailed solution or explanation in other tasks. Time could not have been the problem either because there was no time limit set. They could submit their paper when they were finished with it. The test did not take more than their mathematics class period. Thus, if no explanations or solutions were shown for tasks, which required them, then their answer even if it was correct, was not considered correct.

Correct answers with incomplete or insufficient explanations were included in the analysis of the data. This type of response was coded and formed part of the data in describing a particular growth point. For those students showing more than one solution or explanation, these students were coded at the higher-level solution. As for the possibility that students show a solution different from the one they actually used, I coded the solution reflected on the paper. There was no sure way to tell whether they used a different strategy.

Investigating for Typical Learning Trajectories

The typical learning trajectory in each of the domains of function included in the framework was determined by comparing the percentage of students coded at each of the growth points or by comparing the frequency of students moving from one growth point to the higher growth points between the two data collection period. Although there is not sufficient evidence that the three year levels are comparable, the achievements of students in each of the year levels were also examined. These are discussed in details in the next four chapters of this thesis.

Chapters 4, 5, 6 and 7 consider the four key domains of the framework: Graphs, Equations, Linking Representations and Equivalent Relationships. The discussion in each chapter includes the following:

Assessing the growth points.

This section describes the growth points and the set of tasks. The description of each task includes an explanation of why it was used to assess that particular growth point and in some cases how the task also informed the identification and description of this growth point. Where appropriate, the revisions made and findings related to these tasks are also presented. The procedures of the coding for growth points are discussed along the way.

The trend in the growth points.

This section presents the empirical data supporting the order of the growth points. Other findings of the research are also presented.

Discussion and summary.

This section presents the summary of the chapter and key findings for the particular domain investigated.

Chapter 8 summarises and discusses the results in Chapter 4 to 7. Chapter 9 presents the implications and recommendations of the study.

CHAPTER 4

DOMAIN 1 - EQUATIONS

The approach in the teaching of algebra in most curriculum documents typically starts with linking it with arithmetic, students' previous experience of mathematics. That is, algebra is initially presented as a generalised arithmetic. The teaching of function is also opened through this door, working with individual values to search for patterns and eventually formulate rules. The rules are almost always in the form of formulae or equations. Thus, equations are perhaps the most widely used and preferred representations of functions especially in secondary schools. These representations are versatile in the sense that they naturally lend themselves to process and object interpretations alike. This means, students could think of the equations as procedures for generating individual values, or, students could work on the equations as mathematical objects that could be manipulated or transformed which is what the teaching of algebra is trying to achieve. Thus, the understanding of function represented by equations is one of the major nodes in the network of students' understanding of function.

The first section of this chapter presents the growth points involving equations. The second section presents the results of the data collection. Included in the discussion are the success rate for each task and the data collection results, showing students' typical learning trajectories in terms of the order of the growth points. The third part presents the summary and discussion of the results.

The succeeding chapters present growth points in other key domains of the function concept and growth points involving linking equations with other representations.

Assessing the Growth Points

This section describes the tasks and the identified growth points under Equations. Nine tasks were used to assess the four growth points under this domain.

As had been explained in Chapter 3, initially, a framework of growth points was developed, based on the literature. The initial framework of growth points informed the creation of tasks designed to assess the growth points. Students' performance in the tasks and further analysis of the tasks informed the subsequent refinement of the framework. Thus, the discussion in the present section includes the description of the growth points and the tasks and students' performance on the tasks, which were used to assess the growth points. The purpose is to show the correspondence between the tasks and the growth points.

Growth Point 1: Equations as procedures for generating values

To be coded at Growth Point 1, students had to show that they could at least conceive an equation as a formula from which they could generate values. This conception corresponds to Sfard's (1991) interiorization phase or DeMarois and Tall's (1996) procedure layer in understanding of function. Students need not have a holistic conception of function represented by equations to be able to be classified at this growth point.

There was no specific task designed to assess Growth Point 1. The tasks that were used to assess Growth Point 1 were designed to assess higher growth points. These tasks involve interpreting equations but they do not directly ask students to evaluate the equations or generate values from the equation. However, some students' solutions to these tasks show thinking of equation representation as a procedure for generating values despite the fact that this strategy is the most tedious way of completing the tasks. The tasks used to assess Growth Point 1 are described next.

Task 10.1 and Task 10 were adapted from Moschkovich, Schoenfeld and Arcavi's study (1993). Task 10.1 (in Figure 12) requires students to construct the table of values. The corresponding table of values for the first equation is already given and the students' task is to complete the corresponding table of values for the second equation.

10.1 Examine the two equations shown below. The specific values of $y = x^2 + 3x + 3$ are shown in the table on the left. Fill in the table on the right with values of $y = x^2 + 3x$. Please explain/show how you obtained your answer.

$y = x^2 + 3x + 3$

x	y
0	3
1	7
2	13
3	21
4	31

$y = x^2 + 3x$

x	y
0	
1	
2	
3	
4	

Explanation or solution:

Figure 12. Task 10.1 - Generating values.

Most of the students' solutions to Task 10.1 involved substituting values to the given equation rather than examining the relationship between the two given equations. These solutions were coded Strategy 1 and 2 respectively. Figure 13 shows a typical solution coded Strategy 1. Substituting individual values to the equation is an indication of a point-by-point or point-wise interpretation of the equation, which is reflective more of a process conception of function than an object conception.

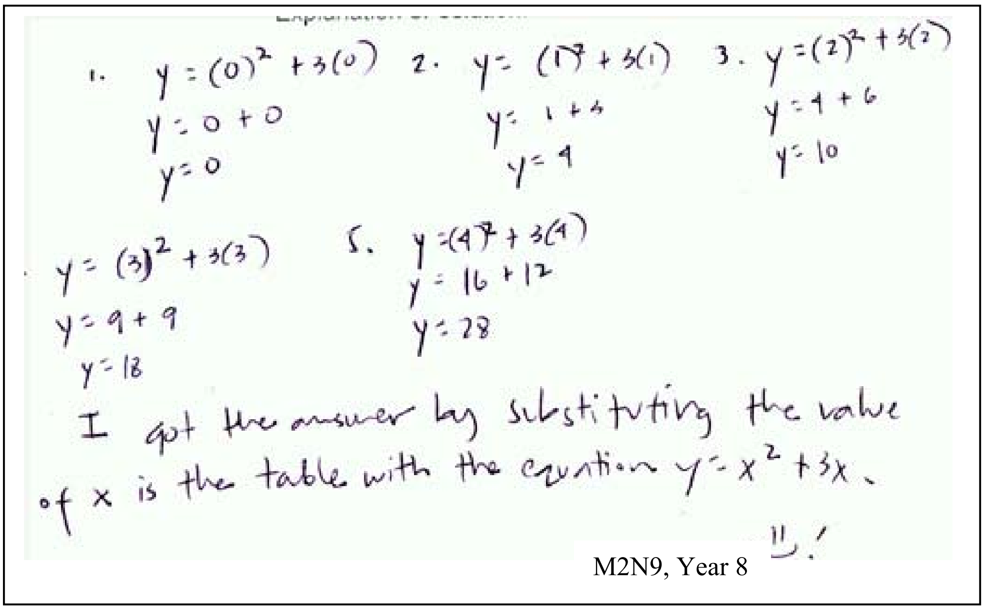


Figure 13. Task 10.1 - Strategy 1. Point-by-point interpretation.

Strategy 2 involves a more holistic interpretation of the equations. This strategy indicated that students were able to interpret the relationship between the two equations and the values in the tables. A typical reasoning coded Strategy 2 is shown in Figure 14.

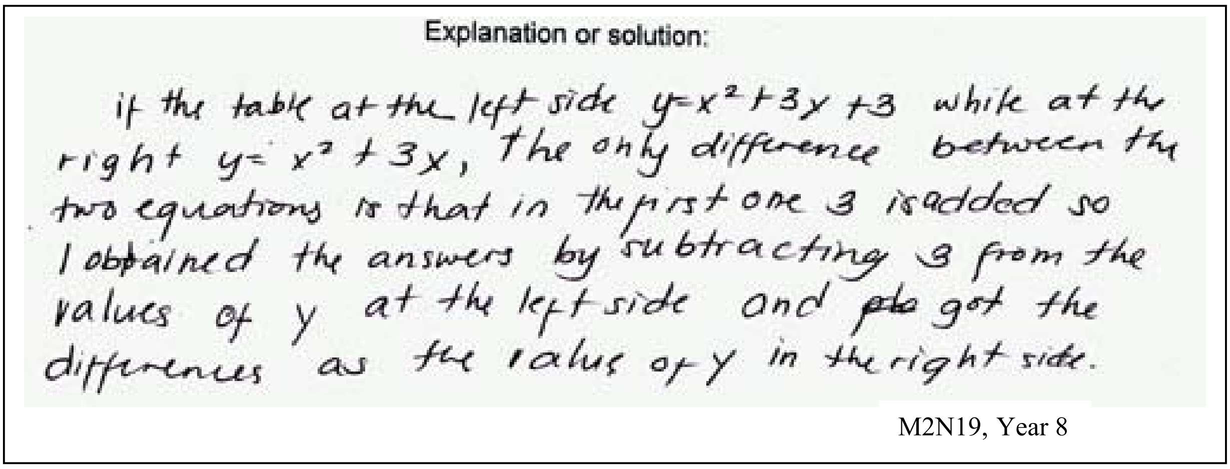


Figure 14. Task 10.1 - Strategy 2. Holistic interpretation of relationship.

Shown in Table 8 are the percentages of students coded as using the particular strategies. D1 stands for data collection 1 and D2 stands for data collection 2. The same students took the test in both data collections.

The data in Table 8 shows that the preferred students' strategy involves evaluating the equation by individual values.

Table 8
Percentages of Students Coded the Strategies for Task 10.1

Strategies	Year 8 (*n* = 149)		Year 9 (*n* = 152)		Year 10 (*n* = 143)	
	D1	D2	D1	D2	D1	D2
Strategy 1: Point-by-point	43.6	77.2	83.6	87.5	86.7	50.4
Strategy 2: Holistic	5.4	13.4	5.9	12.5	9.8	19.6
Incorrect	58.4	9.4	10.5	0.0	3.5	0.0

Data in Table 8 also indicate that at the beginning of the school year, only half of the Year 8 students could complete this task successfully. Five months later, over 90% completed the task, indicating that concepts and skills required to succeed in Task 10.1 are easily acquired. In fact, in the second data collection period, all the Year 9 and Year 10 students gave the correct answers for this task. Results also show that although Task 10.1 involves quadratic relationships or equations in the second degree, it was one of the easiest tasks in this study involving equations.

Task 10 (see Figure 15) is related to Task 10.1, but Task 10 requires students to construct the equation rather than the table of values.

10. Examine the two tables shown below. The set of values in the table on the left shows specific values of $y = 2x^2 + 3$. Please write the equation whose values are shown in the table on the right. Please show or explain how you obtained your answer.

x	y
-1	5
0	3
1	5
2	11
3	21

$y = 2x^2 + 3$

x	y
-1	3
0	1
1	3
2	9
3	19

____

Solution or Explanation:

Figure 15. Task 10 - Generating equation.

It was expected that students would find this task more difficult than Task 10.1, especially students whose understanding of functional equations were still at the level described in Growth Point 1.

Students' solutions to Task 10 could be categorised into three strategy types. The first strategy is called point-by-point interpretation since it involves evaluating the equation using individual values. This was coded Strategy 1. Figure 16 shows a typical solution coded Strategy 1.

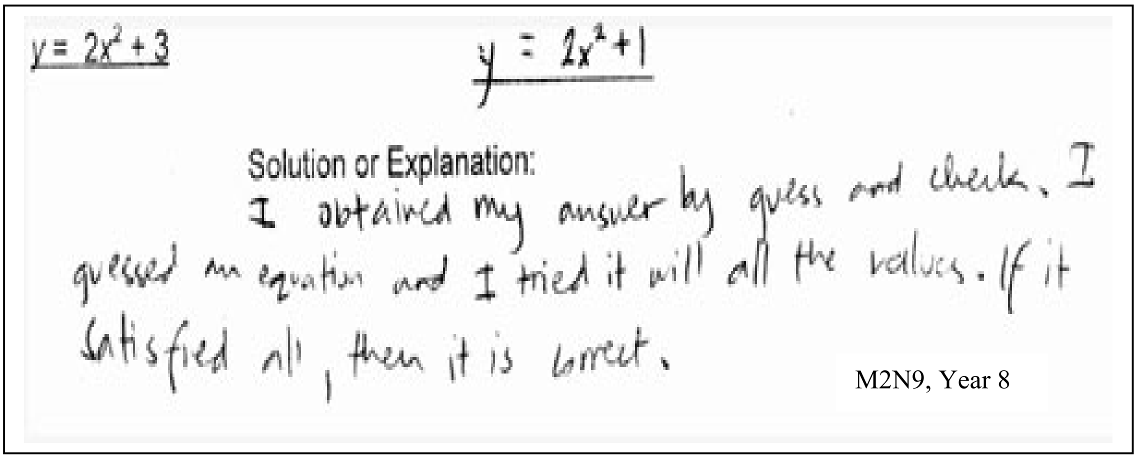


Figure 16. Task 10 - Strategy 1. Point-by-point interpretations.

Some students completed this task by noting the value of the intercept only or using the algorithm for determining the quadratic equations from a set of values. These strategies were coded Strategy 1.5. Students who used this strategy show more holistic understanding than point-by-point understanding of equation reflected in Strategy 1. A sample of student solution is in Figure 17. Student B2D8 understood the role of the intercept and perhaps the sufficiency of the intercept to determine the equation for the particular set of values given.

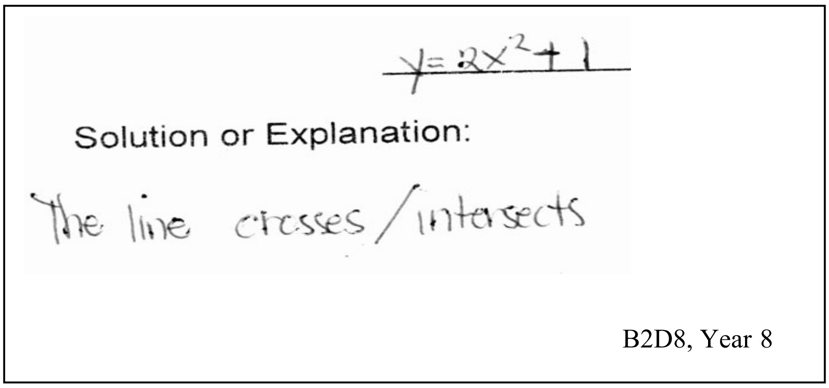


Figure 17. Task 10 - Strategy 1.5. Use of intercept.

The preferred solution for Task 10 involves the holistic interpretation of the relationship between the values in the given tables and then working out the equation by performing operation on the equation as a mathematical object in itself. The present study considers this solution reflective of an object conception of function. Figure 18 shows a typical solutio coded Strategy 2.

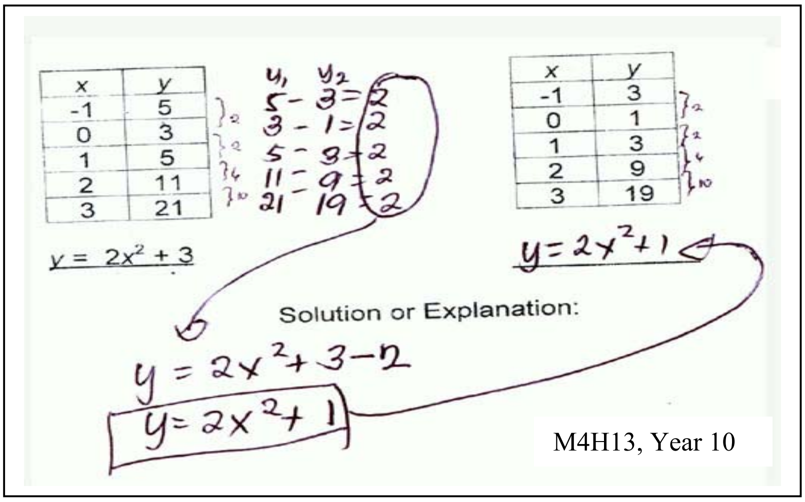


Figure 18. Task 10 - Strategy 2. Performing operation on equations.

Solving Task 10 by point-wise or point-by-point interpretation is tedious but majority of the students' solutions to this task involved this technique. Table 9 shows that more students were coded Strategy 1 than Strategy 2 indicating preference for point-by-point interpretations. Most of the students who were able to complete this task used *guess and check*. That is, they guessed the equation first, and then evaluated the equation using the x values in the table to check if they had the correct equation.

Table 9
Percentages of Students Coded the Strategies for Task 10

Strategies	Year 8 (n = 149)		Year 9 (n = 152)		Year 10 (n = 143)	
	D1	D2	D1	D2	D1	D2
Strategy 1: Point-wise	8.7	53.7	41.4	63.8	59.4	51.7
Strategy 1.5: Some properties	3.4	2.0	3.9	4.6	5.6	6.3
Strategy 2: Holistic	5.4	8.7	8.6	10.5	15.4	26.6
Incorrect	82.6	35.6	46.1	21.1	19.6	15.4

Compared to Task 10.1, however, the proportion of students using Strategy 2 compared to Strategy 1 is much greater indicating that Task 10 is more likely to make students think holistically than Task 10.1. This is because Task 10 makes point-wise analysis tedious thereby forcing students to interpret the relationships between the tables and between tables and equation.

Results in Table 9 also show Year 8 students found this task difficult during the first data collection period. Over eighty percent of the Year 8s failed to complete the task, which is to be expected since they have no experience with quadratics yet at this stage. This percentage decreased to about thirty-five percent in the second data collection period but most of those who did completed the task used point-wise analysis. This trend indicates that students are more likely to make point-by-point interpretation first in working with functional equations. This is probably because evaluating algebraic expressions and sentences is among the first of the skills teachers introduced to students when working with equations and algebraic expressions to show its link with arithmetic.

Preference for point-by-point analysis and the inability to conceive equations as objects where an operation could be applied was further confirmed by the results of students who were given the tasks in purely interview form. In the interview, students were asked first to notice the similarities and the differences between the two tables (in Task 10) and the values in the two equations (in Task 10.1) before they were asked to do the tasks. There were those who noticed the difference but they did not use that knowledge to complete the tasks. The following excerpts were from an interview of a Year 9 student. In the interviews, just as in the written test, students answered Booklet 1 first. Therefore, in this interview, Task 10 was given first.

Interviewer: Look at these two tables, what do you notice about them? (Showed the two tables, covering the question first)

Student: I think in the first table, the first column of the first table or the x values is the same while the y variable in the table (pointing to the first table) are higher value than the variable in second table.

Interviewer: Any other relation between the y values?

Student: Umm… In Table 1, the y variable is greater than Table 2?

Interviewer: By how much?

Student: Two

Interviewer: Is it true to all?

Student: Yes. (Checks the values one by one.)

Interviewer: (Read Task 10 slowly.)

Student: Can I have trial and error?

Interviewer: Sure.

Student: (Student wrote $y = 2x^2 – 2$ then checked by evaluating for $x = 1$. When it did not match the value in the table tried the following equations: $y = 2x^2 – 3$, $y = 2x^2 + 5$ and $y = -2x^2 + 5$ for $x = -1$, getting 3 for this last equation). I think I know it, that's my answer (wrote $y = -2x^2 + 5$).

The following excerpts show the results of the interview with the same Year 9 student for Task 10.1. The interview shows that the student preferred point-wise analysis to complete the task.

Interviewer: (Showed the two tables and equations.) What do you notice about the two equations?

Student: The first equation, there's a 3 added.

Interviewer: (Read Task 10.1 slowly)

Student: I will just compute (evaluated the second equation for $x = 0$ to 4, showing the computations step by step)

Interviewer: Can you think of other ways of doing this?

Student: Here, I can get the *y* variables [values] minus 3.

Interviewer: Why did you not do it?

Student: I wasn't sure if I could get it that way, I might get others [values].

Students who completed Task 10.1, like this Year 9 student, might have noticed that all they needed to do was to subtract 3 from the *y* values in the other table but chose not to because they were not confident enough that they would get the same answer.

To be coded Growth Point 1, students need to complete correctly at least one of Task 10.1 or Task 10. There were students who solved these tasks using Strategy 2. These students were also coded at Growth Point 1 but they have the chance to be coded at higher growth points. The study assumed that students who used Strategy 2 were capable of working out the tasks in terms of Strategy 1.

Growth Point 2: Interpretations based on relationships

Unlike Growth Point 1 where equations were conceived simply as a procedure, Growth Point 2 required students to conceive the equation as representing a relationship between variables. This growth point was not in the initial framework. It was identified and added when results showed that the there was a large difference between the students interpreting equations based on individual values and those interpreting the equations as objects. Tasks 10.1, Task 10, Task 9.1 and Task 9 (see Figure 19) were used as assessment tasks for Growth Point 2.

> 9.1 The relation of *s* with *p* is shown in the equation $s = 5p + 3$. The relation of *p* with *n* is shown in the equation $2p = 6n$. If n = 5, what is s? Please show your solution.
>
> 9. The relation of *s* with *p* is shown in the equation $s = 5p + 3$. The relation of *p* with *n* is shown in the equation $2p = 6n$. From this information, please write the equation that will show the relation of *s* with *n*. Please show your solution.

Figure 19. Tasks 9.1 and 9 - Relating equations.

In Task 9.1, the interpretation may involve only a single value but it also requires an analysis of the relationship between the variables in the two equations given, as well as the interpretation of the relationship between the equations. It does not simply involve a straightforward evaluation of equations.

Task 9 (in Figure 19) is similar to Task 9.1. Both tasks assess students' interpretation of relationship between the varying quantities. However, Task 9 also assesses students' ability of thinking and working with equations holistically or as objects.

Most of the students' solutions in Task 9 involved composition — solving for *p* in terms of *n* and using the resulting equation as input to $s = 5p + 3$. However, there were also a considerable number of students that used "partitioning". This solution was coded Strategy 1.5. This solution still involves repetitive application of a procedure although it does not deal with individual values. An example of solution coded Strategy 1.5 is in Figure 20.

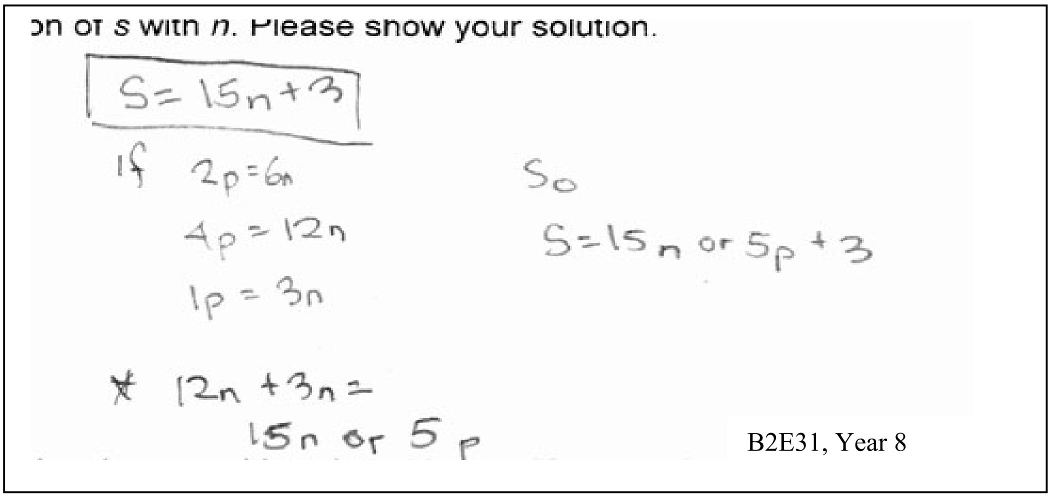
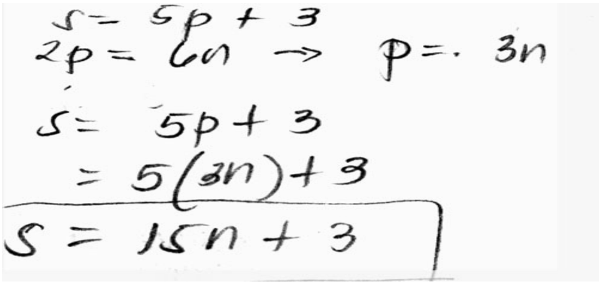


Figure 20. Task 9 - Strategy 1.5. Partitioning.

A solution that shows composition, the preferred solution for Task 9, was coded Strategy 2. A typical solution coded Strategy 2 is in Figure 21.



M3N11, Year 9

Figure 21. Task 9 - Strategy 2. Composition.

Data in Table 10 show the distribution of students' strategies for Task 9. The majority of the students solved the task using the preferred strategy – Strategy 2. This strategy is more reflective of the understanding of function represented by equations as an object that can be manipulated or transformed compared to Strategy 1.5. However, both strategies could be considered reflective of holistic understanding of equations as representations of functions.

Table 10
Percentages of Students Coded the Strategies for Task 9

Strategies	Year 8 (n = 149)		Year 9 (n = 152)		Year 10 (n = 143)	
	D1	D2	D1	D2	D1	D2
Strategy 1.5 – "Partitioning"	4.0	7.4	7.2	11.1	10.5	12.6
Strategy 2 - Composition	0.7	16.8	9.2	34.9	28.0	47.6
Incorrect	95.3	75.8	83.6	53.9	61.5	39.9

Growth Point 2 involves interpretations of relationship between two functions represented by equations. Minimum achievement of those coded at Growth Point 2 represents a combination of holistic and point-wise interpretation of relationship between the variables and between equations. Thus, students should meet at least one of the following conditions to be coded at Growth Point 2.

- Task 9.1 and Task 9;
- Task 9 and Task 10.1 or Task 10;
- Task 9.1 and Task 10.1 (Strategy 2) or Task 10 (Strategy 1.5 or 2).

Growth Point 3: Interpretations based on local properties

Students were coded Growth Point 3 if they could interpret local properties of the function such as rates and intercepts from equations. Task 7c, Task 7b and Task 8 (see Figures 22 and 25) were used to assess the achievement of this growth point.

Task 7a, which involved substituting a single value, was included in the set to make the set of tasks appear "less difficult" and ease the students into the problem but it was not used to assess any of the growth points.

Imagine water flowing through a pipe into a container. The following equations show how the water level or height of the water (w) in the container was related to the number of minutes (t), when the pipe was opened for 10 minutes.

$w = t + 8$ for the first four minutes *($t = 0$ to 4)*

$w = 3 \times t$ for the remaining six minutes ($t = 4$ to 10)

where,

w refers to the water level (height) in centimetres

t refers to the number of minutes

Please use the above information to answer the following questions.

7a. What was the height of the water in the container three minutes after the pipe was opened?

7b. From the given information, do you think the height of the water in the container is increasing at the same rate throughout the 10 minutes? Circle the letter corresponding to your answer.

a) Yes, the water level increases at the same rate throughout the 10 minutes.

b) No, the water level is not increasing at the same rate throughout the 10 minutes.

Please show or explain how you obtained your answer.

7c. From the given information, do you think the container already contains water before the pipe was opened? Circle the letter corresponding to your answer.

a) Yes, the container already contains water before the pipe was opened.

b) No, the container does not contain water before the pipe was opened.

Please state or show how you obtained your answer.

Figure 22. Tasks 7a, 7b, 7c - Intercept and Rate.

Task 7c requires interpreting the intercept. Students could either evaluate the given equation for $t = 0$ or recognise the meaning of the constant in the equation. As with the other questions, incorrect solutions or explanations or no explanations were not considered. Most of the responses involved evaluating the equation for $t = 0$. A typical solution is shown in Figure 23.

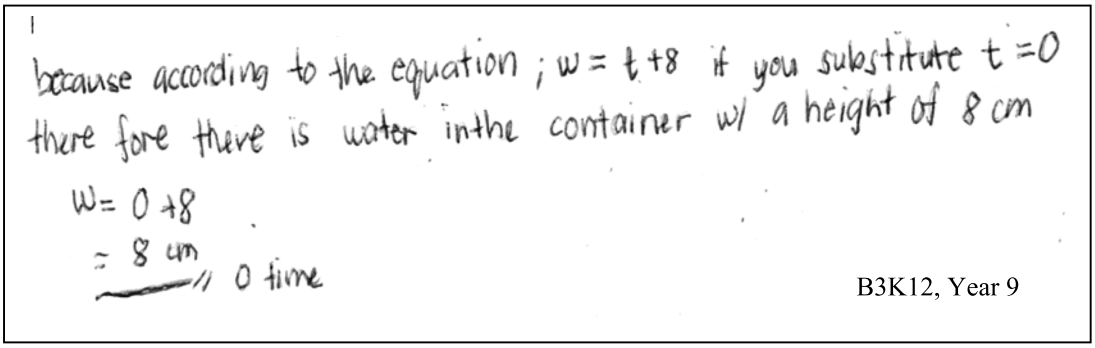

because according to the equation ; w = t +8 if you substitute t =0
there fore there is water inthe container w/ a height of 8 cm
w= 0 +8
= 8 cm
// 0 time
B3K12, Year 9

Figure 23. Task 7c solution.

There were some students who evaluated the equation for $t = 1$ and then reasoned that because the value is 9, which is too big, the container must not have been empty. This kind of answer, though correct, was not considered acceptable for Growth Point 3. The response does not clearly reflect that students knew the concept of intercept or can interpret it from the equation.

Task 7b is the most difficult of the three tasks involving the piece-wise function. Students' correct solutions in Task 7b included evaluating the two equations for several values then comparing the increase or interpreting the slope as showing the rate of change. A typical student solution to this task is in Figure 24. Student B2D8 solution, which involved evaluating the equations for each value of *t*, was written on a separate page.

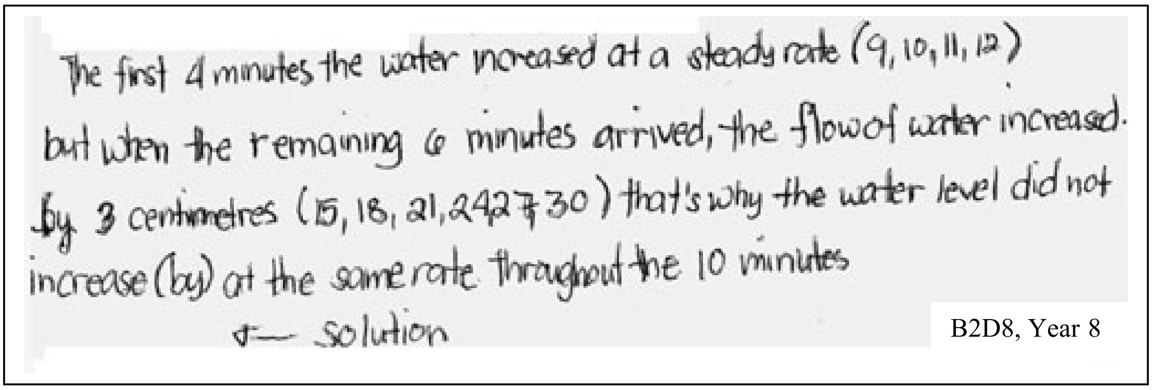

The first 4 minutes the water increased at a steady rate (9, 10, 11, 12)
but when the remaining 6 minutes arrived, the flow of water increased.
by 3 centimetres (15, 18, 21, 24 27 30) that's why the water level did not
increase (by) at the same rate throughout the 10 minutes
← solution
B2D8, Year 8

Figure 24. Task 7b - Solution involving point-wise interpretations.

Point by-point interpretations (see e.g. in Figure 24) are the most tedious way to complete the task but all of the students' solutions were of this type. No student used their knowledge of the slope or interpreted the coefficient of *t* as the basis for their answer. This latter solution is the preferred strategy since it is more straightforward and reflects understanding of the properties of function or the meaning of the parameter (the coefficient of *x*).

Some students selected the correct answer but their explanations were incorrect. For example, students evaluated the two equations for a single value then compared the result. Because the results were different, they concluded that the rates were different also. Other incorrect reasoning involved merely saying that the rates were different because the two equations were also different. This latter solution could be considered a partial explanation, but because the students did not specify which part of the equation tells about the rate, the explanation was not accepted as correct.

Since it is possible that students' difficulty with Task 7b may be due to the fact that it involved a piece-wise function (Markovits, Eylon & Bruckheimer, 1986) or context may have played a part in it, students were given a second chance to show their knowledge of interpreting the growth property from equations. Task 8 (see Figure 25) is similar to Task 7b, except that Task 8 is not in context.

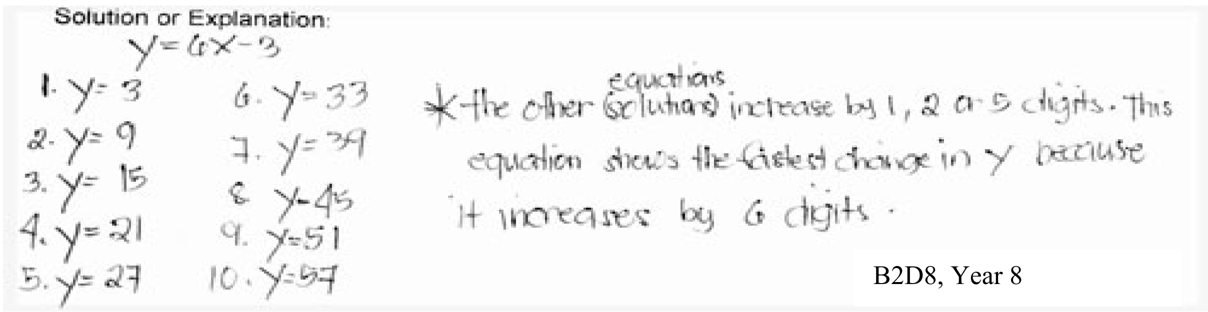
8. Which equation shows the fastest change in y when x takes values from 1 to 10? Please show/ explain how you worked out your answer.

a. $x + y = 100$ b. $y = 6x - 3$ c. $4y = 8x$ d. $y = 75 + 5x$

Solution or Explanation:

Figure 25. Task 8 - Rate.

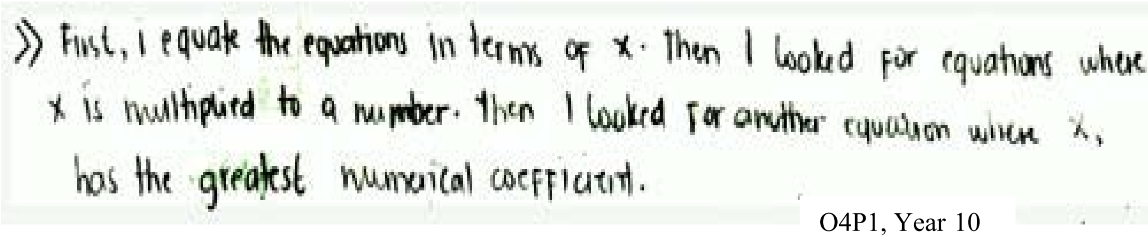
Like Task 7b, Task 8 also involves simple linear equations. It was thought to be easier than Task 7b since the question was more straightforward and did not involve a piecewise function but the result showed that it was just as difficult. The source of error was also in evaluating the equations for a single value and then choosing the equation producing the highest value, a clear sign that the concept of rate was not really understood. The majority of the solutions used evaluating the equations and then comparing the increase in the value of *y*. Figure 26 shows a typical solution for this task.



B2D8, Year 8

Figure 26. Task 8 - Solution involving point-by-point strategy.

The preferred strategy would be the one based on the interpretation of the parameter (numerical coefficient of x) but only three students used the idea: one student from Year 9 and two from Year 10. Figure 27 shows a sample solution.



O4P1, Year 10

Figure 27. Task 8 - Solution involving interpretation of parameter.

In order to be coded Growth Point 3, students needed to be able complete correctly Task 7c and least one of Task 7b or Task 8. This would mean that they could work out function tasks that involved interpretation of these basic properties of function – intercept and rate.

Growth Point 4: Manipulation/ transformation of functions as objects

In the initial framework, students' understanding of function represented by equation as a manipulable mental object was to be assessed in terms of tasks involving linking it with other representations. However, students' understanding involving linking representations was made a separate domain to highlight its importance in the understanding of function. Hence, a growth point, which would assess understanding of function represented by equations as an object, was added to the framework. This is Growth Point 4.

Growth Point 4 requires that students should not only be able to conceive of equation as a representation of a relationship between two variables, but must be able to conceive the function it is representing as an object that can be manipulated or transformed. This was evident in Task 9 where the entire equation became the input itself (composition of function, Strategy 2) or in Task 10 (Strategy 2) where an operation could be performed based on analysis of the set of values generated by the equation. To further assess understanding in this level, Task 11 (see Figure 28) was designed.

11. The relationship between x and y in Table 1 is $y = 2x + 1$. In Table 2, the values of x and y in Table 1 were swapped or interchanged. Please write the equation, which shows the relationship between x, and y in Table 2? Show how you obtained your answer.

Table 1

x	y
0	1
1	3
2	5
3	7
4	9

$y = 2x + 1$

Table 2

x	y
1	0
3	1
5	2
7	3
9	4

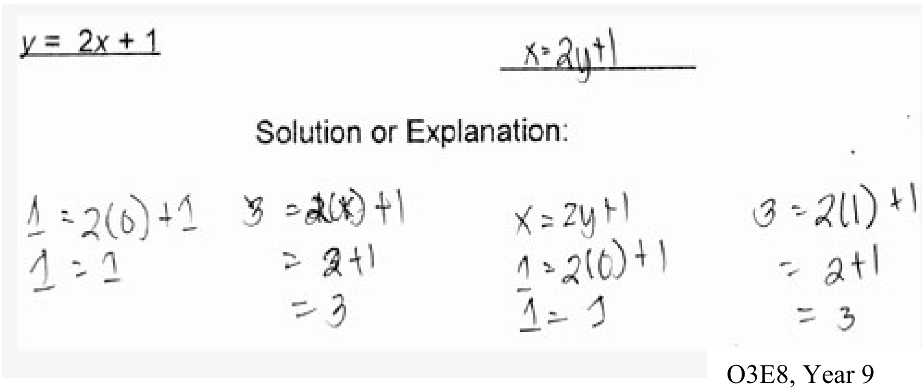
____

Solution or Explanation:

Figure 28. Task 11 - Inverse.

Task 11 involves determining the equation of the inverse of the function given in tables. The task did not use the term "inverse" in order to prevent students from using mechanically learned algorithms for determining the inverse of a function.

Strategies used to solve Task 11 were classified into three. Strategy 1 involved swapping the x and y values in the given equation. Figure 29 shows a typical solution coded Strategy 1.



O3E8, Year 9

Figure 29. Task 11 - Strategy 1. Swapping x and y.

Strategy 2 involved guessing the equation and then checking against the given values. Using the algorithm for deriving linear equations from the values in the table was also coded Strategy 2, since it is still point-wise interpretation in the sense that it involves analysis of specific values in the table. Figure 30 shows an example of a student's solution coded Strategy 2.

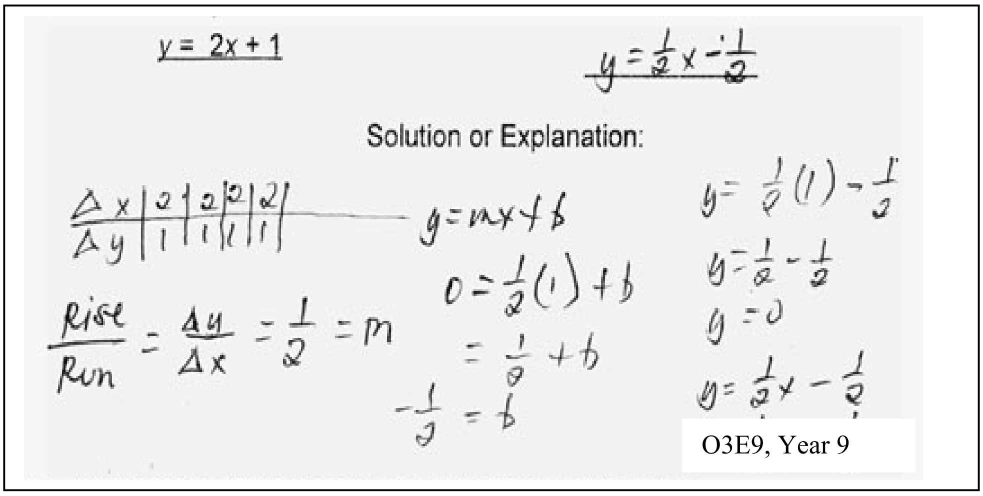


Figure 30. Task 11 - Strategy 2. Using specific algorithm for linear function.

Strategy 3 involved swapping the *x* and *y* variables first in the equation and then solving for *y*. This approach reflected an understanding of function as a manipulable object. A typical solution coded Strategy 3 is shown in Figure 31.

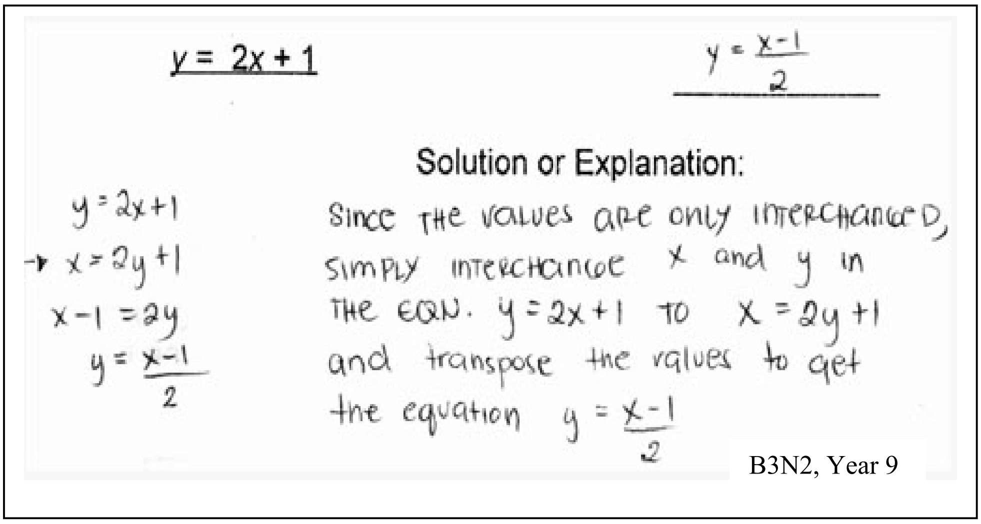


Figure 31. Task 11 - Strategy 3. Equations as manipulable objects.

The distribution of students' strategies for Task 11 is shown in Table 11. Results showed Year 8 students almost exclusively solving this task using Strategy 1. This indicates that students would first solve this task using Strategy 1 only. Performance of Year 9 and Year 10 students shows a steady increase in the proportion of students working out the task using Strategy 3 confirming that students come to conceive functional equation as an object as they gained more experience with mathematics. This also further indicates that Task 11 is a good assessment task for understanding of function represented by equation as a mathematical object.

Table 11
Percentages of Students Coded the Strategies for Task 11

Strategies	Year 8 (n = 149)		Year 9 (n = 152)		Year 10 (n = 143)	
	D1	D2	D1	D2	D1	D2
Strategy 1	22.2	43.6	43.4	45.4	21.7	20.3
Strategy 2	0.7	8.1	4.0	13.8	12.6	15.7
Strategy 3	0.0	2.6	3.9	7.2	21.0	33.3
Incorrect	76.5	45.6	48.7	33.6	44.8	27.3

Students coded Growth Point 4 were able to work with composition of function (Strategy 2 in Task 9), perform operations on the functional equations (Strategy 2 in Task 10) and solve for the inverse of a function by manipulating and transforming the given equation (Strategy 3 in Task 11).

Solutions coded Strategy 2 in Task 11 were not considered for assessing Growth Point 4 for the following reasons. Solutions, which indicated guessing the inverse equation, were not considered for assessing this growth point since it was possible that students used point-by-point analysis. In fact, some did show the checking. A few students used the algorithm for finding the equation of a linear function. This solution, which was also coded Strategy 2, indicated students' facility with computation but this solution is also more complicated. The fact that the student opted for this solution could be interpreted as lack of confidence to work with the function in equation form as an object in itself, which was what Growth Point 4 is about working with function represented by equations as objects.

The use of Strategy 1 could indicate lack of knowledge and skill on the part of the students to perform an operation on the equation. There is a possibility that students do not have the skill to perform an operation on the equation and if students have the skill there is the possibility students were not sure whether they needed to further manipulate the equation. So they settled on simply swapping x and y, which was what the problem says. It could also be due to lack of knowledge of the commonly accepted form of describing function, which was expressing the equation in terms of y, the dependent variable. The fact that there was a larger proportion of Year 8 students who used Strategy 1 than Year 10, who proceeded to solve for y after the swap, shows the likelihood that these could indeed be the reason why students settled on swapping the x and y values as a complete solution for Task 11. However, there is also the possibility that students in the higher year levels did not express the relations in terms of y because they knew that this is not necessary. That is, they understand that functions remain invariant in equivalent forms of the equation. Hence, the data on students' performance in the three tasks assessing Growth Point 4 were revisited including their solution in Task 18, which was used to assess understanding of equivalent forms of function in equations (see Figure 76 in Chapter 7). The result showed that there were only seven students, all from the second data collection period, who were not coded Growth Point 4 because they used Strategy 1 for Task 11. Of the seven students, five were Year 8 and two were Year 9. Only one of the five Year 8 students used solution for Task 18, which reflects understanding of function as an object. This student was coded Growth Point 4 under Equations.

Investigating for Typical Learning Trajectory

This section is divided into two parts. The first part presents the success rate of each task. This gives an overview of students' overall performance on the tasks. The second part presents the data showing the trend in the order of the growth points and other relevant findings.

The assessment tasks were constructed so that they could efficiently be solved by strategies involving holistic interpretation of the equations or its properties. However, as shown in the preceding discussion of the tasks, the majority of the students still opted to use specific values to work out almost all of these tasks. Hence, a large part of the success rates of the tasks comprised solutions using point-by-point analysis or evaluating the equations using specific values.

Success Rates

Table 12 summarises the success rates in the tasks used to assess the growth points identified under *Equations*. D1 refers to the first data collection period and D2 refers to the second.

The results showed that tasks requiring point-wise analysis (generally associated with Growth Point 1) had higher success rates and tasks requiring working with the representations as objects (associated with higher growth points) have lower success rates.

There are slight differences in the order of difficulty of the assessment tasks for Equations in each year level and in both data collection periods. However, in general, the tasks found easy or difficult in the first data collection were still found easy or difficult in the second data collection.

Students' success rate for all tasks increased in the second data collection period. The increasing success rates from Year 8 to Year 10 further confirmed that the assessment tasks measure conceptual understanding that the students develop as their mathematics experiences increase.

All year levels found Task 10.1 the easiest among the tasks. This is true for both data collection periods. In Year 8, however, at the beginning of the school year, less than 50% completed this task compared to almost a hundred percent in Years 9 and 10. Five months later, the percentage of the Year 8s who could work out the task rose from 50% to 90%. This means that the minimum skills, generating values from equations, necessary to complete Task 10.1 are easily learned.

Table 12
Success Rates on Tasks Assessing the Growth Points under Equations (%)

Tasks	Year 8 (n = 149)		Year 9 (n = 152)		Year 10 (n = 143)	
	D1	D2	D1	D2	D1	D2
T10.1: Generating values	49.0	90.6	94.1	100	96.5	100.0
T9.1: Relating equation (point-wise)	41.6	62.4	69.7	78.9	78.3	91.6
T10: Generating equation	17.4	64.4	53.9	78.9	80.4	84.6
T11: Inverse	23.5	54.4	51.3	66.4	55.2	72.7
T7c: Intercepts	18.8	36.2	36.8	50.0	42.7	69.9
T9: Relating equation (Composition)	4.7	24.2	16.4	46.1	38.5	60.8
T8: Rate	6.0	25.5	13.2	35.5	30.8	51.7
T7b: Rate	4.7	20.1	14.5	21.7	23.8	44.8

The second easiest among the tasks was Task 9.1, also a task involving working with equations using specific values. However, Task 9, a version of Task 9.1, which involves working with equation as "wholes", was one of the most difficult tasks for the Year 8 students even though it involves the same equations as Task 9.1. This result further confirmed that concepts, in this case functions represented by equations, are learned first as a process rather than as an object.

The two most difficult tasks involve understanding of rate. These are Task 8 and Task 7b. Both tasks involve linear equations and may be solved by substituting specific values. In fact, nearly all of the students' solutions involved point-by-point interpretations through evaluating the equations with individual values.

Task 11 was one of the tasks, which assessed the last growth points under Equations, yet the success rate shows that students did not find this task as difficult as Task 8 or Task 9. This is because the calculation of the success rate included all correct strategies. In coding for Growth Point 4, only solutions coded Strategy 3, which is reflective of understanding described by this growth point, were considered.

Order of the Growth Points

The framework of growth points that the present study developed aimed to describe typical learning trajectories in students' developing understanding of key domains of the function concepts hence the order of acquisition of the growth points was investigated. The method and results of the investigation are discussed below.

The coding procedure for the growth points under Equations is summarised in Table 13. GP stands for Growth Points. The coding procedure shows that Growth Points 1, 2, and 4 share some assessment tasks. However, for each growth point, the set of tasks was different even though they were based on a common group of questions. The numbers in brackets after the task number represent the strategies accepted for coding the corresponding growth point. If there is no strategy number indicated, then it means all strategies for the tasks discussed earlier are accepted in coding for the growth points.

Table 13
Procedures for Coding the Growth Points under Equations

Growth Points	Coding Procedure
GP 1: Equations as procedures for generating values	Any one of T10.1, T10, and T9.1
GP 2: Equations as representations of relationships	1. T9.1 + T9 2. T9.1 + T10.1 [2] or T10 [1.5 or 2] 3. T9 + (T10.1 or T10)
GP 3: Interpretations based on local properties	T7c + (T7b or T8)
GP 4: Equations as objects that can be manipulated and transformed	T9 [2]+T11 [3]+T10 [2]

Of course, if different sets of tasks were used in assessing the growth points, there is a possibility that students would be coded at other growth points. Thus, one way to show the trend in the order of acquisition of the growth point is through comparing the percentage of students coded at the growth points. In this approach, the growth point with the highest percentage of students could be assumed the first to be acquired since the majority of the students could work at this level. The growth point with the lowest percentage could be assumed the hardest to reach and could therefore be considered the last growth point.

The positions of GP 1, GP 2 and GP 4 could be established theoretically. That is, based on the process-object perspective, which states that in general individuals conceive of mathematical concepts as a process first before they are conceived as objects, GP 1 would indeed be the first growth point. GP 1, GP 2 and GP 4 are reflective of Sfard's (1991) interiorization- condensation-reification stages toward conceiving a concept as a mental object or Breidenbach, Dubinsky, Hawks and Nichols' (1992) action-process-object levels described in Chapter 2 of this report. GP 3 is based on Slavit's (1997) property-oriented perspective on understanding of function, which looks at students' understanding of function based on its properties. GP 3's place in the order of the growth points was not very clear at the beginning of the study. In the following discussion, data confirming and establishing the order of the growth points under Equations are presented.

Table 14 shows the percentage of students coded at the growth points identified for Equations. GP 0 stands for growth point zero. Students coded at this growth point were those not coded in any of the other four growth points.

Table 14
Percentages of Students Coded at the Growth Points under Equations

Growth Points	Yr 8 (*n* = 149)		Yr 9 (*n* = 152)		Yr 10 (*n* = 143)	
	D1	D2	D1	D2	D1	D2
GP 0	33.6	7.4	2.0	0.0	0.0	0.0
GP 1	66.4	92.6	98.0	100.0	100.0	100.0
GP 2	10.1	28.9	23.0	52.6	44.8	68.5
GP 3	5.4	20.1	14.5	27.6	24.5	55.2
GP 4	0.0	0.7	1.3	0.7	3.5	12.6

GP 1, which was about understanding of functional equation as a procedure for generating values is the first growth point in students' developing understanding of function represented by equations. The large percentage of students coded at this growth point confirmed this. Initially, only about 66% of 149 Year 8 students were coded at GP 1. Five months later, there was a big jump in the percentage of students coded GP 1 to 93% in Year 8. This jump was largely due to Task 10.1, which was not initially accessible to Year 8 students. Students considered in the study may not have been introduced to quadratic functions but these students have experiences with equations in second degree because they learn algebraic expressions and sentences earlier in the year. The high percentage of students at GP 1 indicates that this point-wise understanding of function represented by equations is easily understood even at this year level.

That GP 2 is the next growth point in students' developing understanding of function was indicated by the second largest percentage of students coded GP 2. GP 2 is about interpretations based on relationship both involving point-wise and holistic interpretations.

The percentage of students coded GP 3, which was about interpretation of properties of the function such as rate and intercept, show that it is the third growth point. The last growth point was indeed GP 4 as shown by the smallest percentage of students coded at this growth point.

The order of the growth points, which was from GP 1 to 4, is true for all year levels and for both data collection periods. Of course, this does not imply that this learning trajectory is true for all students. However, for the majority of them, the order is appropriate.

Other Observations

Other observations that could be gleaned from the data on the growth points are discussed in the following sections. Figure 32 shows the relationship among the growth points in visual form.

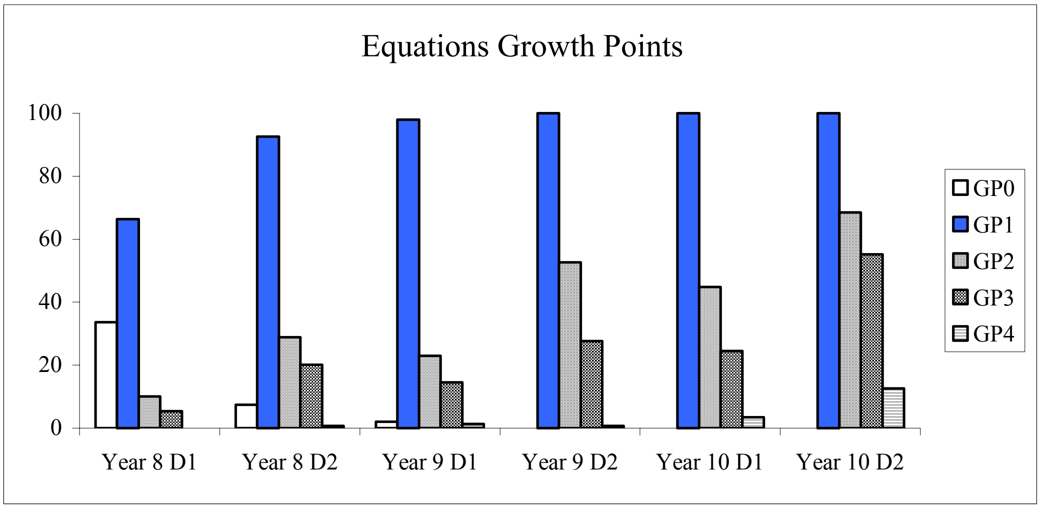


Figure 32. Percentages of students at each growth point under Equations.

The large difference in the percentage of students between growth points within the same year level confirmed that the identified growth points were indeed big ideas. In Year 9 during the second data collection period for example, while all of them could work in terms of GP 1, only half could work in terms of GP 2, about a quarter in GP 3 and almost none in

GP 4. The difference in the percentage however is closing in, in Year 10 based on the result of the second data collection period. This is to be expected since these students have considerable experience with working with function.

The percentage of students in the growth points, except GP 1, differed considerably between year levels with the Year 10 showing higher achievement compared to Year 9 students and the latter compared to Year 8 students. For example in the second data collection period, about 30% of the Year 8 students could work in terms of GP 2, while the figure was about 50% and 70% in Year 9 and Year 10 respectively. The percentage of students coded GP 3 was about 20%, 30% and 60% for Years 8, 9 and 10 respectively. At GP 4, the percentages were zero, two and eleven for the year levels, respectively. This trend is ideal, for indeed students with more experience with mathematics should achieve higher growth points. These large differences also indicate a wide gap in the level of understanding among the year levels as far as understanding of functions represented by equations is concerned.

Relationships between Growth Points

The relationship between the growth points under Equations was investigated further. Does achievement of growth points imply understanding in terms of lower growth points? Table 15 shows the distribution of students achieving the growth points.

In the table, GP 0 means students coded Growth Point 0. GP 1 means students coded GP 1 only. GP [1, 2] means students coded GP 1 and GP 2. Likewise, GP [1, 2, 3] means students coded GP 1, GP 2 and GP 3 and GP [1, 2, 3, 4] means students coded GP 1, GP 2, GP 3 and GP 4. The shaded area means it did not fit the "cumulative" pattern of the growth points just listed. For example, GP [1, 3] means students coded in GP 1 and GP 3 but missed GP 2.

It was expected that those coded in GP 4 would also be coded in GP 2 and GP 1 as well, since conditions satisfying GP 4 satisfy GP 2. Also, conditions satisfying GP 2 also satisfy GP 1 (see Table 13 which shows the coding formula). What needs to be investigated therefore is how GP 3 is related to the other growth points. That is, could students coded in GP 3 also work in terms of GP 2 and did students coded GP 4 show understanding in terms of GP 3?

Table 15
Frequencies of Students at the Growth Points under Equations

Growth Points	Year 8 (*n*=149)		Year 9 (*n*=152)		Year 10 (*n*=143)	
	D1	D2	D1	D2	D1	D2
GP 0	50	11	3	0	0	0
GP 1	78	85	95	61	63	34
GP [1, 2]	13	22	30	47	40	28
GP [1, 2, 3]	2	20	3	30	19	52
GP [1, 2, 3, 4]	0	1	0	1	2	16
GP [1, 3]	6	10	19	11	16	11
GP [1, 2, 4]	0	0	2	2	3	2

Rows GP [1, 2, 3] and GP [1, 2, 3, 4] in Table 15 show the number of students coded GP 3 but also those coded GP 2. Row GP [1, 3] shows the number of students coded GP 3 but not GP 2. Comparing these frequencies, it appears that at the beginning of the school year there were more students coded at GP 3 who were not coded at GP 2: six out of eight students in Year 8 and 19 out of 22 in Year 9. However, in the middle of the school year during which the second collection of data occurred, 21 out of 31 Year 8 students and 30 of 42 Year 9 students or over 60% of students coded at GP 3 could now also work in terms of GP 2. The percentage is even higher in Year 10 with over 85% or 68 out of 79 students coded at GP 3 who were also coded at GP 2. These results indicate a low chance that students achieving GP 3 early in the school year could also work in terms of GP 2 but this chance is greatly improved in the middle of the school year. The chance of course is greater in Year 10 than in Year 9 and definitely than in Year 8 because students in the higher years have more experiences with mathematics and other families of functions.

An investigation of the performance of those who were at GP [1, 3] or those coded at GP 3 but not GP 2 was made. It appeared that these students did not completely miss the conditions required for GP 2. Most of them were not coded GP 2 because they did not meet the requirement of at least a code of Strategy 1.5 or 2 which was about holistic interpretation of the relationship represented by the equations assessed by Tasks 9, 10 or 10.1. They completed at least one of these tasks but using a point-by-point strategy. Therefore, students who were coded GP [1, 3] does not necessarily mean they were not able to interpret the relationships between equations. They were just not capable yet of interpreting the function represented by equations in a more holistic way.

Rows GP [1, 2, 3, 4] and GP [1, 2, 4] in Table 15 show the number of students coded in GP 4 and the other growth points. No Year 8 student was coded GP 4. Data for Year 9 show that two of the 152 students were coded GP 4 but were not coded in GP 3. There was one student in Year 9 during the second data collection period that was coded GP 4 and GP 3 as well. This proportion is somewhat the same with Year 10 students in the first data collection period. However, in the second data collection period, 17 of the Year 10 were coded GP 4 and 15 of these were coded GP 3 as well. This indicates students who could work in terms of GP 4 may be expected to work in terms of GP 3 as well. But because of the limited data, it is suggested that the assessment task be given to Year 11 (equivalent to first year college in the Philippines) to further confirm this observation.

The distribution of students in Table 15 also shows that the majority of the students were coded GP 1 only. It was only during the second data collection period in Year 9 where there were now more students coded at other growth points as well. In fact, it was only in Year 10, during the second data collection period, where the distribution shifted. By this period, the Year 10 students coded at GP 1 only and those coded the first two growth points (GP [1, 2]) decreased: from 63 to 34 for GP 1 only and from 40 to 28 for GP [1, 2]. The frequency of students coded the first three growth points, GP [1, 2, 3], jumped from 19 to 52 in this period. There was also a sharp increase in the frequency of Year 10 students coded at all the growth points from two to 16. These movements support the order of acquisition of the growth points. It also shows a very slow movement from one growth point to the next. This slow movement confirms that the identified growth points were indeed big ideas. That there was a considerable number of students in Year 10 who achieved GP 1 only should be a cause of concern. It seems that while students were indeed learning higher-level mathematics content, for the majority, the level of analysis remained point-wise. Students' achievement in the growth points particularly under Linking Representations and Equivalent Relationships domains presented in Chapter 6 and 7, respectively further confirms this observation.

Discussion and Summary

As mentioned in Chapter 3, the growth points and the development of the tasks assessing the growth points informed each other. Although the development of the tasks were based on a tentative list of growth points, the growth points were revised and the descriptions refined after further analyses of the tasks and students' responses and strategies used to work out the tasks.

The typical learning trajectory in this domain proceeds from Growth Point 1 to Growth Point 4. The growth points identified under Equations were:

Growth Point 1. Interpretations based on individual values (interpretations of functional equations as procedure for generating values)

Growth Point 2. Interpretations based on relationships.

Growth Point 3. Interpretations based on local properties.

Growth Point 4. Manipulation and transformation of equations seen as objects

Growth Points 1, 2 and 4 may be located within the process-object path with Growth Point 1 at the process end and Growth Point 4 near the object end. Growth Point 3 belongs to the property-oriented path. The interpretations of local properties (Growth Point 3) such as intercept and slope comes after Growth Point 2. This is because the local properties considered were properties of the relationship or the function; hence understanding of the equation as a representation of relationship between two variables should indeed be acquired first.

There were some students coded Growth Point 3 but not Growth Point 2. Further examination of these students' performance in the tasks assessing GP 2 showed that these students were able to make interpretations based on relationships but could only work in terms of interpreting the relationship point-wise. Growth Point 2 requires both point-wise and holistic interpretations. This implies that the growth points identified are not discrete. That is, a student may already be operating in a higher growth point but has not fully understood the previous growth point.

Except for a small number of students, the students coded Growth Point 4 were coded Growth Point 3 as well. This means that those students who could work in terms of Growth Point 4 are more likely to work in terms of Growth Point 3 also. However, further investigation involving students in the higher years is needed to check the relationship between Growth Point 3 and Growth Point 4, because there were only ten percent of the students who achieved Growth Point 4 in the study. Of course, achieving both Growth Point 4 and Growth Point 3 does not necessarily imply that understanding of Growth Point 4 builds on an understanding of Growth Point 3.

Students' success rates in individual tasks show students' preference for point-by-point interpretations. This preference was also apparent in the strategies they used to work out the tasks. It seems a big step for the majority of the students working with the equation representation of function to move from point-by-point interpretation to a more holistic interpretation. This observation is true also for the other three domains considered in the framework of growth points. Students' difficulty in working with the equation representation of function beyond point-wise analysis may be because equations themselves could easily be interpreted point-wise. In addition, experiences with this representation provided in their mathematics class may have been limited to merely point-wise interpretations. Teachers therefore need to design learning experiences where students are encouraged to interpret the relationships represented by equations holistically.

Students had difficulty in interpreting the rate or growth property of the function from equations as shown by their performance in Task 7b and Task 8. This may be because interpretation of rate involves analysing and comprehending the changes not only between x's and y's but also between y_m and y_{m+1} and x_m and x_{m+1} (see Confrey & Smith, 1994; Slavit, 1997). Sierpinska (1992) also argued that students have difficulty identifying the changing quantities in a functional relationship. Rate is one of these changing quantities, which explains students' difficulty with the concept. This difficulty needs to be overcome in order to understand function fully.

Nearly all those who correctly completed Task 7b and Task 8 used substituting individual values to the given equation, a very tedious approach, instead of interpreting the parameter m or using the idea of slope to determine which of the equations shows fastest change. This may be because a parameter demands thinking at a general, abstract level. It is a higher-level variable; a change in its value affects not just one value of the function but the entire function itself (Drijvers, 2001).

Rate is a complex concept and the present study only assessed a part of it. Another aspect of rate for example, is rate of change, which is a fundamental concept in Analysis. In addition, students' difficulty with this concept is not confined to equation representation but also in the graphical representation of function (see Chapter 5). It is suggested therefore that a study identifying the growth points in students' understanding of this concept in terms of level of abstraction involved be made to enhance the framework of growth points that the present study developed.

I have presented in this chapter the theoretical and empirical evidence of the growth points under Equations. I have described and analysed the tasks including students' strategies for each task used to assess and describe the growth points, arguing particularly on the correspondence between the tasks and students' strategies and, the growth points they were assessing.

The frequency of students who have achieved the growth points established empirically the order of the growth points. Frequency of students at the growth points was decreasing from Growth Point 1 to Growth Point 4. Results also showed that the majority of the students who have achieved a higher growth point also achieved the lower growth points. The order of the growth points is the same for both data collections, which were about five months apart, the first given at the beginning of the school year. The pattern is also true to all year levels. Furthermore, the order of the growth points also supports the theory that students are more likely to think of function as a process rather than an object, initially.

The next chapter presents the growth points in students' understanding of function represented by graphs.

CHAPTER 5

DOMAIN 2 - GRAPHS

The graph is one of the most widely used visual representations of relationships between variables. Traditionally, students are introduced to graphs in the *x*-*y* plane about the same time as the concept of function (Goldenberg, Lewis, & O'Keefe, 1992). The graph packs in information about the relationship it is representing from relationship between individual values to the way the relationship is decreasing or increasing and from local, global to invariant properties of the relationship. Understanding of functions represented by graphs could therefore be considered one of the major nodes in the network of students' understanding of function.

The growth points under Graphs presented in this chapter deal with the aspects of understanding of function represented by graphs alone. Aspects of understanding of function that could be demonstrated by linking graphs with other representations are described in other domains in the Framework.

The discussion in this chapter includes the description of the growth points and the tasks used to assess them. This is presented in the first part of the chapter. The purpose is to show the correspondence between the tasks and the growth points. Included in the discussion are the success rates for each task. The second part of the chapter presents the results of the data collections showing students' typical learning trajectories in terms of the order of the growth points and other relevant findings. The third part gives the summary of the chapter.

Assessing the Growth Points

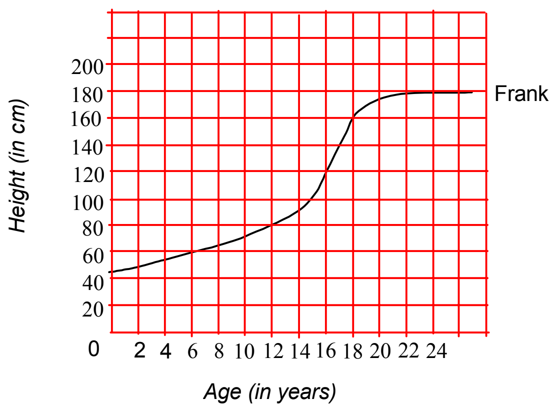
This section describes the growth points and the set of tasks. The description of each task includes an explanation of why it was used to assess that particular growth point and in some cases, how the task informed the identification and description of this growth point. Where appropriate, the revisions made and findings related to these tasks are also presented.

There were four growth points identified under Graphs. Eight tasks were used to assess these growth points. The tasks involved interpreting individual points, intervals, properties, and relationships.

Growth Point 1: Interpretations based on individual points

All that is required of students at this growth point is that they know that graphs consist of points and that each point describes a correspondence between two values. That is, they should be able to interpret individual points in the graphs. Three tasks involving interpreting individual points assessed Growth Point 1. Task 1 is about reading off a single value in the graph. Task 2 is about calculating amount of change. Task 1 and Task 2 are shown in Figure 33. Task 4 (see Figure 35) involves interpreting intersections, hence also requiring interpretation of individual points.

The graph below shows Frank's height and age.



Use this graph to answer questions 1-3.

1. When Frank was 140 cm, how old was he?

2. About how many centimetres did Frank grow between the age 6 and 12?

Figure 33. Tasks 1 and 2 - Reading values and calculating amount of change.

The study assumed that answering at least any two of Tasks 1, 2 and 4 correctly would indicate students' understanding of graphs as consisting of points and that each point describes two values, which is what is required in Growth Point 1. The majority of the students coded at Growth Point 1 however, completed all three tasks. In Year 8 alone, during the second data collection period, of the 147 students who were coded Growth Point 1, 123 students completed all three tasks although some correctly interpreted only one of the intersections in Task 4.

Growth Point 2a: Interpretations involving rates

Some of the local properties of a function studied in high school include rate, symmetry, intercepts, etc. One of the earliest properties of the function and the most fundamental of all properties taught is *growth* or *rate*. Tasks 3a and 3b, in Figure 34, involve identifying the given time period that shows the fastest growth (Task 3a) and the slowest growth (Task 3b) in height.

(Please refer to the graph in Figure used in T1 and T2.)

3a. In which period was Frank's rate of growth the fastest? Circle the letter corresponding to your choice and explain why you chose this answer.
a) Age 6 to 12 years b) Age 12 to 16 years c) Age 16 to 18 years
d) Age 18 to 22 years e) Age 22 to 24 years
Explanation:

3b. In which period was Frank rate of growth the slowest? Circle the letter corresponding to your choice and explain why you chose this answer.
a) Age 6 to 12 years b) Age 12 to 16 years
c) Age 16 to 18 years d) Age 18 to 22 years

Explanation:

Figure 34. Tasks 3a and 3b - Fastest and slowest growth.

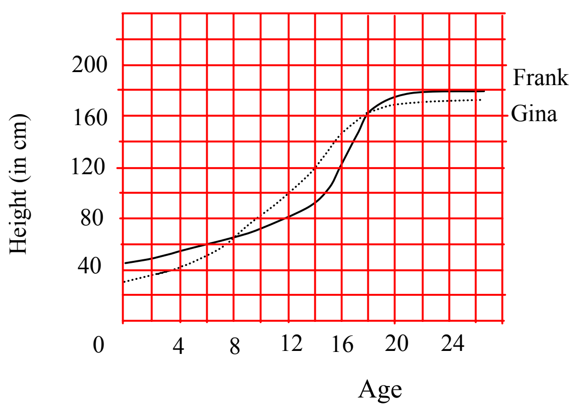
Most students were able to answer Task 3a. A large number of students' explanations included calculation of the average rate and a few reasoned visually, that is in terms of steepness of the graph. Based on the recorded discussion of pairs of students who took the test together, and from observations of those who took the test in interview form, students would identify the interval from the graph first (visual interpretation). Then, realising that they were asked to explain, they wrote the calculation of the average rate. There were a few who reasoned in terms of the steepness of the interval. Task 3a however is a standard textbook task. It was included in the set of tasks in order to make the set of tasks appear more familiar and therefore less difficult, especially to the Year 8 students.

Task 3b was designed to further assess students' understanding of rate in graphs. This task could be completed by looking at the shape of the intervals or by calculating the average rate of the intervals. There was evidence to suggest that students relied on the shape of the graph but failed to consider the length of the intervals. Most of the incorrect answers were choice *d*. In fact in the first data collection, there were more students who chose *d* than the correct answer *a*, 254 to 182 students. In the second data collection the ratio was 216 students choosing *d* compared to 213 students who got the correct answer. If these scores were combined, it would approximate the number of students who answered T3a correctly. These students who chose *d* probably based their answer on the shape of the interval, incorrectly associating *slowest growth rate* with the *levelling* of the graph in choice *d*, which in fact indicates slowing growth rate. A cross-tabulation of students' performance showed that over 95% of students that could complete Task 3b could also complete Task 3a. The instrument could therefore be simplified by deleting Task 3a. However, Task 3a was also useful because it made the test less difficult especially for the Year 8s. To be coded at Growth Point 2, students had to complete both Task 3a and Task 3b.

Growth Point 2b: Interpretations based on continuous property

Growth Point 2b was added to the framework when the results showed a wide difference in the success rate between the tasks assessing growth (Tasks 3a and 3b) and the task assessing another property – continuity (Task 5). Task 5 (see Figure 35) was initially conceived as one of the tasks to assess Growth Point 2a which was originally described as *interpretations based on properties*. Task 5 requires interpreting interval values and conceiving a graph as made up of continuous points. The task requires two answers, one involving exact endpoints and the other needing extrapolation.

The second graph (in broken line) shows also the height of Frank's twin sister, Gina.



Use the graphs to answer questions 4 and 5.

4. When did they both have the same height?
5. When was Frank taller than Gina?

Figure 35. Task 4 and 5 - Intersections and intervals.

Task 5 proved to be one of the most difficult tasks under Graphs. One can argue that this may be because Task 5 is an open-response type rather than multiple-choice type like Tasks 3a and 3b. That is, students have to write the time-interval as compared to Task 3a and Task 3b where time-intervals are given. If this were the case then many students would leave the task unanswered. However, students provided answers to the task but they were incomplete. Some of the teachers in the participating schools were asked to suggest why students' had difficulty with Task 5. One reason provided was that this task required two-part answers. Hence, students' responses to this task were further investigated. Table 16 show Year 8 students' responses for Task 5 in the second data collection. The responses were categorised into four types.

Table 16
Distribution of Year 8 at the Different Responses for Task 5 (%)

Responses	Year 8 (*n* = 149)
A. Incorrect	18.1
B. Single values or one interval only	18.8
C. Two intervals	50.7
D. Correct responses	13.4

Responses in Category A were the incorrect answers. This included any number from 8 to 18. Students who left the task unanswered were also included in Category A. Responses in Category B were incomplete answers. This means students only listed the numbers in the rigid intersections or had only written one of the two correct intervals. Responses in Category C were also incomplete answers. These responses included the two intervals but failed to interpret one of the endpoints correctly. Examples of responses in this category are "0 to 8 and 19 to 26" and "1 to 7 and 19 to 24". Responses in category D were those accepted as correct answers. Answers accepted for Task 5 included "birth to 8 and from 18", "before reaching 8 and from 18 onwards", "0 to 8 and 18 to 26", or other similar responses. If the second interval was only up to 24 or any number before 26, then it was assigned in category C because the graph clearly showed that it was up to 26. The ideal answer would have been the one that included the fact that Frank would be taller than Gina from 18 onwards and not just up to age 26, though this extrapolation requires knowledge of context.

With over half of the total number of Year 8 students in Category C, the non-completion of the task could not be attributed to Task 5 requirement of two-part answers. It could not also be attributed to what Leinhardt, Zaslavsky and Stein (1990) called point/interval confusion in which “students often narrow their focus to a single point even though a range of points (an interval) is more appropriate” (p. 37), because, of the 444 students in the second data collection, only 33 students gave single value answers or enumerated values in the rigid intersection. There is however a possibility that students’ difficulty with Task 5 was due to the word “when”. Leinhardt, Zaslavsky and Stein, reporting on students’ difficulties in interpreting graphs argued that the word “when” is ambiguous. It might indeed be because the word “when” does not specify a range so any range will do. The 262 students out of the 444, who gave only part of the interval, mostly leaving out the endpoint, were probably thinking that giving only a part of the interval would already be considered correct.

While the majority of the students answered in terms of intervals, most of these responses did not include the endpoints or interpreted them incorrectly, indicating lack of knowledge of interpreting the continuous property of the graph.

The last two growth points just described were labelled 2a and 2b to indicate that although they are separate growth points and not related conceptually, both described understanding involving properties

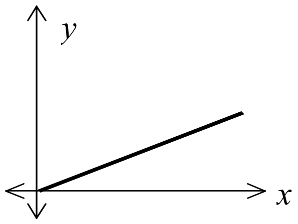
Growth Point 3: Interpretations based on holistic analysis of relationships

Growth Point 3 requires students to conceive of function represented in graphs holistically. This requires an understanding of graphs both as sets of related points and as single visual entities representing relationships between variables.

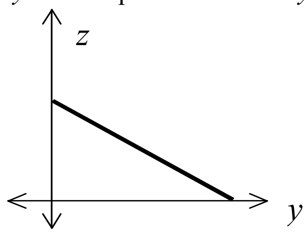
Growth Point 3 was added to the framework when pilot studies showed that students could reason in terms of the relationships represented in graphs holistically. A task was then designed to assess if students could interpret holistically functions represented by graphs and be able to coordinate relationships between variables as well as create the representation of this relationship in graphical form.

Tasks 6 and 6.1 in Figures 36 and 41, respectively, were developed to assess this growth point. Both tasks involved interpreting the relationship between the variables and then coordinating the relationship between three variables x, y and z and constructing the appropriate graph. Both tasks were more of a construction than simply an interpretation task. The only difference between the two tasks was the use of grids. Grids were not used in Task 6 to encourage students to reason in terms of relationship or interpret the graphs as “wholes” and not by individual points. As with other tasks, students' answers were only marked correct if the explanation made sense.

6. Graph 1 shows how *x* is related to *y* and Graph 2 shows how *y* is related to *z*.

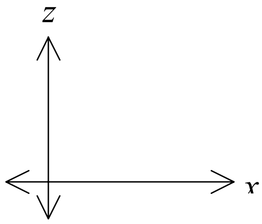


Graph 1



Graph 2

On the axes below, please draw the graph that shows how *x* is related to *z* based on the information from the two graphs above.



Please explain how you worked out your answer.

Figure 36. Task 6 - Relating graphs.

Students' correct explanations or solutions for Task 6 could be categorised into four strategies. Solutions coded Strategy 1 involved point-wise analysis and are reflective of a conception of function as a process. A typical example is shown in Figure 37.

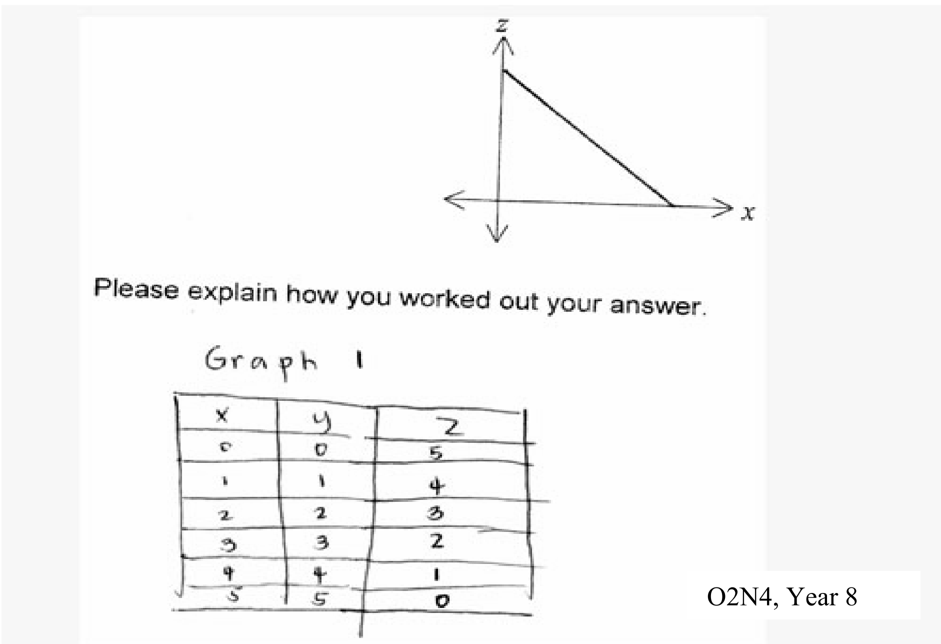


Figure 37. Task 6 - Strategy 1. Assigning specific points.

Solutions coded Strategy 2 involved a more holistic interpretation of the graph. It is clear in this strategy that students could interpret the relationship between the variables involved. However, they used words rather than symbols. The use of words may mean that they are not yet confident in using mathematical symbols or they know there is no need for symbols. In some cases, the reasoning reflects use of transitive property in equations. For example, students reasoned that since y is directly related to x and z is not directly related to y then z must not also be directly related to x. Some also apply their knowledge of proportion. A sample of this type of reasoning is shown in Figure 38.

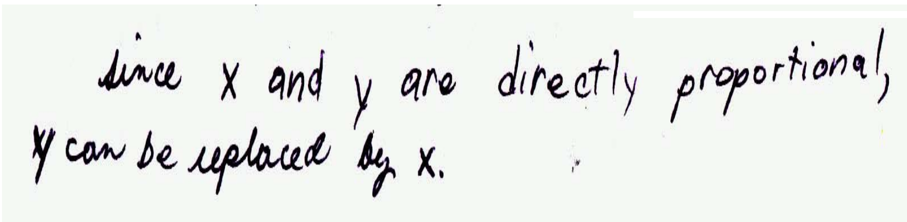


Figure 38. Task 6 - Strategy 2. Reasoning in terms of relationships (in words).

Other solutions in Task 6, which reflect holistic interpretations of the graphs, involve the use of symbols to explain their thinking. Those who assigned specific equations were coded Strategy 3 and those who used symbols for direct and inverse relationships were coded Strategy 4. An example of solution coded Strategy 3 is shown in Figure 39.

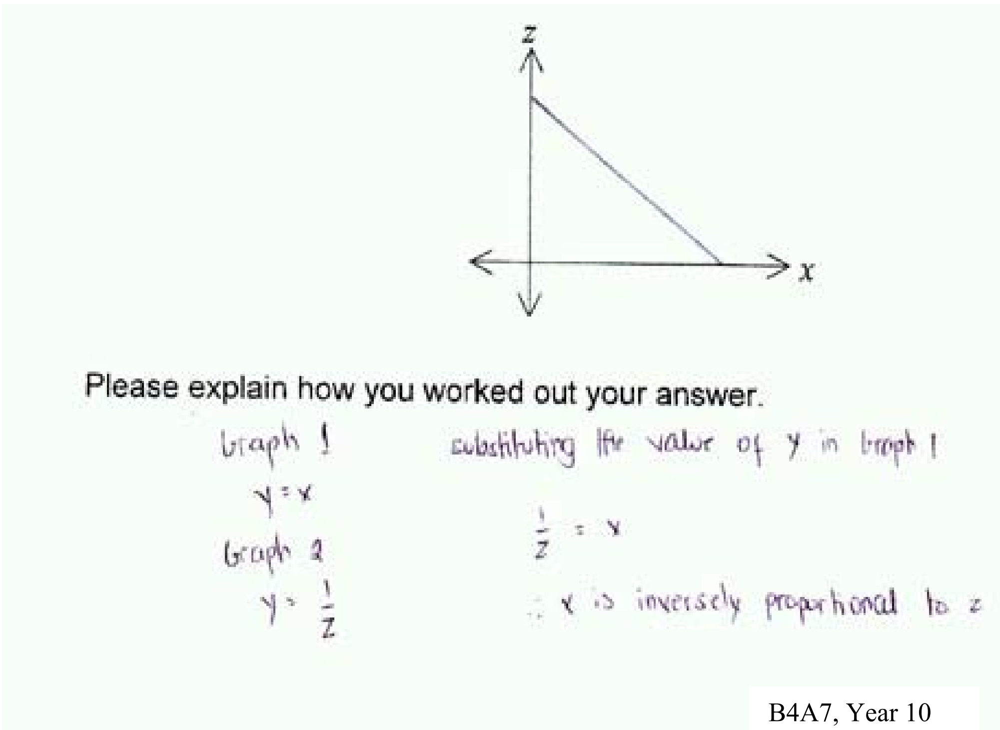


Figure 39. Task 6 - Strategy 3. Assigning specific equations.

Solutions coded Strategy 4 involved more general thinking than that coded Strategy 3. Students coded Strategy 4, demonstrated confidence in using symbols for proportional relationships. An example of solutions coded Strategy 4 is shown in Figure 40.

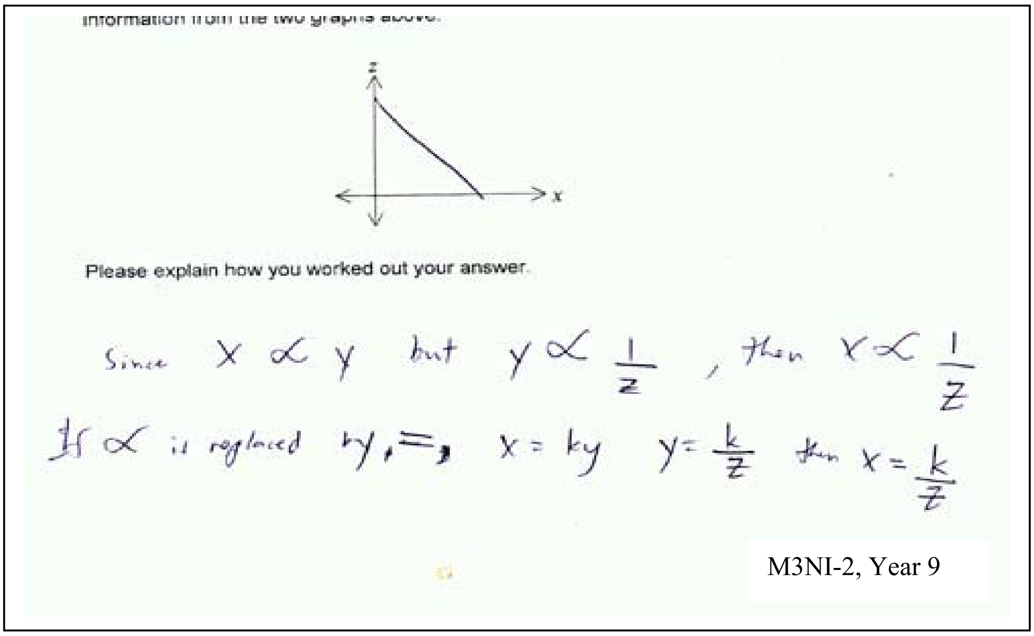


Figure 40. Task 6 - Strategy 4. Reasoning in terms of relationships (in symbols).

The distribution of students coded for the different strategies for Task 6 is shown in Table 17. D1 stands for the first data collection period and D2 stands for the second data collection period.

Table 17
Distributions of Students at the Different Strategies for Task 6

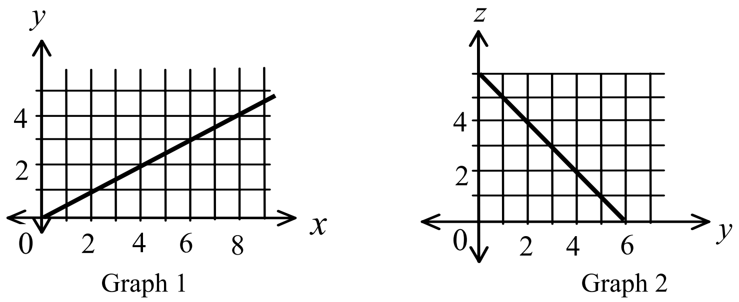
Tasks	Year 8 (n = 149)		Year 9 (n = 152)		Year 10 (n = 143)	
	D1	D2	D1	D2	D1	D2
Strategy 1: Point-wise	0	2	0	1	1	11
Strategy 2: Words	8	0	15	26	33	41
Strategy 3: Equations	0	2	1	1	5	5
Strategy 4: Symbols	0	0	0	1	6	14
Total	8	4	16	28	45	71

Results in Table 17 show that the students' most frequently used strategy was Strategy 2, which involved holistic interpretation of the graph but using words to explain their thinking. This highlights the importance of the use of words or verbal descriptions in describing relationships in the teaching of function.

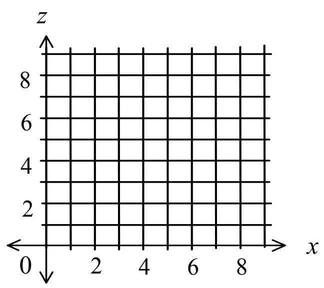
The pilot tests of Task 6 showed that only very few students in Year 8 and Year 9 could answer Task 6. Task 6.1 (see Figure 41) was then designed to assess if students could answer the same task with the graphs on grids. This means that they could make a point-by-point interpretation of the graph.

It was initially assumed that students would find Task 6.1 easier because the graphs were on numbered grids and they can do a point-by-point analysis, which was the most frequently used strategy in almost all the tasks. However, this proved otherwise. This may be because students have to coordinate the individual values of the three variables. Coordinating individual values seemed an added constraint. Studies have shown that students have difficulty coordinating information relating to two variables and two axes (Bell, Brekke & Swan, 1987). Task 6.1, which involves more than two variables and three axes, must indeed be difficult for many.

6.1 Graph 1 shows how x is related to y and Graph 2 shows how y is related to z.



On the axes below, please draw the graph that shows how x is related to z based on the information from the two graphs above.



Please explain how you obtained your answer.

Figure 41. Task 6.1 - Relating graphs.

Students' strategies in Task 6.1 were classified in two: Point-by–point interpretations and reasoning in terms of invariant properties of the linear relationship. Solutions involving point-by-point interpretations were coded Strategy 1. There were two types of solutions categorised Strategy 1. The first involved obtaining the equations of the given graphs and then working out the equation for *x* versus *z*, then generating pairs of values to plot the graph (see Figure 42).

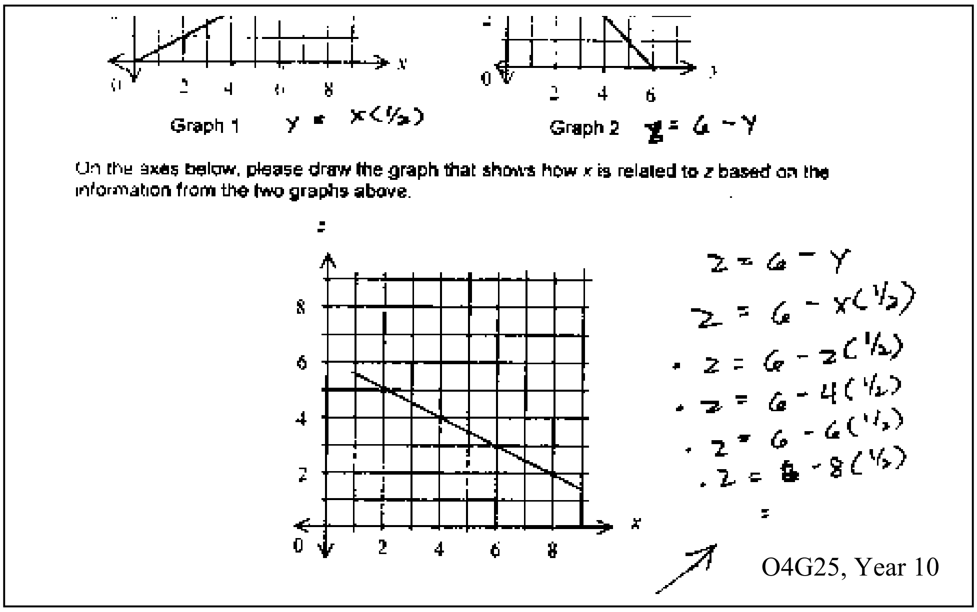


Figure 42. Task 6.1 - Strategy 1. Point-by-point analysis via equations.

The second type of solution, also coded Strategy 1, involved more direct interpretations of the individual pairs of values (see Figure 43).

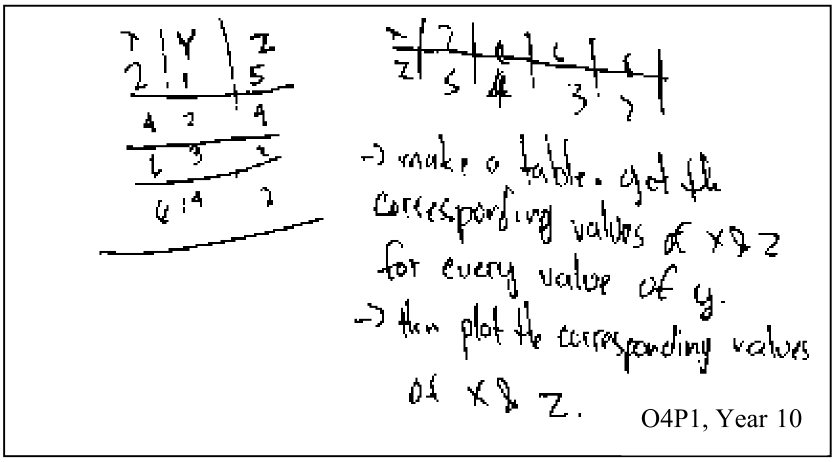


Figure 43. Task 6.1 - Strategy 1. Point-by-point analysis.

It was not very clear in the solution in Figure 42 that the student could interpret the graphs. They interpreted the relationship of the three variables in the equation rather than in the graphs. This may be due to the almost exclusive use of equations to analyse functions in many mathematics classes. Hence, although both strategies in Figures 42 and 43 were coded Strategy 1, the solution in Figure 43 is considered more elegant than the one in Figure 42.

A more elegant solution than the one in Figure 43 is shown in Figure 44.This solution used knowledge of the invariant property of linear relationships. This solution was coded Strategy 2.

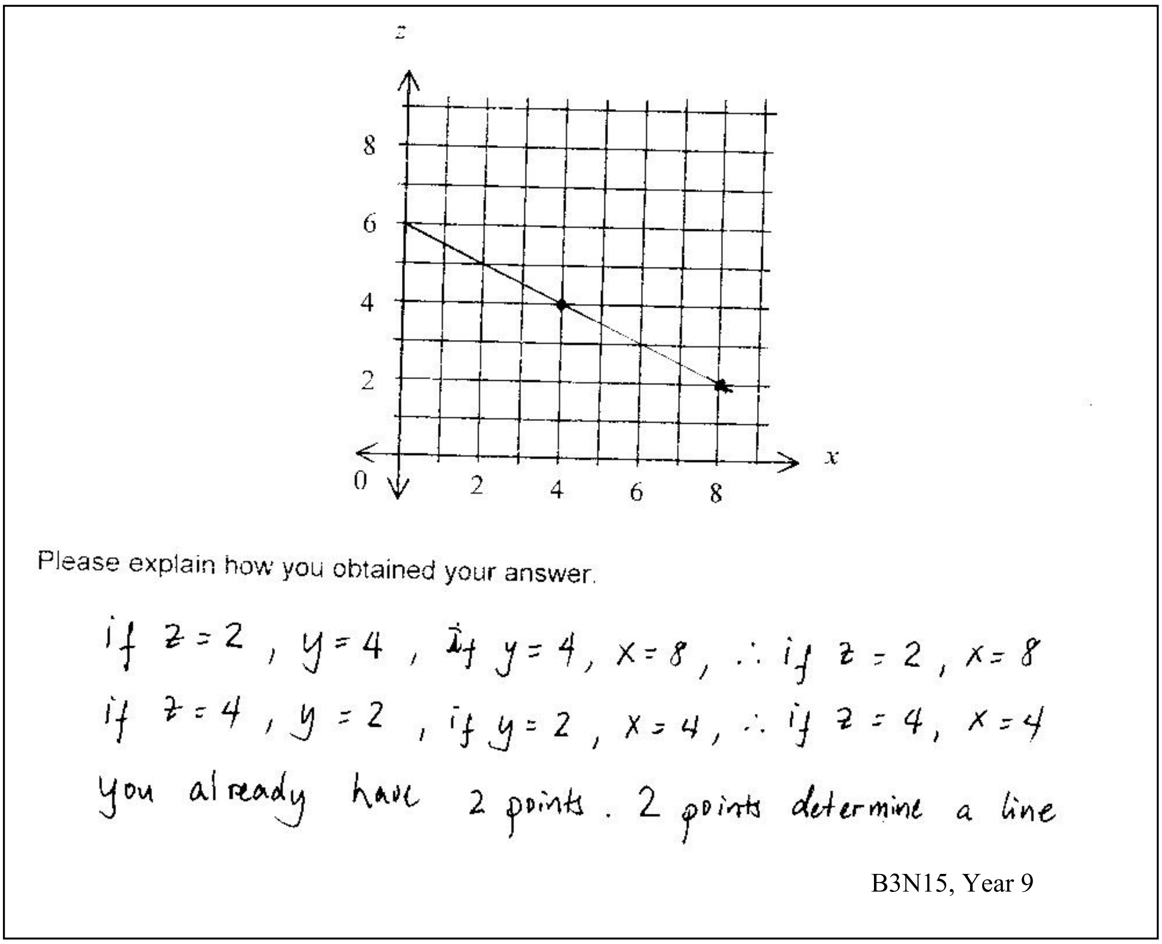


Figure 44. Task 6.1 - Strategy 2. Use of invariant property of linear function.

Table 18 shows the distribution of students at Strategy 1 and 2. Data show that Task 6.1 was a difficult task even to Year 10 students. Students who completed the task used point-by-point interpretations.

Only one student reasoned in terms of the invariant properties of linear relationship. This is cause for alarm. Almost all the Year 10 classes who participated in the study were doing analytic geometry, where they find equations of lines from two points and surely making sketches of lines, yet the students who were able to complete Task 6.1 still solved the problem using the strategies shown in Figure 42 and 43.

Table 18
Distributions of Students at the Different Strategies for Task 6.1

Tasks	Year 8 (n = 149)		Year 9 (n = 152)		Year 10 (n = 143)	
	D1	D2	D1	D2	D1	D2
Strategy 1: Point-wise	0	7	3	9	17	25
Strategy 2: Property	0	0	0	1	0	0
Incorrect	149	142	149	142	126	118

The second part of the chapter describes the procedure of analysis and results of the investigation about the order of the growth points.

Investigating for Typical Learning Trajectory

As stated earlier, the framework of growth points that the present study developed also aimed to describe a typical learning trajectory in students' developing understanding of key domains of the function concepts; hence the order of acquisition of the growth points for majority of the students was investigated.

The first part of this section presents the data in terms of success rates in each task under Graphs. The success rates already provided an overview of the trend in the growth points. The second part of this section presents and analyses empirical evidence in relation to the order of the growth points.

Success Rates

Results showed that tasks requiring point-wise analysis (generally associated with Growth Point 1) had higher success rates, and tasks requiring working with the representations as objects (associated with higher growth points) have lower success rates. Table 19 shows the success rate in the tasks used to assess growth points under Graphs. D1 stands for data collection 1 and D2 stands for data collection 2. The same students took the test in both data collections.

The first two tasks, Task 1 and Task 2 both involve interpretations of individual points and were used to assess Growth Point 1. Task 6 and Task 6.1, which were designed to assess the growth point on holistic interpretation of graphs, had the lowest success rates.

Table 19
Success Rates on Tasks Assessing the Growth Points under Graphs (%)

Tasks	Year 8 (*n* = 149)		Year 9 (*n* = 152)		Year 10 (*n* = 143)	
	D1	D2	D1	D2	D1	D2
T1: Reading off values	87.9	91.3	88.2	96.1	90.2	95.8
T2: Amount of change	87.9	91.3	92.1	96.1	91.6	97.2
T3a: Fastest growth	88.6	88.6	88.8	89.5	87.4	90.2
T4: Interpreting intersections	57.7	72.5	69.1	73.7	63.6	74.8
T3b: Slowest growth	35.5	39.6	44.1	54.6	42.7	49.7
T6: Relating graphs (no grid)	5.4	9.4	10.5	19.1	31.5	50.3
T5: Interval and continuity	4.0	13.4	8.6	13.2	10.5	15.4
T6.1: Relating graphs (with grid)	0.0	4.7	2.0	6.6	11.9	18.2

The data in Table 19 also indicated that tasks found difficult or easy in the first data collection were still found difficult or easy in the second data collection. That is, the ranking of the tasks was almost the same in both data collection periods. This is also true across year levels. That is, tasks found easy or difficult for one year level were also found the same by students in other year levels except for Year 10 who found Task 6.1 easier than Task 5. The consistency in the degree of difficulty showed that the assessment tasks for Graphs are reliable. The data also showed increasing success rates from Year 8 to Year 10 for each task, which is to be expected since students in higher year levels have more experience with mathematics and function in particular, than students in the lower year level.

In general, the ranking of the tasks in terms of success rates is consistent in both data collection period and in all year levels. Students' success rates for all the tasks increased in the second data collection. The success rates were also increasing from Year 8 to Year 9 to Year 10.

Investigating the Order of the Growth Points

Data that confirmed the order of some of the growth points, which were established theoretically, and the data that informed the position of other growth points, which were not very clear yet at the beginning of the study, are presented and analysed in the following discussion. The procedure in coding for the growth points described earlier is summarised in Table 20. GP 1 refers to Growth Point 1; GP 2 refers to Growth Point 2 and so on.

Table 20
Procedures for Coding the Growth Points under Graphs

Growth Points	Coding Procedure
GP 1: Interpretations based on individual points	Any two of T1, T2, and T4
GP 2a: Interpretations based on rates	T3a + T3b
GP 2b: Interpretations based on continuous property	T5
GP 3: Constructions based on holistic analysis of relationship	T6 + T6.1

Theoretically, GP 1, which describes understanding of a graph as a set of individual points, is the first among the other four growth points. GP 4 is also expected to be the last growth point because it is more on conceiving relationships represented by graphs as objects representing unique relationship between the variables. The position of Growth Points 2a and 2b was hard to predict since no previous study was found which looked into the relationship between these concepts. Positions of Growth Points 2a and 2b were established empirically.

The coding procedures presented in Table 20 show that distinct sets of tasks were used to assess the growth points. Thus, it is possible to show the trend in the growth points by determining the percentage of students coded at the growth points. The assumption is that the growth point with the highest percentage of students coded indicates that it would be the first growth point that students are likely to acquire first in their developing understanding of function in Graphs, since the majority of them are able to work in terms of it. Of course, this is not necessarily the case for every student.

Table 21 shows the percentage of students who were coded at each of the growth points. A growth point zero (GP 0) is added in the table to also show the percentage of students who were not coded at any of the growth points. Since the tasks assessing GP 1 were the easiest of the tasks under Graphs, those in GP 0 were those who were not able to meet the condition for GP 1.

Results in Table 21 show that the percentage of students coded at the growth points is decreasing from GP 1 to GP 3. This trend is true for all year levels and for both data collection periods.

Table 21
Percentages of Students in each of the Growth Points under Graphs

Growth Points	Year 8 (*n* = 149)		Year 9 (*n* = 152)		Year 10 (*n* = 143)	
	D1	D2	D1	D2	D1	D2
GP 0	1.3	1.3	1.3	0.0	2.1	0.7
GP 1	98.7	98.7	98.7	100.0	97.9	99.3
GP 2a	32.2	39.6	38.2	50.0	37.8	46.2
GP 2b	4.0	14.1	8.6	13.2	11.2	16.1
GP 3	0.0	0.7	1.3	4.6	6.3	10.5

The large percentage of students coded at GP 1 indicates that an understanding of graphs described in GP 1 was accessible to almost everyone. This implies that most of the students, even those in Year 8, have an understanding of graphs described in GP 1 at the beginning of the school year. This further implies that of the growth points identified in this domain, GP 1 is the initial stepping-stone towards an understanding of function represented in graphs for most students.

GP 2a had the second largest frequency and therefore may be considered as the next growth point to GP 1. The number of students coded at GP 2b was much smaller than GP 2a, indicating that GP 2b comes much later than GP 2. Results also showed that only ten percent of Year 10 students were coded GP 3. This figure is even lower in Year 9 and in Year 8 where only one out of 149 students was coded GP 4. This shows that understanding of function at this level must be very difficult to achieve.

Other Observations

Other observations that could be gleaned from the data are discussed in the following sections. The data in Table 21 are shown in visual form in Figure 45 to show clearly the difference in the percentage of students coded at the growth points. In the figure, D1 stands for data from the first data collection and D2 stands for data from the second data collection.

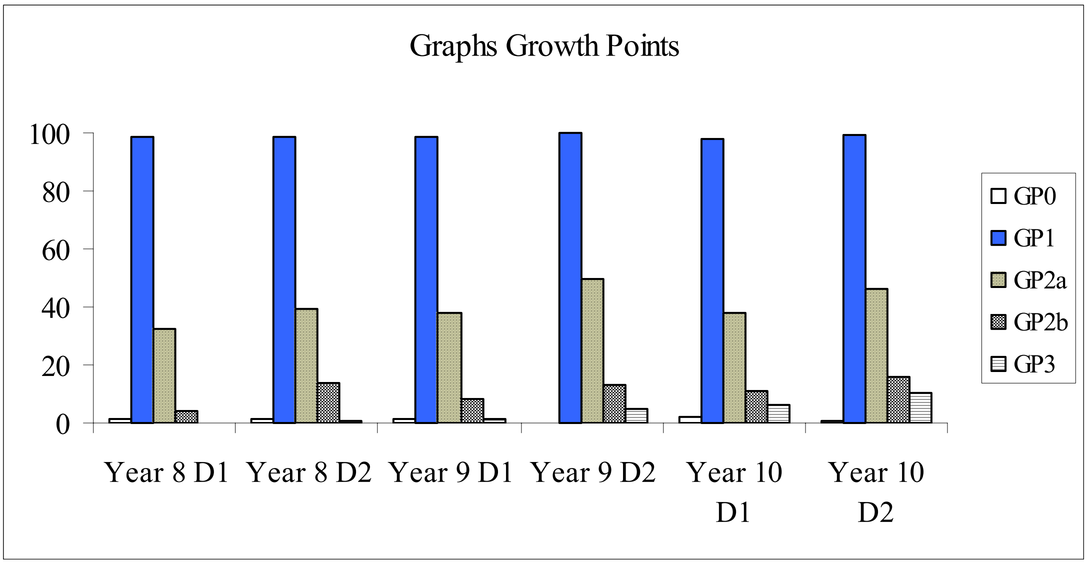


Figure 45. Percentages of students at each growth point under Graphs.

The bar graph in Figure 45 shows a big difference in the percentage between growth points within the same year level in both data collection periods. In Year 9 for example, while all the students were coded GP 1, only half of them were coded GP 2a and again less than a quarter were coded GP 2b and only five percent GP 3. This implies that it is a big jump for the majority of the students to go from one growth point to the next. This also provide evidence that the growth points identified in the Framework indeed represent "big ideas" or major nodes in the network in understanding function represented by graphs.

Another notable observation one could make from the data is on the percentage of students in the growth points across year levels. Data show that there is not much difference in the percentage of students at each growth point between year levels as compared to the data of the growth points under Equations (see Table 14 and Figure 32 in Chapter 4). While it may not be surprising that almost all the students in all year levels are coded GP 1, since this could be interpreted as GP 1 being understood even in Year 8, having almost equal percentages in the other growth points is a cause of concern. This could mean that students have almost the same level of understanding of these aspects of graphs regardless of their mathematical experiences or that the mathematical experiences provided in schools do not contribute to the understanding of these aspects of graphs. This is especially true in GP 2b and to some extent in GP 2a. The number of students achieving GP 2b for all year levels during the second data collection period only differed by less than three students. One could argue of course that maybe these groups are not comparable. However, this was not the case with Equations where there was a marked difference in the performance in each year level with the Year 10 outperforming the Year 9 and the Year 9 outperforming the Year 8 in all the growth points. Hence, a plausible reason would be that interpretations of graphs involving continuous property or in general, the interpretations of domains and range of the function represented by graphs are not given emphasis in the teaching, with the teacher perhaps assuming that students would understand these concepts by themselves in the course of working with graphs. These findings clearly point to a need to revisit the mathematics syllabus and teaching practices the students in this study are exposed to. In addition, further investigation should be made to determine whether this difficulty is indeed *didactically-related* or *epistemologically-related* (Artigue, 1992).

The next level of analysis investigated the relationship between the growth points, especially on the extent to which achievement of one growth point predicts achievement of lower growth points. To investigate these, comparisons in the frequency of students who have achieved GP 1 only, GP 1 and 2a only, GPs 1, 2a, and 2b only, and so on were made. This is shown in Table 22.

Table 22
Frequencies of Students at the Growth Points under Graphs

Growth Points	Year 8		Year 9		Year 10	
	D1	D2	D1	D2	D1	D2
GP 0	2	2	2	0	3	1
GP 1	97	81	86	71	75	60
GP [1, 2a]	44	45	49	56	41	47
GP [1, 2a, 2b]	2	13	9	14	6	10
GP [1, 2a, 2b, 3]	0	1	0	2	1	2
GP [1, 2b]	4	7	4	4	8	10
GP [1, 3]	0	0	2	1	3	6
GP [1, 2a, 3]	0	0	0	4	5	7
GP [2b]	0	0	0	0	1	0
Total	149	149	152	152	143	143

In the table, GP 0 means students coded Growth Point 0 or GP 0. GP 1 means students coded Growth Point 1 only. GP [1, 2a] means students coded GP 1 and GP 2a. Likewise, GP [1, 2a, 2b] means students coded GP 1, GP 2a and GP 2b and GP [1, 2a, 2b, 3] means students coded GP 1, GP 2a, GP 2b and GP 3. The shaded area means it did not fit the "cumulative" pattern of the growth points just listed. For example, GP [1, 2b] means students coded in GP 1 and GP 2b (missed GP 2a).

The data in Table 22 show there was only one student who was not coded GP 1 but coded at higher growth point (see GP [2b]). This implies that GP 1 is a prerequisite growth point for all the other growth points in this domain. This further indicates that if a student could work in terms of GP 2a, or any of the higher growth points, they could work in terms of GP 1.

When a student is coded GP 2b, does this mean that he or she could also work in terms of GP 2a as well? Data in rows GP [1, 2a, 2b] and GP [1, 2a, 2b, 3] in Table 22 show the number of students who were coded GP 2b as well as GP 2a. Data in row GP [1, 2b] show the number of students coded GP 2b but not GP 2a. There were only about 15% of the students in each level who were coded GP 2b (see Table 21). The figure is even lower in Year 8 at the beginning of the school year where 6 out of 149 or about four percent were coded at this growth point, although the figure rose to the same level as Years 9 and 10. Estimating the extent to which understanding in terms of GP 2b predicts GP 2a is based on these limited data and therefore may be inconclusive. Using this limited data, it could be said that understanding in GP 2b would not completely guarantee understanding in GP 2a, which could indeed be the case since they are not conceptually related. GP 2a involves rate while GP 2b involves continuous property.

Rows GP [1, 2a, 2b, 3], GP [1, 2a, 3], and GP [1, 3] in Table 22 show the frequency of students coded GP 3. Of these rows, GP [1, 2a, 3] had the highest frequency. It seems that students coded in GP 3 are more likely to work in terms of GP 2a but not GP 2b as well. Also, the number of students in GP [1, 3] is higher than those in GP [1, 2a, 2b, 3], especially in Year 10 during the second data collection period, which could be an indication that understanding in GP 3 does not necessarily build on understanding in terms of GP 2a and GP 2b.

These results do not necessarily contradict the trend established earlier that students' understanding typically goes from GP 1 to GP 2a to GP 2b to GP 3. However, it provides a clear picture of the relationships among the growth points especially on the likelihood that achievement of higher growth points guarantees achievement in lower growth points.

The data in the Table 22 also showed that a large majority of the students were coded at GP 1 *only*. In Year 10 for example, 75 out of 143 (or about half of the students) were in Growth Point 1 in the first data collection period. Although this percentage decreased to about 40% in the second data collection period, this is still very high considering that they were Year 10 students.

Equations and Graphs

Task 9, which involves composition of function, and Task 6 (see Figure 36), which assesses an understanding of function in terms of holistic interpretations of the relationships, represented by equations and graphs respectively were compared. Task 9 is restated below:

> The relation of s with p is shown in the equation $s = 5p + 3$. The relation of p with n is shown in the equation $2p = 6n$. From this information, please write the equation that will show the relation of s with n.

These two tasks are comparable in terms of structure and content. The tasks involve linear relationship and performing operation on the representations. The results from the second data collection in Year 9 and Year 10 were used to compare the achievement in these tasks because the success rate was greater at this period than in the first data collection period. There was 156 out of 295 Year 9 and Year 10 students who completed Task 9 correctly while there were only 99 out of the 295 who completed Task 6 correctly. This indicates that it is easier for students to interpret the relationship or the function represented in equations than in graphs. This result is consistent with the finding of Bell, Brekke and Swan (1987) that students find it hard to conceive the graph as a representation of two varying quantities. That it is easy also for students to conceive the equation as a representation of relationships between two quantities may be because students' experiences with functions were more in equation form than graphical form.

GP 3 under Graphs was further compared with GP 4 under Equations, which describes an understanding of function represented by equations as objects that can be manipulated or transformed. There were 23 students coded at GP 3 under Graphs and 22 students were coded at GP 4 under Equations. However, only four of these students were coded at these growth points in both domains. This indicates that there seem to be no relations between the two domains as far as the highest growth point is concerned. This indicates that the achievement of one cannot be used to predict the achievement of the other. However, this should be investigated further since 23 and 22 are small numbers compared to the total, which is 444 students.

Discussion and Summary

This chapter presented the growth points under Graphs and the tasks assessing the growth points. As mentioned earlier, the identification of the growth points and the tasks assessing the growth points informed each other.

The decreasing percentage of students from Growth Point 1 to Growth Point 3 confirmed that the order of the growth points was indeed as claimed. This order of the growth points was the same for both data collections, which were about five months apart, the first given at the beginning of the school year. The order is also consistent for all the year levels.

There were only two growth points exclusively under Graphs in the initial framework: the first involved point-by-point interpretations and the second involved interpretations based on local properties. Because of the large differences in the success rates in tasks assessing local properties, the growth point involving properties was split into two. One of the growth point involved rates and the other, continuous property. These two properties are also not conceptually related; hence they were treated as separate growth points. A fourth growth point was added since there were students who could work on the function represented by graphs holistically which could be considered more reflective of conceiving function as an object than as a procedure or process.

The four growth points identified under Graphs were:

Growth Point 1. Interpretations based on individual points.

Growth Point 2a. Interpretations based on rate

Growth Point 2b. Interpretations based on continuity

Growth Point 3. Interpretations based on holistic analysis of relationships

Growth Point 1 which involves interpretations based on individual points and Growth Point 4 which involves interpretation and construction of graphs holistically are reflective of process and object conceptions of functions respectively, while Growth Points 2a and 2b are ideas involving local properties of functions and graphs.

The students' acquisition of the growth points proceeds in general from Growth Point 1 to Growth Point 3. There were of course other students whose understanding of the growth points would not follow this trend, but for the majority, the trend holds.

Growth Points 2a and 2b both describe understanding of some local properties of function, rate and continuity, respectively. These are not related concepts, which explains why achievement of Growth Point 2b does not necessarily entail achieving Growth Point 2a.

Students had difficulty with Growth Point 2b. Some teachers were asked why students found interpretation of continuity presented in Task 5 difficult. The reason provided was that maybe the tasks involve a two-part answer. However, analysis of the distribution of responses of Year 8 students showed that this was not the case. The result of the analysis of students' responses indicated that it was not because the task was open-ended. In fact, the teachers I had conversation with regarding Task 5 expected their students to answer the task correctly.

The majority of the students were not able to interpret the endpoints of the intervals in Task 5 correctly. It may be that this was not given attention in teaching. Teachers may have made the assumption that this aspect of the graph is obvious, something students would naturally learn as they work with graphs. It is therefore recommended that more learning experiences should be provided which involve the concept of continuity, interpolation and extrapolation. It is also possible that the task, which assessed this growth point, was ambiguous. Firstly, the context used involves age, which for some students would only involve whole numbers. Secondly, the study framed the question *When was Frank taller than Gina*?. Leinhardt, Zaslavsky and Stein (1990) reported that the word "when" creates ambiguity. Students may have thought that they would be given credit even with partially correct answer only.

Students had little difficulty in identifying the interval showing the fastest growth, but a large majority of these students had difficulty identifying the interval showing the slowest growth rate. It appears that interpretation of fastest growth and slowest growth in graphs represent "big ideas" in students' understanding of this property. It is therefore suggested that there should be more study that would identify growth points of students' understanding of rate not only under equations but with graphs as well.

It was no surprise that Growth Point 3 would be the hardest to achieve among the four growth points. In fact, the majority of the students coded at this growth point were Year 10 students. This growth point not only requires holistic interpretation of the function represented by graphs but also construction of graphs. It requires mental flexibility in conceiving the function represented by the graph as a process and mental object. It corresponds to Growth Point 4 under Equations. This *flexibility* in thinking of function as a process and as a mental object should be the aim of instruction (Moschovich, Schoenfeld, & Arcavi, 1993).

The growth points under Graphs provided a picture of a typical learning trajectory of students' understanding of function in graphs. There are surely other growth points that could be identified in students' understanding of function represented by graphs apart from the four growth points described in the Framework. For example, there is a large difference between the number of students in GP 1 and GP 3. Both these two growth points are along the process-object route. Hence, growth points between them could be identified. Sierpinska (1992) argued that overcoming epistemological obstacles is an act of understanding, so growth points in students' understanding may be identified in the process of overcoming obstacles in the understanding of graphs. She identified epistemological obstacles in the understanding of function in graphical form. An example of this epistemological obstacle is that the graph as a geometrical model of functional relationship "need not be faithful. It may contain points (x, y) such that the function is not defined in x" (p. 52). Schwarz and Dreyfus (1995) have also identified ambiguities inherent in the graphical representation of function such as it being only a partial representation of function. Addressing these ambiguities may result in the identification of "big ideas" in students' understanding of function.

Since students are more likely to interpret graphs visually, learning experiences should be provided so that students would not consider only one aspect of the graph in the interpretations of rate. For example, as shown by the result of the study, the students only consider the levelling of the graph to determine the interval which shows slowing growth rate when in fact they should also consider the length of the interval.

The next two chapters, Chapter 6 and 7, present the growth points in students' understanding of function, which involve linking representations.

CHAPTER 6

DOMAIN 3 - LINKING REPRESENTATIONS

To understand function is to understand that the properties of the function are not unique to the representation but to the concept itself. This means that the different representations, graphs, equations and tables should not be understood as separate entities but as one entity representing a single object — the function (see, e.g., Schwarz & Dreyfus, 1995; Sierpinska, 1992). Thus, to be able to link the different representations of function is probably the most important node in the network of students' understanding of this concept.

The understanding of the link between the different representations of function was initially a growth point in students' understanding of function under each of the domains, Graphs, Equations and Tables. However, the richness and range of students' strategies reflected different levels of understanding and this indicated that linking representations should be considered as a key domain.

Linking representations requires more than translating from one representation to another. Students who can graph a given set of values in a table do not necessarily understand the link between these representations. In the present study for example, there were several instances where students were able to graph the table or equation correctly but failed to identify the correct graph matching the given table or equation from among the choices.

Linking representations requires higher-level skills in the sense that it goes beyond simple functional skills such as substituting a value into a formula, or reading coordinates in a graph or table. It requires understanding of the connection between the ways in which different representational systems exhibit the same properties of the concept of function. Thus, the inclusion of Linking Representations as a domain in the final framework of growth points enhanced the scope and depth of the framework in describing students' understanding of function.

Linking representations are of two types. The first involves linking between representational systems and the second involves linking within the same representational system (Kaput, 1989). The domain, Linking Representations, discussed in the present chapter, refers to linking between representational systems. The domain, Equivalent Relationships discussed in the next chapter used linking within the same representational system to assess students' understanding of equivalent functions.

There were five growth points identified under Linking Representations. The following discussion describes the growth points and the tasks used to assess them.

Assessing the Growth Points

The aim of the framework was to provide teachers with a map and direction towards which students were to be guided in their understanding of function. This direction is towards conceiving the function as a mathematical object. Tasks involving linking representations were designed so that they would draw out the most sophisticated strategy reflective of this level of understanding. Students at this level of understanding could reason in terms of invariant properties of the function or perform operations with functions as mathematical objects. These tasks were also so designed that a point-by-point analysis would be tedious but could still be used if students could not solve the tasks in any other way.

Of the three representations of function, graphs have the potential to make salient the nature of function as an entity (Schwartz & Yerushalmy, 1992). That is, graphs, especially when not on grids, encourage students to focus on the interpretations of relationship of the variables holistically rather than on the individual values. Because of this, four of the growth points under Linking Representations involve linking graphs with other representations.

Functions represented by equations could be interpreted as objects or by point-by-point interpretations, depending on students' level of knowledge of functional equations. Tables also naturally lend themselves towards point-wise interpretations. Because equations and tables lend themselves to point-by-point interpretations, it is no surprise that the first growth point in students' understanding of Linking Representations is about linking equations and tables. This is discussed below.

Growth Point 1: Linking equations and tables

This growth point requires students to work with tables of values in relations to equations and vice versa. Task 10.1, Task 10 and Task 11 in Figures' 12, 15 and 28 discussed in Chapter 4 were used to assess Growth Point 1. All three tasks are not simple tasks in the sense that they do not involve simple equations, and were so constructed not to be mere translation tasks. Task 10 and 10.1 involve quadratic relationships. In Task 10.1, a table of values and its corresponding equation are shown. A second equation, which is three less than the first equation is given. The students' task was to complete the corresponding table of values for the second equation. The given x values in the second table are the same as in the first table. Task 10 is similar to Task 10.1. In Task 10, a table of values and its corresponding equation are also shown. A second table is given showing the same x values as the first table but with y-values two more than the y values of the first. The students' task was to write the corresponding equation of the second table. Task 11 involves linear relationships, but it requires interpreting an inverse relationship in equations and tables. It was clear from the solutions of students in these tasks that they could not have answered these tasks correctly without knowing that the equations and tables are related. Hence, students were coded Growth Point 1 if they could get at least two correct answers from these three tasks using any strategy. Most of those coded at this growth points completed Task 10.1 and Task 11 correctly.

Growth Point 1 could conceivably be subdivided into two growth points. The first growth point would involve point-by-point strategy and the second would involve working with the representations as objects that could be manipulated or transformed holistically. These distinctions are not made any more because Growth Point 1 and Growth Point 4 under the Equations domain already assess these levels of understanding.

The other four growth points under Linking Representations domain involved linking graphs with other representations.

Tasks Assessing Growth Points 2, 3 and 4

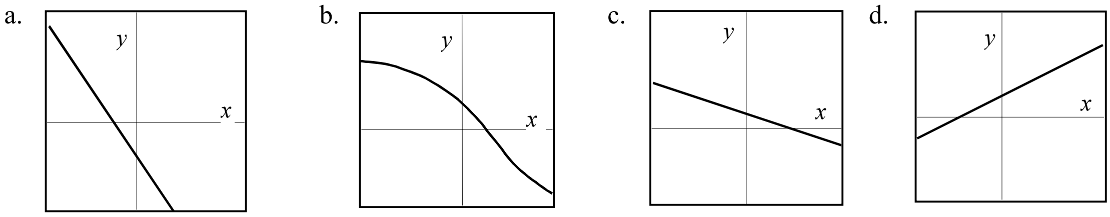
There were four tasks assessing Growth Points 2, 3, and 4. In all the four tasks, the graphs were not on grids, in order to encourage holistic interpretations. Two of the tasks involved linear relationships while the other two involved quadratic relationships.

In all the four tasks, students' strategies that led to the correct answer were coded Strategy 1 if they involved point-by-point interpretations, and coded Strategy 2 if the solution involved use of trends and patterns or if it involved the use of properties like slope and intercept. Solutions were coded Strategy 3 if they involved the use of invariant properties of the function to link graphs with other representations. These three strategies are arranged according to abstraction and generality involved. Strategy 1 shows evidence of conceiving function more as a process than a permanent construct because students coded as using this strategy still rely on individual points. Strategy 2 shows evidence of conceiving function still more of a process but shows understanding of the role of properties and Strategy 3 shows evidence of conceiving function as a permanent construct.

The students found Task 15 (see Figure 46) the easiest among the four tasks. Task 15 requires identifying the graph, which matches the given table. The function involved is a linear function.

15. Which graph can match the set of values in the given table? Please show/explain what you did to determine your answer.

x	-4	-2	0	2	4
y	10	8	6	4	2



Solution or Explanation:

Figure 46. Task 15 - Lines and tables.

The majority of the students used Strategy 1 or point-by-point analysis to complete this task. Figure 47 shows a typical solution coded Strategy 1.

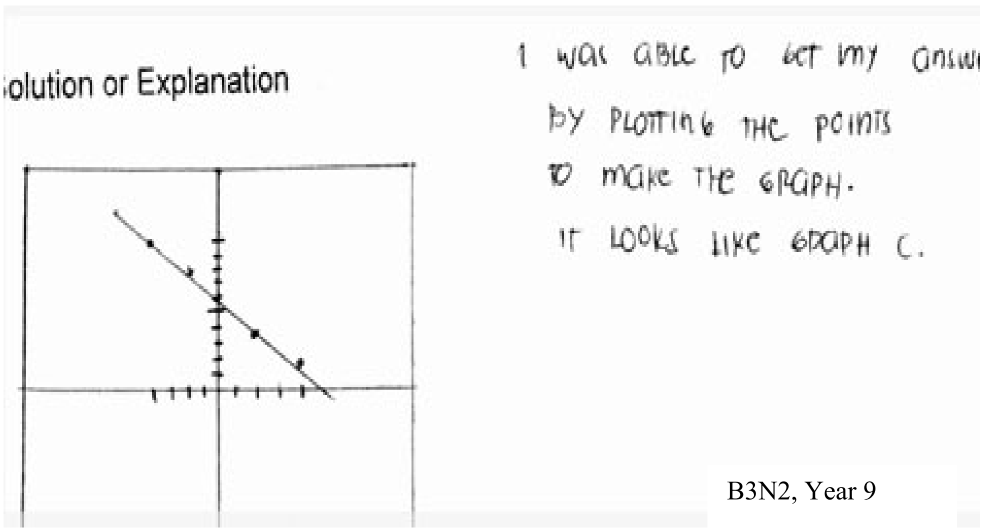


Figure 47. Task 15 - Strategy 1. Point-by-point interpretations.

Some students used the trend in the values of *x* and *y* explaining the table shows decreasing *y* values as *x* increases. There were also some students who used properties such as the intercept and the slope or the constant difference in the values to identify the graph matching the table of values. Those who used any of these reasoning and identified the correct graph as well were coded under Strategy 2.

Figure 48 shows a solution coded Strategy 2. Student B3K12 used individual points and slope to solve the problem. However some students coded Strategy 2 using trend or some properties did not indicate that they used other points. Students' solutions coded Strategy 2 are of course insufficient, but the fact that they were able to identify the correct graph shows they have some level of understanding. Maybe these students used point-wise analysis as well, like student B3K12, but did not acknowledge it.

A few solved the corresponding equation first and analysed the properties of the function from this before selecting the appropriate graph. This solution was also coded Strategy 2.

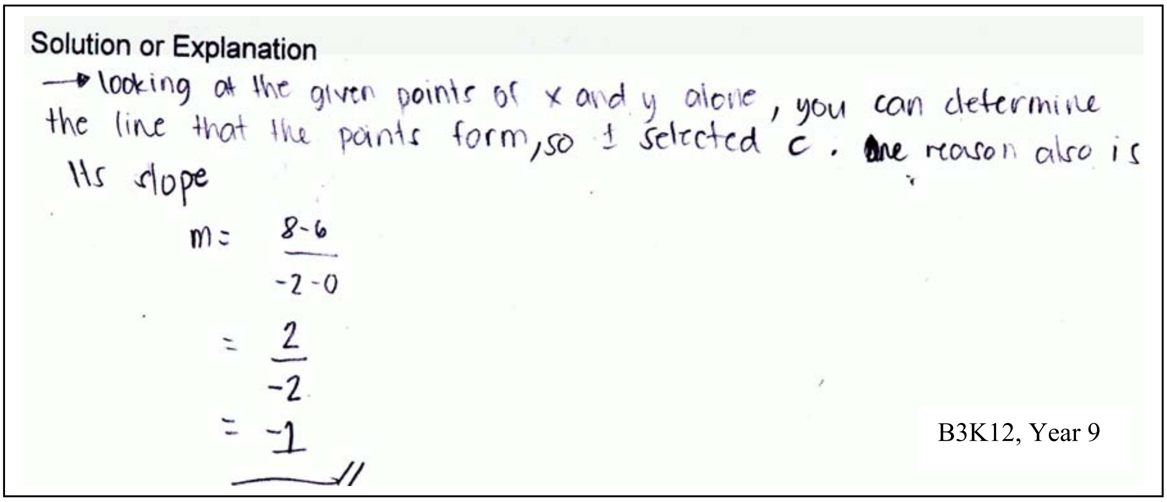


Figure 48. Task 15 - Strategy 2. Use of pattern, trend or some properties.

Those who used Strategy 3 were very few but among the tasks on Linking representations, Task 15 had a good number of students who reasoned in terms of the invariant properties of linear functions to work out the task. Figure 49 shows a sample of a student's reasoning coded Strategy 3. Student O4P1's solution in Figure 49 clearly demonstrated confidence in the knowledge that the *y*-intercept and slope are sufficient conditions to define linear function.

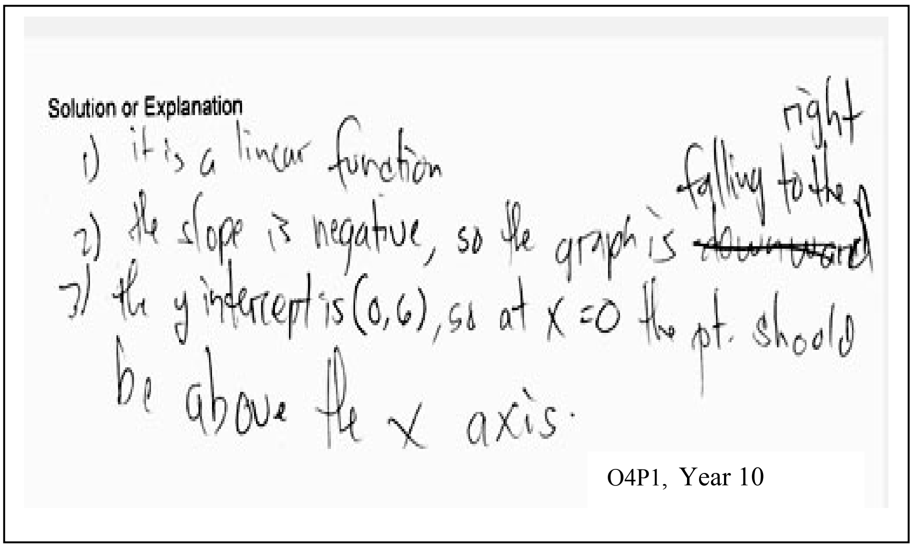


Figure 49. Task 15 - Strategy 3. Reasoning in terms of invariant properties.

The majority of the students' solutions involved point-by-point analysis. Table 23 shows the distribution of students in the different strategies. Students coded "No answer" were those who left the task unanswered. Those coded "No explanation" were those who selected the correct answer, but provided no explanation. All other incorrect responses were under the "Incorrect response" category. A more detailed distribution of incorrect responses is in Appendix C.

Table 23
Distributions of Students at the Different Responses for Task 15

Responses	Year 8		Year 9		Year 10	
	D1	D2	D1	D2	D1	D2
Strategy 1 – Point-wise	13	55	49	60	71	86
Strategy 2 – Some Properties	5	5	9	8	17	14
Strategy 3 –Invariant Properties	0	3	0	18	5	17
No answer	29	2	10	4	2	1
No explanation	10	3	1	2	2	1
Incorrect responses	92	81	83	60	46	24
Total	149	149	152	152	143	143

The distribution of students using each strategy shows that the majority favoured Strategy 1, followed by Strategy 2 then Strategy 3. Although during the second data collection there was more Year 9 and Year 10 students who were coded Strategy 3 than Strategy 2, there were more students coded at Strategy 2 than Strategy 3 in the first data collection period. The decreasing trend in the number of students from Strategy 1 to Strategy

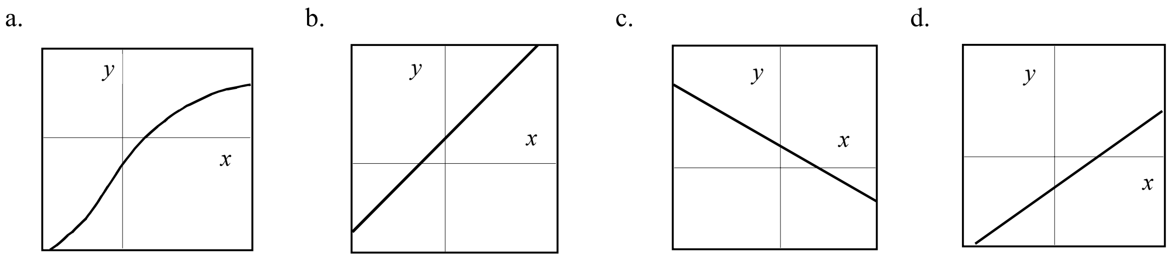
3 is an indication that initially students are more likely to reason point-wise, then start to conceive of the function holistically in terms of some of its properties, then eventually understand the properties that distinguish a function from another function which are the invariant properties.

Except for the Year 8s in the first data collection period, very few students left the task unanswered or provided no explanation for a correct answer, which indicates there were only very few instances where the basis for the coding was not clear.

Choices *b* and *d* were the most popular among the incorrect choices (see Appendix C for the distribution of incorrect responses). The works of Year 8 students who chose *b* and *d* in the second data collection were revisited to determine how students who have just been introduced to linear relations reasoned. Of the 29 students who chose *b*, all 29 plotted the points correctly. Of the 32 who chose *d*, 18 also did point-plotting techniques. These show that students who could plot points do not necessarily understand the connection between graphs and tables.

Task 14 (see Figure 50) involves linking linear equations and graphs. The given equation was in slope-intercept form already, so students would just interpret the slope and the intercept from the equation. The graphs were also not on grids to discourage students from doing point-by-point or *point-wise* analysis.

14. Which graph or graphs can be the graph of $y = 2x - 3$? Please explain how you obtained your answer(s).



Solution or explanation:

Figure 50. Task 14 - Lines and equations.

Samples of students' strategies in solving Task 14 are described below. A solution involving point-by-point analysis was coded Strategy 1. A sample solution is in Figure 51. Sometimes students did not show the plotting of the points but have shown calculations involving evaluating the given equation. These students were still coded Strategy 1.

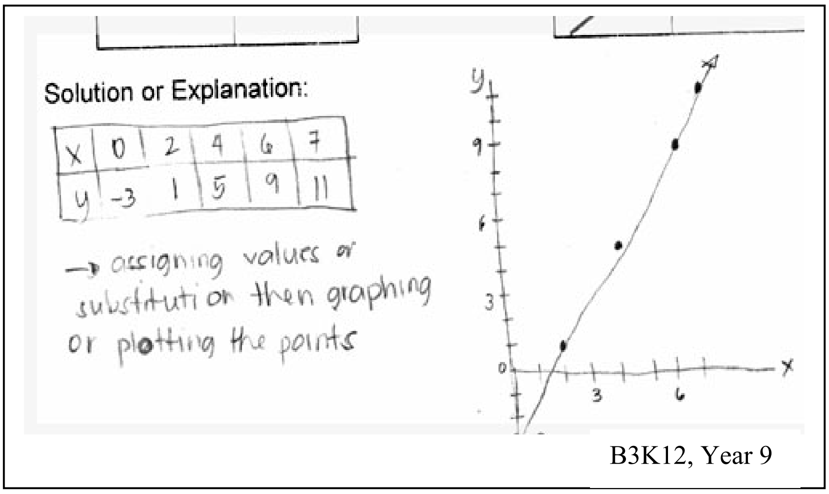


Figure 51. Task 14 - Strategy 1. Point-by-point interpretations.

Solutions involving use of trends and some properties were coded Strategy 2. A typical solution is in Figure 52.

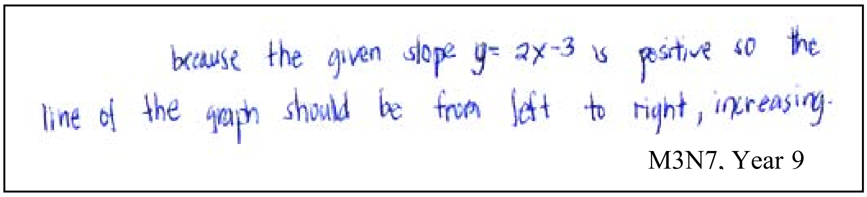


Figure 52. Task 14 - Strategy 2. Some properties.

There were also a few who reasoned in terms of the invariant properties of the linear function. The solution in Figure 53 used the idea of slope while the solution in Figure 54 used the idea of *x* and *y* intercepts.

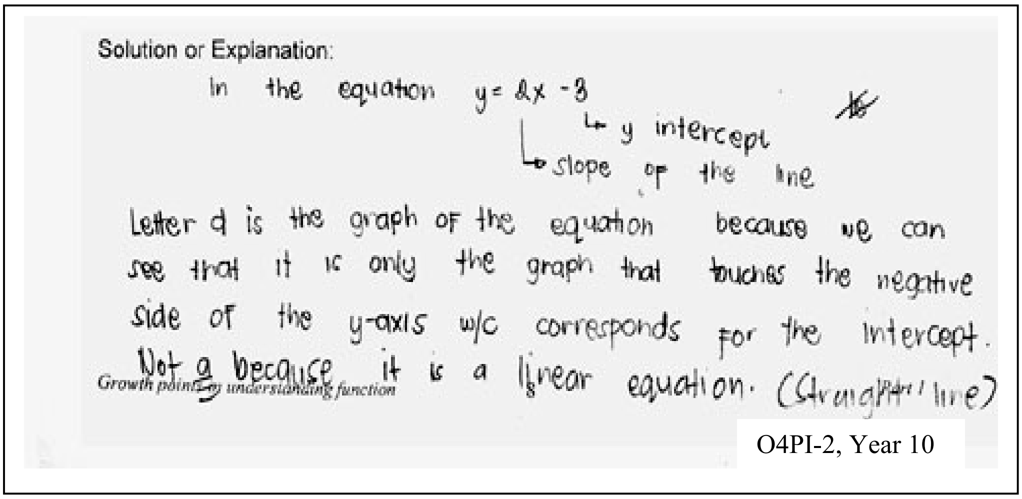
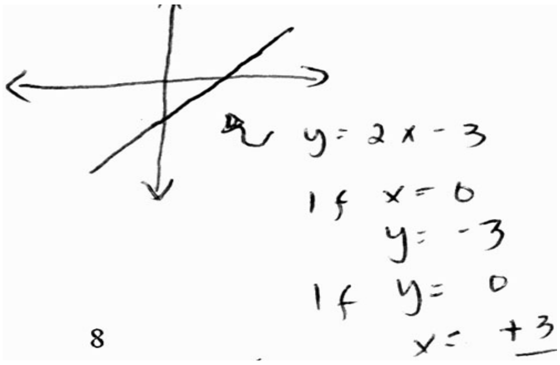


Figure 53. Task 14 - Strategy 3. Invariant properties: slope and intercept.



O2N4, Year 8

Figure 54. Task 14 - Strategy 3. Invariant properties: *x* and *y* intercepts.

While it may appear that students are using solutions that they had been taught, they are applying these methods on a task different from standard classroom or textbook tasks.

There were some students who chose the correct answer, *d*, but the explanation was that the *x* intercept is positive (some would write 2 in the intersection of the graph and the *x*-axis) and the *y* intercept is –3 so the answer is *d*. This is one of the reasons why students with a correct response who did not indicate how they solved the task were not considered in the coding. Table 24 shows the distribution of students coded for the different strategies. Results show students' preference for point-by-point analysis in completing the tasks.

Table 24
Distributions of Students at the Different Responses for Task 14

Responses	Year 8		Year 9		Year 10	
	D1	D2	D1	D2	D1	D2
Strategy 1 – Point-wise	3	25	11	15	54	73
Strategy 2 – Some Properties	0	4	5	12	5	6
Strategy 3 –Invariant Properties	1	2	3	16	6	7
No answer	62	24	41	21	12	4
No explanation	7	10	6	3	5	3
Incorrect responses	76	84	86	85	61	50
Total	149	149	152	152	143	143

Data in Table 24 also show that the Year 9 students performed better than Year 10 students did during the second data collection period. However, nine of the 12 Year 9 students coded Strategy 2 and ten of those coded Strategy 3 in Year 9 were from the same school and taught by the same teacher. When the second data collection period occurred, the

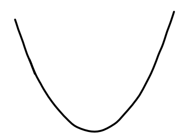
Year 9 classes in this school were just starting with quadratic functions so there is a possibility that the teacher may have conducted some review of linear relations.

As in Task 15, the majority of those who selected the correct answer in Task 14 used point-by-point interpretations.

There was a considerable number of students who left the task unanswered, especially in Year 8 and in Year 9. Most of the Year 8 students had not been introduced to linear functions when the first data collection period occurred so it is understandable that they would leave the task unanswered. The reason why the Year 9s left the task unanswered could not be clearly determined although there is a possibility that because the graphs were not on grids, they may have viewed the task as difficult, something unfamiliar to them. There is also the possibility that because Task 14 was the last task in Booklet 1, students may have been tired when they got to the last task. However, results for Task 12 show that almost the same number of Year 9s who left Task 14 unanswered also left Task 12 unanswered (see Table 25). So the reason that a considerable number of Year 8 and Year 9 students did not answer Task 14 could not be attributed entirely to it being the last task in the booklet. These students may have also perceived the task difficult and hence did not try to answer it.

The third task assessing linking representations is Task 12. Task 12 (see Figure 55) requires students to identify the table matching a given parabola.

12. Which table of values can be part of the graph on the right? Circle the letter corresponding to your choice and explain what you did to get your answer.



a.

x	1	2	3	4	5
y	0	8	12	10	4

b.

x	1	2	3	4	5
y	1	2	8	16	32

c.

x	1	2	3	4	5
y	0	2	4	6	8

d.

x	1	2	3	4	5
y	8	5	3	2	2

Explanation:

Figure 55. Task 12 - Curves and tables.

The parabola was used instead of a general curve in the tasks so that it could as well assess whether students would use their knowledge of the properties of quadratic relationships. However, the majority of the students solved this task by point-by-point analysis. Some used the patterns or trend from the table of values, and a few used the second constant difference property of quadratic relationship to justify their choice. These strategies were coded Strategy 1, 2, and 3 respectively.

Strategy 1 involves point-by point interpretations through plotting each pair of values. A typical solution is shown in Figure 56.

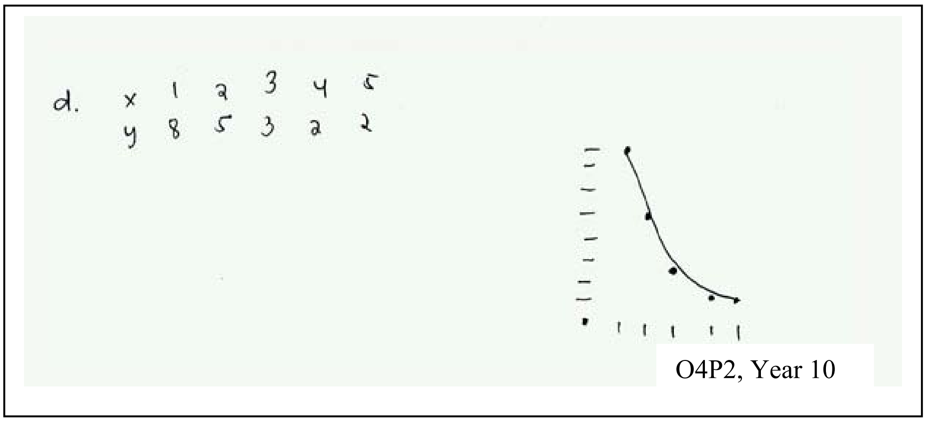


Figure 56. Task 12 - Strategy 1. Point-by-point interpretations.

Strategy 2 involves global interpretations through observing the trend in the values in the table. As in the previous tasks, Tasks 15 and 14, students' explanations in Task 12 coded Strategy 2 may be insufficient, but the fact that they were able to identify the correct table shows that these students have some understanding of the link between graphs and tables of values. This also indicates that there are students who would interpret the representations in more global terms than analytically. A typical student solution coded Strategy 2 is in Figure 57. The student M3N19 explanation in Figure 57 that the *y* values decreases and then increases show that this student understands that the values shown in choice *d* could be extended and would correspond to the points of the curve.

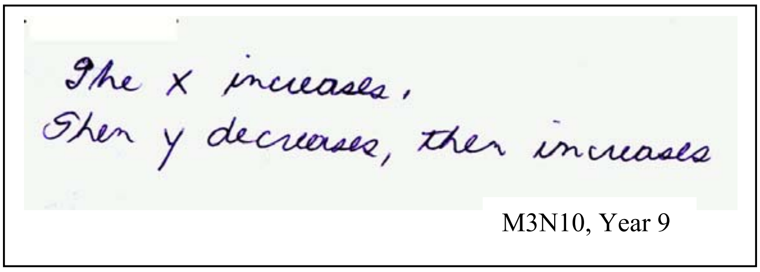


Figure 57. Task 12 - Strategy 2. Use of trends.

Strategy 3, which involved the use of second constant difference in the values of *y* was considered evidence of using invariant properties. Figure 58 shows a sample of a student solution coded Strategy 3.

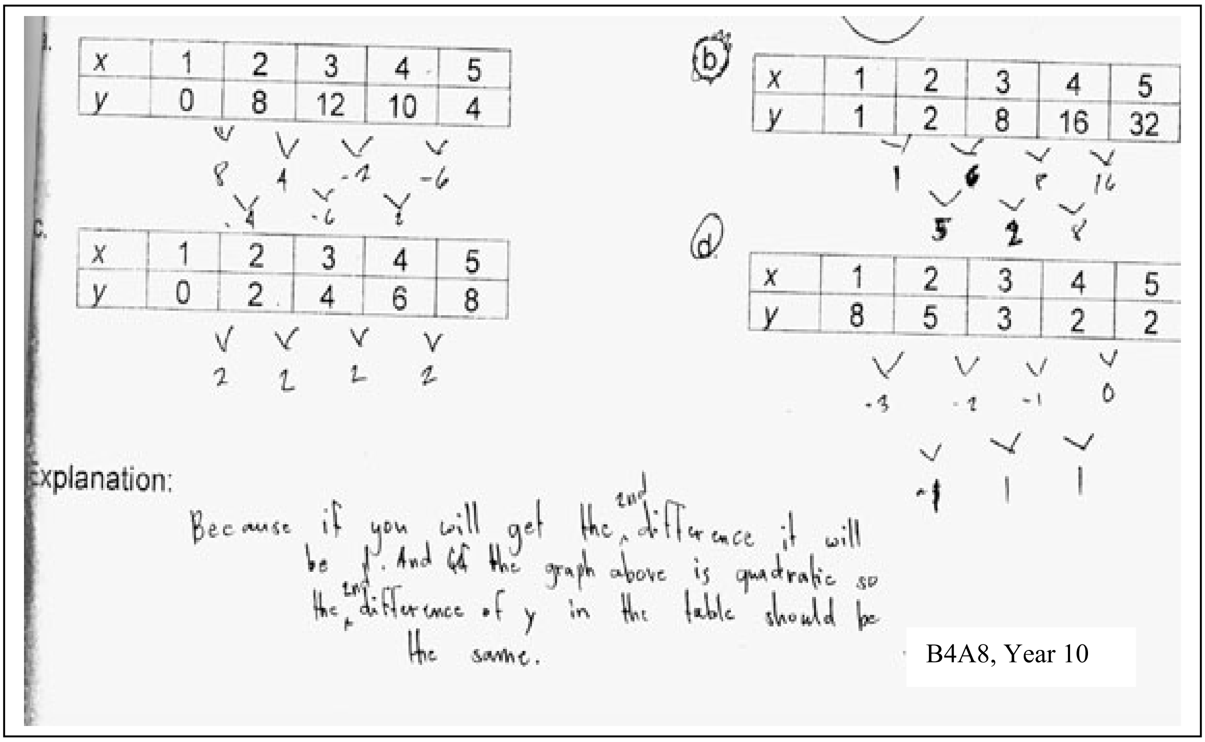


Figure 58. Task 12 - Strategy 3. Use of invariant properties.

Table 25 shows the distribution of students across strategies in Task 12. Like Task 15 and Task 14, the majority of the students used point-by-point interpretations.

Table 25
Distributions of Students at the Different Responses for Task 12

Responses	Year 8		Year 9		Year 10	
	D1	D2	D1	D2	D1	D2
Strategy 1 – Point-wise	4	12	25	47	31	33
Strategy 2 – Some Properties	5	15	9	7	6	6
Strategy 3 –Invariant Properties	0	0	0	3	3	13
No answer	88	23	43	14	7	1
No explanation	11	4	2	0	0	1
Incorrect responses	41	75	73	81	96	89
Total	149	149	152	152	143	143

No Year 8 student reasoned in terms of the second constant difference, which was understandable because their experience with function is still limited to linear relationships at this stage. By the second data collection period, there were now some Year 9 students who used Strategy 3. This is to be expected since in the Philippines, quadratics and "second difference tables" are introduced in Year 9.

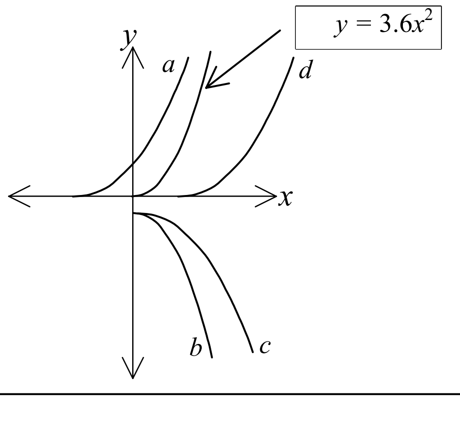
The majority of the Year 8 and Year 9 students, especially in the first data collection period, left this task unanswered, which is an indication that this task is indeed unfamiliar to them although, theoretically, they should be expected to complete this task correctly because they can plot points. The fact that the parabola was not on a grid must have made the task appear difficult.

The breakdown of students' incorrect responses for Task 12 is in Appendix C. Choice *c*, which shows linear relations, was the most popular answer among the incorrect responses in Year 8 and Year 9. In a study conducted by Markovits, Eylon and Bruckheimer (1986), it was observed that when students were required to give examples of functions, there was an excessive adherence to linearity. This may be because among the choices, linear relationship was the most familiar to the students.

In Year 10, the most popular response was choice *b*, which when plotted or when the trend of the values in the table is observed has curvature a little like that of half of a parabola. However, *b* is the graph of an exponential function. Task 12 also requires that students need some familiarity with the graph of a quadratic function, at least with nonlinear graphs.

The last task assessing Growth Points 2, 3 and 4 is Task 13.1. Task 13.1 involves linking curves and equations (see Figure 59).

13.1 Part of the graph of $y = 3.6x^2$ is labelled in the figure below. Which may be the graph of $y = -1.5x^2 - 4$? Circle the letter corresponding to the graph of your choice and explain how you determined your answer.



Explanation or Solution:

Figure 59. Task 13.1 - Curves and equations.

Students need to be able to interpret the effect of the constant term and the coefficient of x^2 on the graph. Students' solutions to the tasks were also categorised in three: Point-by-point analysis, use of some properties and holistic interpretations of the parameters.

Figures 60 to 63 show typical students' strategies for Task 13.1. Solutions involving evaluating the given equation with individual values or point-plotting technique were coded Strategy 1. A sample of a student's reasoning coded Strategy 1 is shown in Figure 60.

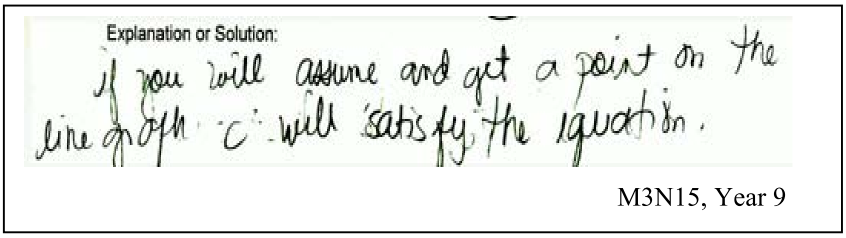


Figure 60. Task 13.1 - Strategy 1. Point-by-point interpretations.

Solutions coded Strategy 2 include solutions involving properties such as the intercept, but where the explanation is unclear or insufficient. An example of solution coded Strategy 2 is shown in Figure 61.

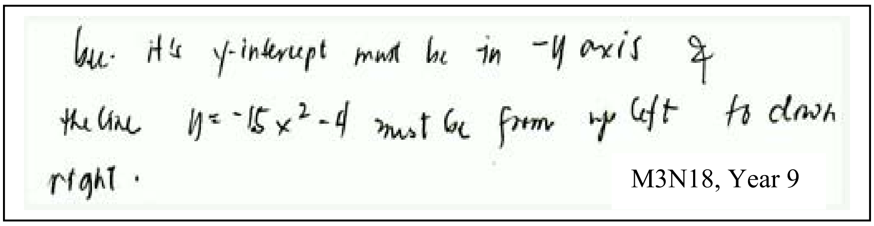


Figure 61. Task 13.1 - Strategy 2. Some properties.

Solutions involving interpretation of the coefficient of x^2 and the constant term in the equation were coded Strategy 3. The student's solution in Figure 62 for example, showed understanding of the effect of the constant term and the coefficient of x^2 on the graph of the function. The level of reasoning shown in the student's solution is reflective of an understanding of function as an object, as a permanent construct.

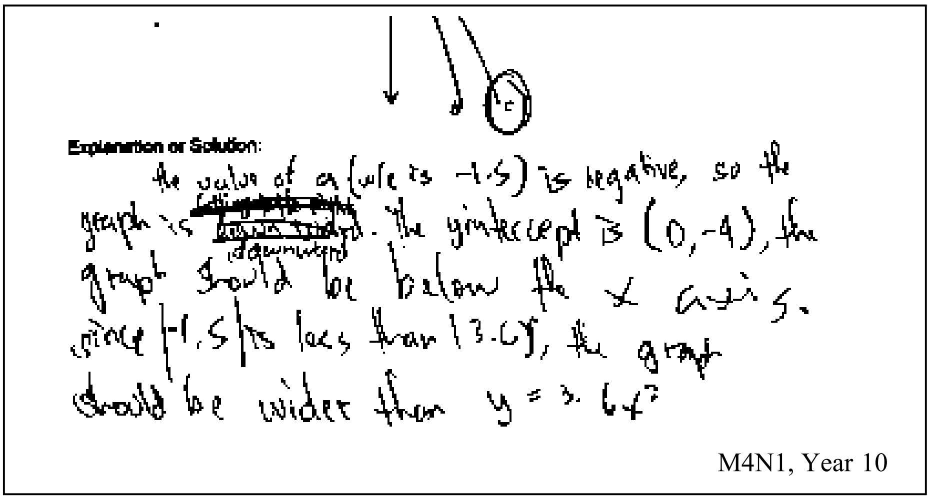


Figure 62. Task 13.1 - Strategy 3. Interpretation of parameters.

The solution shown in Figure 63 was also coded Strategy 3. This solution of a Year 8 student also reflected an object conception of function because it showed correct interpretations of the parameters, although the student applied knowledge of linear relations to the given quadratic relation.

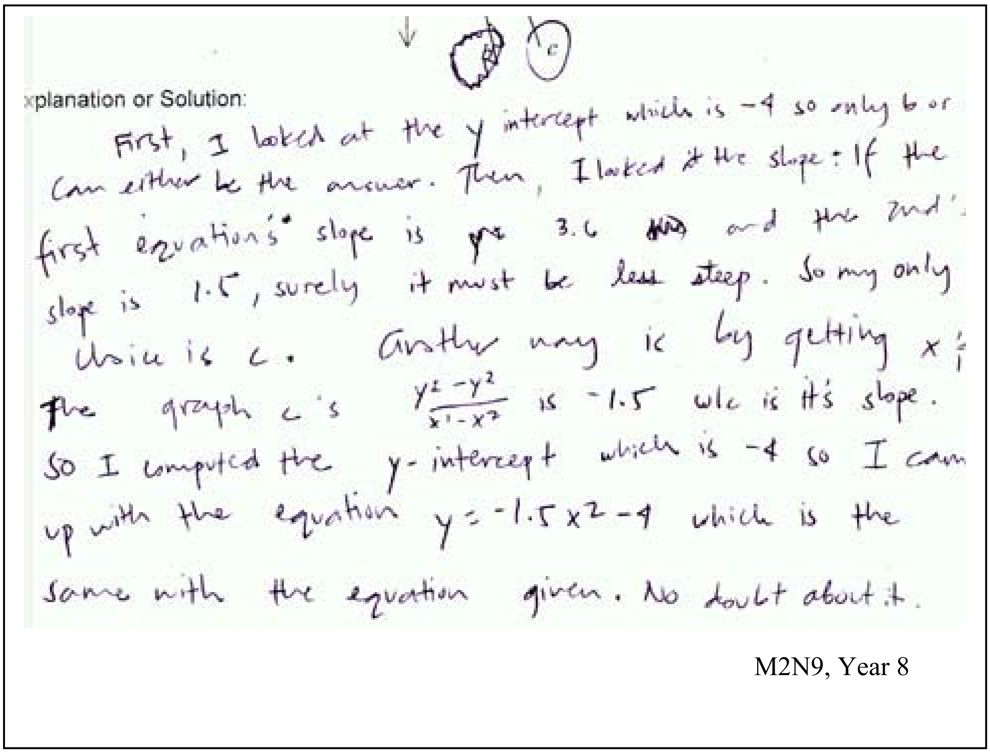


Figure 63. Task 13.1 - Strategy 3. Interpretation of parameters.

Table 26 shows the distribution of students coded at the different strategies. There were few Year 8 students who completed this task. This is because the topic has not been introduced to them.

Table 26
Distributions of Students at the Different Responses for Task 13.1

Responses	Year 8		Year 9		Year 10	
	D1	D2	D1	D2	D1	D2
Strategy 1 – Point-wise	1	6	5	6	14	22
Strategy 2 – Some Properties	2	5	4	13	9	11
Strategy 3 - Invariant Properties	0	1	1	14	4	12
No answer	40	17	25	17	11	12
No explanation	9	5	6	7	4	4
Other incorrect responses	97	115	111	95	101	82
Total	149	149	152	152	143	143

There was a sharp increase in the number of Year 9 students completing this task in the second data collection period. There were also an almost equal number of students who used Strategy 2 and Strategy 3 in this year level. This may be because this topic was recently discussed in Year 9.

The majority of the students coded Strategy 1 generated values from the given equation and then chose *c*. This strategy, evaluating the equation with individual values, is reflective of point-by-point strategy. It was not reliable because sometimes the graph produced looked like graph b. In fact, in Year 8, about 47% and about 40% in Years 9 and 10 chose *b* although not all of these used point-plotting techniques. Some of the reasons provided for choosing *b* involved the intercept or the negative constant term in the equation.

The most popular incorrect responses were choices *a* and *b* with 101 and 145 respectively out of the 444 students in the first data collection period. In the second data collection period, there were now only 62 students who chose *a*. However, those who chose *b* rose to 188 (see complete distribution of students' incorrect responses in Appendix C). These seem to indicate movement from an initial confusion between x and y intercepts to correctly interpreting the y-intercept.

All these tasks, Tasks 15, 14, 12 and 13.1, are interpretations tasks and involved graphs linked with either equations or tables. Students needed to complete at least two of the tasks to be coded the Growth Points 2 or 3, and at least three correct answers to be coded Growth Point 4. Assessing the growth points using at least two of the four tasks would allow Year 8 students whose experience with function may be limited to linear relationships only, to be coded at these growth points as well. Year 8 students' achievement would provide a good baseline data for students' initial understanding of function.

The following discussion describes the rest of the growth points under the Linking representations domain.

Growth Point 2: Point-wise linking of graphs with other representations

Students were coded Growth Point 2 if they could link graphs with other representations using point-by-point analysis or comparing individual values of each representation. Evidence of this would be at least correctly answering any two of Tasks 15, 14, 12, and 13.1 by point-wise interpretation. This means students should get at least two codes of Strategy 1 to be assigned at Growth Point 2.

Growth Point 3: Linking by trends/patterns or some properties

Students were assigned at Growth Point 3 if they used trends, patterns or some properties of the function to link graphs with other representations. This means they have at least two codes of Strategy 2. In the case where a student has two codes of Strategy 1 as well, the student would be coded at this growth point, Growth Point 3. However, there was no case of this sort.

Growth Point 4: Linking by invariant properties

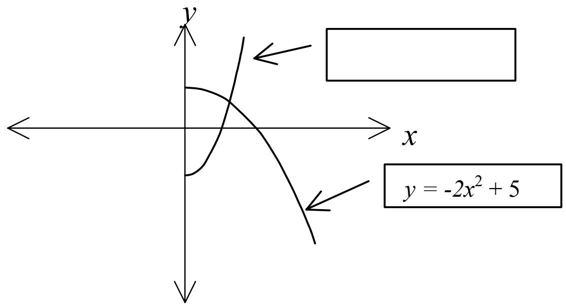
Growth Point 4 requires students to conceive function as a permanent construct, a concept possessing properties that would distinguish it from other concepts. Students were assigned Growth Point 4 if they could use invariant properties to link representations. Students should be able to complete at least three of the four tasks with at least two tasks solved using invariant properties, and the third task not by point-by-point strategy. That is, at least two codes of Strategy 3 and a code of Strategy 2 were required.

Growth Point 5: Linking graphs and equations seen as objects

This growth point requires students to conceive the functions involved as a mathematical object. That is, they could interpret a concept in terms of its properties and could think of it as an entity that could be manipulated or transformed without going through a specific, repetitive algorithm. This last condition, performing action on the function as an object, could best be assessed by tasks that would involve *construction* of representations. This last condition also differentiates this growth point from Growth Points 3 and 4, which involve linking representations in terms of interpretations only. Growth Point 2 involves action but is specific and repetitive.

Task 13 (see Figure 64) was designed to assess further understanding beyond Growth Point 4. It was the last task included under the Linking Representations domain, when pilot studies showed that there are a few students who could operate with function as an object.

13. Part of the graph of $y = -2x^2 + 5$ is labelled in the figure below. What may be the equation of the other graph in the box? Please explain how you worked out your answer.



Explanation:

Figure 64. Task 13 - Curves and equations.

Task 13 is similar to Task 13.1. However, Task 13 involves constructing rather than interpreting representations. A point-by-point analysis is also difficult to apply in this task. Task 13 requires conceiving both the graph and the equation as objects. In this task, students should recognise that the curves are part of a parabola and so they should suspect that the corresponding equation must be second degree. Students should see the relationship between the two curves and know the corresponding effect of the coefficient of x^2 on the steepness of the curve.

Those who completed Task 13 used specific values for the coefficient of x^2. No one wrote $y = kx^2 – 5, k > 2$ or $y = kx^2 – c, k > 2, 4 < c < 6$ which was the ideal answer. Samples of students' strategies are shown in Figures 65 and 66.

The student's solution in Figure 65 shows an understanding of the structure of the equations of the quadratic functions and the effect of the coefficient of x^2 on the graph of the function. A solution solving for the x-intercept or a zero of the function shows the student was probably attempting to solve the problem analytically.

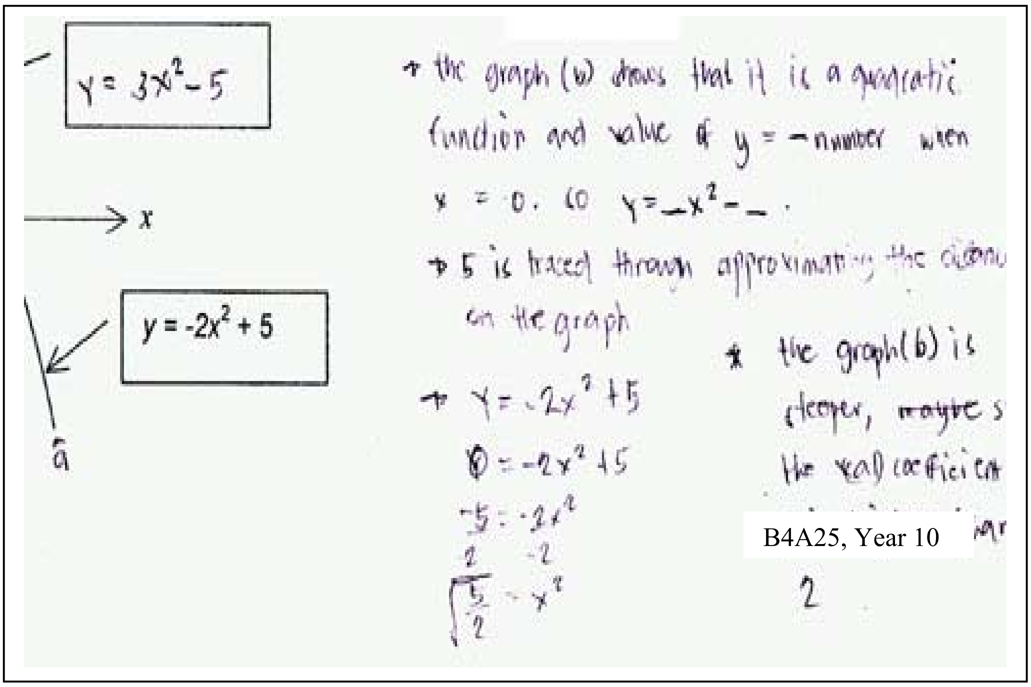


Figure 65. Task 13 - Solution 1.

Student O4P1 in Figure 66 solved the problem by assuming values for the point of intersection.

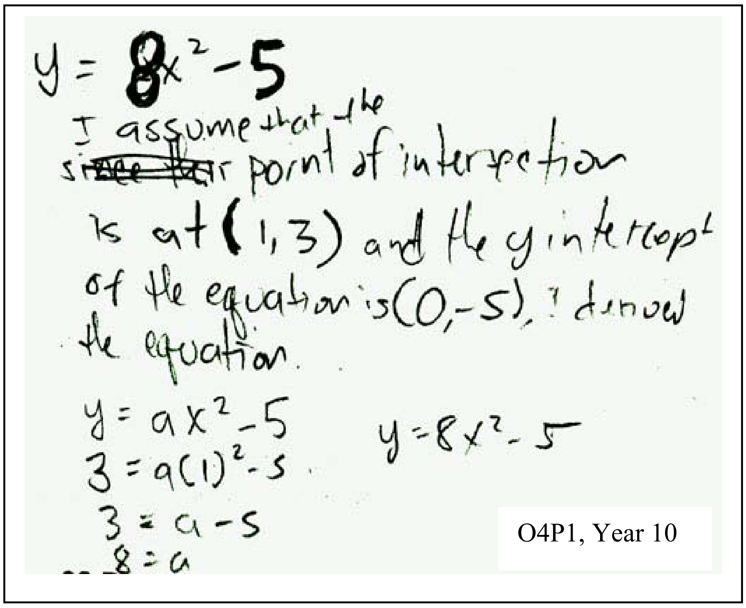


Figure 66. Task 13 - Solution 2.

Out of 444 students, there were only three and all of them Year 10 students from the second data collection period and one from the first data collection, who were able to complete Task 13. All these students were coded Growth Point 5. Students were coded Growth Point 5 if they could at least work out three of Task 15, Task 14, Task 12 and Task 13.1, but not using point-by-point analysis in any of them. These students should also complete Task 13. One of the three students who completed Task 13 was coded Strategy 3 in all of Tasks 15, 14, 12 and 13. The second student was coded Strategy 3 in Tasks 15, 12, 13.1 and Strategy 1 in Task 14. The third student was coded Strategy 3 in Task 13.1 and Strategy 2 in Task 15 and Task 14 but missed Task 12. This third student was also the only student who was coded Growth Point 5 in the first data collection period. The fact that those who were able to solve Task 13 also solved at least three of the four tasks without using point-by-point analysis suggest that Task 13 could be used as a sole assessment task for Growth Point 5.

The following section presents the statistics supporting the order of the growth points.

Investigating for Typical Learning Trajectory

The first part of this section presents the success rates of each task. The second part presents the statistics supporting the order of the growth points.

Success Rates

Table 27 shows students' success rates on tasks used to assess the growth points on Linking Representations.

Table 27
Success Rates on Tasks Assessing the Growth Points under Linking Representations (%)

Tasks	Year 8 (*n* = 149)		Year 9 (*n* = 152)		Year 10 (*n* = 143)	
	D1	D2	D1	D2	D1	2
T10.1: Generating values	42.8	88.9	89.5	100	90.8	100.0
T10: Making equation	17.2	63.2	53.9	78.9	75.7	79.6
T11: Inverse	23.1	53.3	63.1	66.4	51.9	68.5
T15: Lines and tables	12.1	42.3	38.2	56.6	65.0	81.8
T12: Curves and tables	6.0	31.5	22.4	37.5	28.0	36.4
T14: Equations and lines	2.7	20.8	12.5	28.3	45.5	60.3
T13.1: Curves and equations (Interpretation)	2.7	8.1	6.6	21.7	18.9	31.5
T13: Curves and equations (Construction)	0.0	0.0	0.0	0.0	0.7	2.1

The top three tasks, Tasks 10, 10.1 and 11 assessed Growth Point 1. The fact that they occupy the top slots shows that Growth Point 1 is learned indeed easily.

The next easiest tasks in Table 27 were Task 15 and Task 12. This is to be expected since both tasks involve linking graphs and table of values. Although the graphs were not on grids, the table of values corresponding to them would be easier to determine since the tables lend themselves naturally to point-by-point interpretations, which was the preferred strategy of the students in all the tasks.

The results also showed that more Year 8 and Year 9 students found Task 12 easier than Task 14 even if the latter only involve simple linear relationships while the former involve a quadratic relationship. This may be because Task 12 involves tables. Point-by-point interpretations could be made more directly with tables than with equations.

Task 13 as expected, was the most difficult of all the tasks in this domain because a point-by-point strategy is almost impossible to do to complete this task. Task 13.1 is an easy version of Task 13. Task 13.1 could be interpreted point-wise so it was not surprising that more students were able to complete this task compared to Task 13. This also indicates that conceiving function, as an object is not accessible to many students in the study.

In general, tasks found easy or difficult in the first data collection were still found to be the same in the second data collection period, indicating consistency and hence reliability of the assessment tasks.

Investigating the Order of the Growth Points

This section presents the data showing the order at which growth points under Linking Representations are likely to be acquired. A summary of the coding procedure is shown in Table 28.

Table 28
Procedures for Coding the Growth Points under Linking Representations

Growth Points	Coding Procedure
GP 1: Linking equations and tables	Any two of T10.1, T10, and T11
GP 2: Point-wise linking of graphs with other representations	At least two codes of Strategy 1 in T12, T14, T15, T13.1
GP 3: Linking of graphs with other representations by trends/patterns/local properties	At least two codes of Strategy 2 in T12, T14, T15, T13.1
GP 4: Linking of graphs with other representations by invariant properties	Three correct answers and at least two codes of Strategy 3 in T12, T14, T15, T13.1
GP 5: Linking of graphs with other representations seen as objects.	T13 and any three of T12, T14, T15, T13.1 but not coded Strategy 1 in any one of these

A two-part analysis of the data was made in order to determine the order of the growth points. The first involves the percentage of students coded at each growth point and the second involves the movement of the students between the growth points within a five-month period.

Using percentage of students coded at the growth points.

Table 29 shows the distribution of students in the growth points. The percentages do not add up to 100 because some students achieved more than one growth point. The coding procedure for Growth Points 2, 3 and 4 used the same set of tasks hence students coded at any one of them could not be coded at the other two. However, a different set of tasks was used for GP 1 and GP 5, so it is possible to be coded at three growth points in all.

Table 29
Distributions of Students at the Growth Points under Linking Representations

Growth Points	Year 8 (*n* =149)		Year 9 (*n* = 152)		Year 10 (*n* = 143)	
	D1	D2	D1	D2	D1	D2
GP 0	78.8	26.2	33.6	6.6	8.4	2.8
GP 1	24.8	73.2	63.2	86.2	84.6	89.5
GP 2	2.0	24.8	15.1	27.6	35.7	53.8
GP 3	0.7	4.0	1.3	5.9	7.0	9.1
GP 4	0	0	0	7.2	2.1	3.5
GP 5	0	0	0	0	0.7	2.1

The decreasing percentage of students from GP 1 to GP 5 in each year level and in both data collection periods indicates that the order of the growth points proceeds from GP 1 to GP 5.

The results also showed that except for Year 8 at the beginning of the year, Growth Point 1 is accessible to the majority of the students. The big jump in the percentage of Year 8 students in GP 1 between data collection 1 and data collection 2 showed that this level of understanding is acquired easily.

The next growth point likely to be acquired is GP 2, which is about linking graphs with other representations through point-by-point interpretations.

The results showed that understanding of linking representation in terms of Growth Points 3, 4 and 5 are difficult to acquire. In fact, less than ten percent of Year 10 students in the middle of the school year (second data collection period) could work in terms of GP 3. Only about four percent of them could work in terms of GP 4 and only two percent or three students could work in terms of GP 5. No Year 8 students could work in terms of Growth Points 4 and 5 and no Year 9 student could work in terms of GP 5.

The graph in Figure 67 shows visually the distribution of students coded at the highest growth point achieved during the second data collection period.

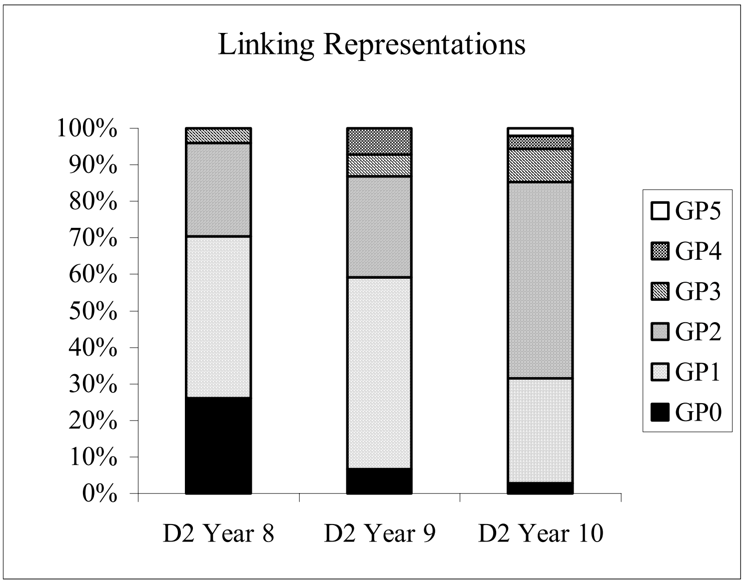


Figure 67. Distributions of *s*tudents coded at the highest growth points achieved.

The trend in the increase or decrease in the percentage of students across the year levels shown in Figure 67 further confirms the order of the growth points. For example, the percentage of students coded at GP 0 was decreasing from Year 8 to Year 10, which is to be expected since students in the higher year levels have more experiences with mathematics hence are more likely not to remain at GP 0. There was also a slight increase in the percentage of students coded at GP 1 in Year 9, which is to be expected because of the decrease in the percentage of students at Growth Point 0.That the percentage of students coded at GP 1 should decrease in Year 10 is also to be expected since most of them were likely to work in terms of the higher growth points. The increasing percentage of students from Year 8 to Year 10 at Growth Points 3, 4 and 5 indicates that as students gain more experiences in working with function, they are more likely to work in more general terms. However, the increase is only very small. There were only very few students coded at these growth points even at Year 10 considering the fact that the Year 10 students, when the second data collection period occurred, have at least completed their study of the polynomial functions and exponential and logarithmic functions. They were already doing analytic geometry and would be doing topics in basic Analysis later in the year, yet a large majority of them were still working with function in terms of point-wise analysis.

Tracing Movement between the Two Data Collection Periods.

As mentioned earlier, the same set of tasks was used to assess GP 2, GP 3 and GP 4. This means that a student coded GP 4 for example, was not coded GP 3 or GP 2. With this coding procedure, it is possible that more students could be coded in GP 4 than GP 3 or GP 2. In this method, the growth point with the highest percentage could not be interpreted as the first growth point. It could mean that the majority of the students were on that level of understanding and this would not necessarily be the first growth point. Thus, using the decreasing percentage of students coded at the growth points as evidence that the growth points were ordered as they were, is not applicable at all times for the coding procedure such as the one followed for GP 2, GP 3 and GP 4, where only one set of tasks was used.

To verify further the order of the growth points, the present study investigated students' movement between the growth points. That is, if the majority of the students coded GP 1 in the first data collection for example, moved to GP 2 rather than to GP 3 in the second data collection, this is interpreted that GP 2 is indeed the next growth point after GP 1. However, even with this method, it is still possible to have more students moving from GP 1 to GP 3 for example, rather than to GP 2. However, this did not happen in any of the year levels, which means that the growth points are not learned easily within this five-month period. This is also an indication that the identified growth points were indeed big ideas.

Tables 30, 31 and 32 show the cross-tabulation of students coded for each growth point in the two data collection periods. The broken line was drawn to highlight the number of students remaining at each growth point. D1 stands for the first data collection and D2 stands for the second data collection.

Table 30
Cross-tabulation of Year 8 Students Achieving the Growth Points under Linking Representations in Data Collections 1 and 2

		D2				
	Growth Points	GP 0	GP 1	GP 2	GP 3	Total
	GP 0	33	49	26	2	110
D1	GP 1 only	5	17	9	4	35
	GP 2	1	0	2	0	3
	GP 3	0	0	1	0	1
	Total	39	66	38	6	149

Table 31
Cross-tabulation of Year 9 Students Achieving the Growth Points under Linking Representations in Data Collections 1 and 2

		D2					
	Growth Points	GP 0	GP 1	GP 2	GP 3	GP 4	Total
	GP 0	6	30	11	2	2	51
D1	GP 1 only	4	41	21	4	6	76
	GP 2	0	9	10	2	2	23
	GP 3	0	0	0	1	1	2
	Total	10	80	42	9	11	152

Table 32
Cross-tabulation of Year 10 Students Achieving the Growth Points under Linking Representations in Data Collections 1 and 2

		D2						
	Growth Points	GP 0	GP 1	GP 2	GP 3	GP 4	GP 5	Total
	GP 0	2	7	3	0	0	0	12
	GP 1 only	1	23	27	2	3	0	56
D1	GP 2	1	9	43	6	2	0	61
	GP 3	0	2	3	3	0	2	11
	GP 4	0	0	1	2	0	0	2
	GP 5	0	0	0	0	0	1	1
	Total	4	41	77	13	5	3	143

The data in Tables 30, 31 and 32 show that there were more students who advanced to the next growth point than were those who jumped to higher growth points. For example, among those who were initially at GP 0, the majority of the students moved to GP 1 rather than to GP 2 or higher growth points. This implies that Growth Point 1 is indeed the first growth point. The movement of students who were initially at GP 1 further confirms this. The number of students who were coded GP 1 that advanced to GP 2 is higher than those that advanced to GP 3 and much lower than to GP 4.

Results also showed that most of the students who were coded GP 2 at the beginning of the year remained at that growth point. This could be interpreted that GP 2 is indeed a major growth point since it would take more than five months to advance to GP 3. This also indicates that GP 3 is not acquired easily.

No Year 8 student was coded GP 4 in both data collection periods. In Year 9, no student was coded GP 4 at the beginning of the year and only five were coded this growth point in the second collection. These are indications that it is a big leap for students to go to GP 4. This further implies that GP 4 represents a big idea apart from GP 2 and GP 3.

No Year 8 or Year 9 students were coded GP 5. In Year 10, only one student was coded GP 5 in the first data collection and three in the second data collection indicating that this indeed is the most difficult to attain of all the five growth points.

Because it takes the majority of the students some time to move from one growth point to the next, each growth point represents a big idea. In fact, because there were only very few students coded at the growth points beyond GP 2, it seems that the majority of the students in the study would leave high school (Year 10) only able to link representations by point-by-point interpretations.

Linking Equations and Tables versus Linking Graphs with Other Representations

Results also showed that linking graphs with other representations is much more difficult than linking tables and equations. Table 33 compares the number of students who could link equations and tables (GP 1) and those who could link graphs and other representations (GP 2 to GP 5).

Table 33
Percentages of Students who could Link Equations and Tables and those who could Link Graphs with Other Representations

Growth Points	Year 8 (*n*=149)		Year 9 (*n*=152)		Year 10 (*n*=143)	
	D1	D2	D1	D2	D1	D2
GP 0	73.8	26.2	33.6	6.6	8.4	2.8
Linking equations w/ tables (GP 1)	24.8	72.5	63.2	86.2	84.6	89.5
Linking graphs w/ other representations (GP 2-5)	2.7	29.5	16.4	40.8	52.4	68.5

Data show the big difference in the achievement between linking equations and tables and linking representations that involves graphs. As mentioned earlier this may be because of the very nature of the representations. That is, tables and equations could easily be interpreted point-wise, while graphs not on grids make point-by-point interpretations very tedious. This indicates the potential of graphical representations as a means to develop thinking of function in a holistic way.

Linking Representations Domain versus Equations and Graphs Domains

The Linking Representations domain involves understanding function as represented by tables, equations and graphs. Hence, it was interesting to consider how students who achieved higher growth points (at least GP 3) under the domains of Equations or Graphs would work with tasks on linking representations and vice versa. Table 34 shows the frequency of students who achieved the higher growth point in each of the three domains in the second data collection period. The results from the second data collection period were used to compare achievements in these growth points because there were more students coded at the higher growth points than in the first data collection.

Table 34
Cross-tabulation of Students' Achievements at the Higher Growth Points under Linking Representations, Equations and Graphs

Linking Representations	Equations		Graphs
	GP 3: Local Properties (n = 122)	GP 4: Relationship/ Object (n = 22)	GP 3: Relationship/ Object (n = 23)
GP 4: Linking by invariant (n = 15)	9	0	6
GP 5: Invariant and as object (n = 3)	3	2	0

There were 15 students coded at GP 4 under Linking Representations. Of the 15 students, nine students achieved GP 3 under Equations. The rest achieved growth points lower than GP 3. GP 3 under Equations describes understanding in terms of interpreting local properties; hence the majority of the students who could link representations by invariant properties are indeed to be expected to interpret these properties in equations. However, those who achieved GP 3 under Equations could not necessarily be expected to link equations with other representations by invariant properties. Of the 122 students who could interpret local properties such as rate and intercepts, only 12 could apply these concepts as invariant properties to link representations (GP 4 and 5).

Of the three students who could link graphs and equations both seen as objects (GP 5), two could also work with equations as manipulable objects (GP 4 under Equations). While the number is very small to draw any definite conclusions or generalisations, the data seem to show that students who could link representations as objects are more likely to work with functions represented by equations as objects as well. However, of the 22 students who were at GP 4 under Equations, only two could link representations as objects (GP 5 under Linking

Representations). This is to be expected since linking representations as objects, requires understanding of the invariant properties of the function, which was not a part of the GP 4 under Equations.

There were only six out of the 18 students who could link graphs with other representations by invariant properties (GP 4 or GP 5 under Linking Representations), who were able to conceive of the graph as objects (GP 3 under Graphs). In addition, of the 23 students who were at GP 3 under Graphs, only six could link graphs with other representations by invariant properties. This is an indication that these growth points are distinct. Indeed, they are because GP 3 under Graphs does not deal with the invariant properties of function.

Discussion and Summary

There were five growth points identified under Linking Representations domain. These are:

Growth Point 1: Linking equations and tables

Growth Point 2: Linking graphs with other representations through point-by-point analysis

Growth Point 3: Linking by trends/ patterns or some properties

Growth Point 4: Linking by invariant properties

Growth Point 5: Linking representations as objects

The descriptions of the growth points were based on the strategies students used to complete the tasks correctly. Two method of analysis of the data supported the order of the growth points. The first method involved the frequency of students coded at the growth points. The decreasing percentage of students from Growth Point 1 to Growth Point 5 confirmed the order of the growth points in spite of the discrete assignment to Growth Point 2 to 4. The decreasing percentage of students from Year 8 to Year 10 in the lower growth points and the increasing percentage of students through the year levels at the higher growth points also confirmed the trend.

The second method involved tracing students' movements from the first data collection period to the second data collection period. There were more students moving to the next growth point than other growth points. The results showed that the order was indeed as predicted.

The order of the five growth points involving linking representations is consistent with the process-object theory, which states that in general, students conceive a mathematical concept first as a procedure or as a process before they are conceived as object. All five growth points could be located within the process-object continuum with Growth Point 1 nearest the process end and Growth Point 5 nearest the object end. Growth Points 3, 4 and 5 are also reflective of property-oriented conception of function, with Growth Point 3 describing understanding in terms of global properties, while Growth Points 4 and 5 are more on describing understanding of function in terms of its invariant properties. Growth Point 4 involves merely interpretations of invariant properties while Growth Point 5 involves interpretations of properties as well as operations with the representation as object.

Although the percentage of students at the growth points was increasing from Year 8 to Year 10, which was to be expected, the majority of the students only achieved Growth Points 1 and 2. Both these growth points involved point-wise thinking. This finding seems to suggest that advanced students continue to operate using point-by-point interpretations, despite their experiences with other functions. It seems a big step for the students to think beyond this level, which suggest that acquiring an object conception of function does not come naturally with more experience with other families of function. Instruction therefore should be designed so that students are given more opportunity to think at this level.

One way to do this is to use graphing calculators and computer software designed for teaching functions. Some studies have shown that the use of these technologies encourages students to conceive of functions holistically (Penglase & Arnold, 1996) because these technologies facilitate working with the different representations at the same time. Studies have also shown that the use of a numberless numberline helps pupils to acquire a generalised strategy in operating with numbers (Gravemeijer, 1994) so there is a strong possibility that the use of more gridless graphs to represent and analyse function would help students to conceive of functions more holistically. It should be remembered however that continued emphasis on visualising functions without drawing out the underlying algebra would not improve students' understanding of the links between the graphs and its symbolic equivalent and thus would do little to strengthen students' understanding of function (Chinnappan & Thomas, 2003).

Linking graphs with other representations is much more difficult than linking equations and tables. This is expected since both equations and tables naturally lend themselves to point-by-point interpretations, the most frequent strategy that students used in almost all the assessment tasks. Another reason would be the high frequency of use of ordered pairs and rules for functions in Year 7 and 8 textbooks which is also to be expected since this period is a transition period from arithmetic to algebra (Mesa, 2001).

Of those students who could link graphs with other representations, the majority favoured point-by-point interpretations. This may be because students' experiences with graphs may have been limited to plotting points only or interpretations based on individual points.

The next chapter discusses students' understanding of equivalent relationships. It also involves linking representations. However, the focus is more on students' understanding of equivalent relationships within the same representational system.

CHAPTER 7

DOMAIN 4 - EQUIVALENT RELATIONSHIPS

The Linking Representations domain included in the Framework did not directly address understanding of equivalent functions or relationships. Thus, the Equivalent Relationships domain, which explicitly addresses students' understanding of equivalent functions, was added as a separate domain to the Framework. To assess understanding in this domain, tasks were designed that required students to determine the representations showing equivalent relationships among the choices instead of asking students how they would define equivalent relationships or of their understanding of this concept. Students' reasoning or strategies to complete these tasks were used to assess their understanding of the concept of equivalent relationships.

Because the Linking Representations domain, discussed in Chapter 6, already involved linking *between* representational systems, understanding of equivalent relationships was assessed in terms of students' strategies in identifying equivalent relationships *within* the same representational system.

The organisation of this chapter is similar to the other three domains. The first part describes the assessment tasks and the growth points. The second part presents the data supporting the order of the growth points. The third and last part presents a summary of the findings of the study.

Assessing the Growth Points

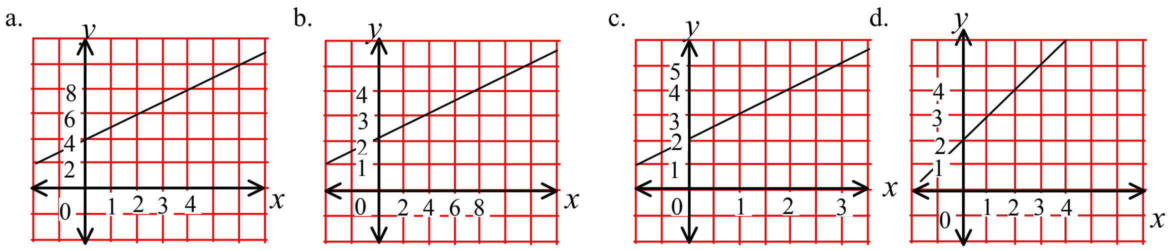
There were three tasks used to assess understanding of equivalent relationships. Task 16 assessed understanding of equivalent relationships in graphical form, Task 17 in tabular form and Task 18 in equation form. All three tasks involved linear relationships of the form $y = ax + b$. The tasks were designed with the domain and range assumed to be the set of real numbers for all the functions. Students seemed to have made this assumption too, since no student made comment on the absence of specified domain and range or reasoned that the tasks provided insufficient information for them to conclude which of the representations showed the same relationships.

The Assessment Tasks

Task 16 (see Figure 68) was designed to assess students' knowledge of equivalent relationships represented in graphs. Grids were used to facilitate reasoning in terms of the invariant properties of linear functions. With the numbered grid, the intercept is visible and determining the gradient or slope is straightforward. The use of the grid however afforded point-by-point interpretation, so to make it less straightforward, different scales were used. The change in the scales also made it possible to have graphs that looked the same.

Students' tendency to consider the shape of the graphs first in working with the task was observed during the interviews. In fact in the written test, most of the students' incorrect responses involved the graphs *a*, *b* and *c* which all look the same.

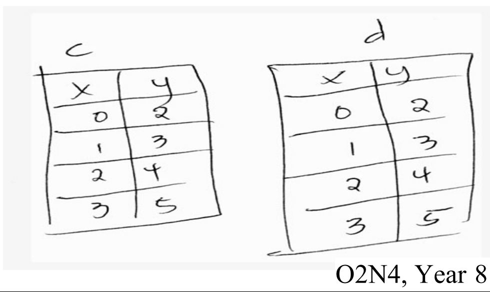
16. Select the graphs showing the same function or relationship. Please circle the letter corresponding to your choices. Please explain how you determined your answers.



Solution or Explanation:

Figure 68. Task 16 - Graphs.

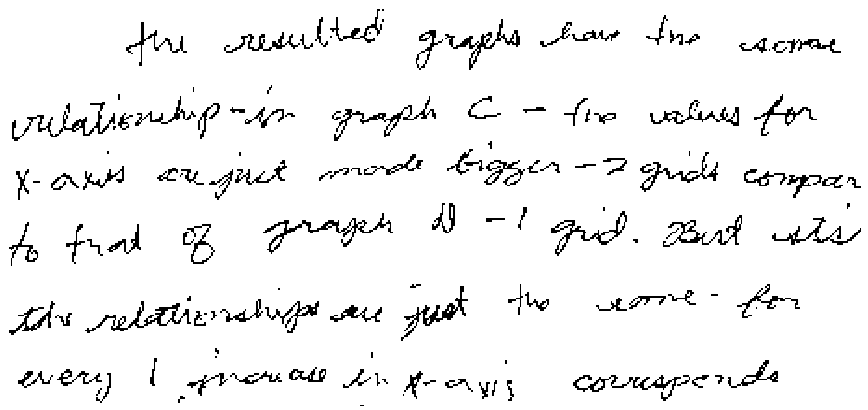
Students' solutions to Task 16 that led to the correct answer could be classified into three groups. The first involved point-wise analysis. Point-wise analysis involves showing that the graphs have the same set of points. This strategy was coded Strategy 1. A typical solution is shown in Figure 69.



O2N4, Year 8

Figure 69. Task 16 - Strategy 1. Point-by-point analysis.

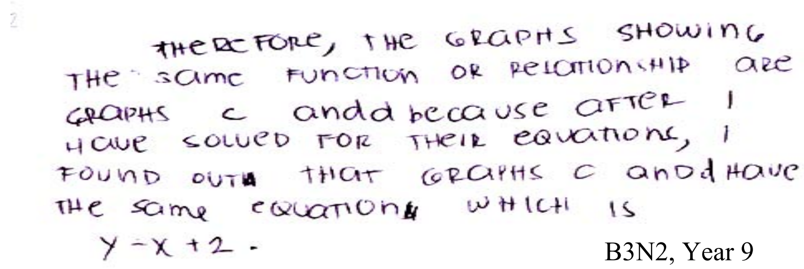
The second category involved the use of some patterns or properties of the function or showing that the graphs have the same equations. These kinds of solutions were coded Strategy 2. In this strategy, students show understanding and knowledge of equivalent relationships and some properties of function, but they have not fully understood the concept of invariance, at least with linear functions. An example of a solution coded Strategy 2, which uses trend or pattern, is shown in Figure 70.



M4N7, Year 10

Figure 70. Task 16 - Strategy 2. Use of some properties or patterns.

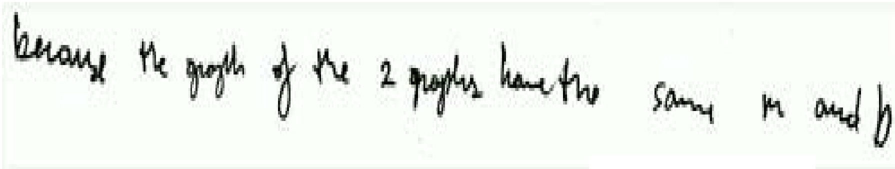
An example of justification showing that the graphs have the same equation is shown in Figure 71.



B3N2, Year 9

Figure 71. Task 16 - Strategy 2. Same equation.

Those who justified their answers using the invariant properties of the linear relationship were coded Strategy 3. Strategies using invariant properties included a) showing that the graphs have the same intercept and gradient; b) showing that they have the same *x* and *y* intercepts; and, c) showing that they have two common points with a comment that it is enough because the relationship was linear. An example is shown in Figure 72. In the sample, the student used *m* and *b.* In the Philippines, *m* is used as letter symbol for slope and *b* for *y*-intercept.



M3N20, Year 9

Figure 72. Task 16 - Strategy 3. Use of invariant properties.

Table 35 show the distribution of students coded at the various strategies. The shaded portion represents incorrect responses.

Table 35
Distributions of Students at the Different Responses for Task 16: Graphs

Responses	Year 8 (n = 149)		Year 9 (n = 152)		Year 10 (n = 143)	
	D1	D2	D1	D2	D1	D2
Strategy 1 – Point-wise	5	30	17	35	35	61
Strategy 2 – Some Properties	0	10	10	18	11	27
Strategy 3 - Invariant Properties	1	1	0	1	0	0
No answer	21	2	13	6	8	3
No explanation	0	1	2	0	0	0
Other incorrect responses	122	105	110	92	89	52

Results in Table 35 show that the most favoured strategy was Strategy 1, followed by Strategy 2. Very few students worked out the task using Strategy 3 indicating that the level of understanding needed for Strategy 3 is not yet accessible for almost all of them, even the Year 10 students.

More than 60% of the Year 10 students in the second data collection period completed the task correctly, but none of them used Strategy 3. These students know that two points determine a unique line because they were already doing analytic geometry but they were unable to transfer this knowledge to solve Task 16, which indicates that this task is not familiar to them. This also shows students' tendency to use point-by-point interpretations even if it is tedious when faced with an unfamiliar task.

There were very few students who left the task unanswered or selected the correct answer but did not provide any explanation. This means that there were only very few instances where the basis for assessing students' understanding was not very clear.

Note that in Task 16, the graphs in *a*, *b* and *c* look the same. The only differences are the scales in *x* and *y* axes. Students must have relied on the appearance of the graph since the top two incorrect answers in Task 16 were "*a* and *b*" and "*b* and *c*" respectively in all year levels and in both data collection periods. For a complete distribution of students' incorrect responses in Task 16, see Appendix C.

Task 17 (see Figure 73) is similar to Task 16 apart from the fact that the representations involved in Task 17 were tables. To encourage students to consider the properties of the function in interpreting the tables, the values in the tables were set so that point-by-point analysis would not be straightforward. Students would still need to interpolate or extrapolate.

At least two tables among the choices share some properties like scale, intercepts, constant difference, gradient and trend. All these properties are easy to determine from the tables. Students need to know which of these properties should be shared by equivalent relationships.

17. Select the tables showing the same function or relationship. Circle the letter corresponding to your choices. Please show/explain how you obtained your answer.

a.

x	4	6	8	10
y	9	11	13	15

b.

x	2	3	4	5
y	9	10	11	12

c.

x	0	1	2	3
y	5	7	9	11

d.

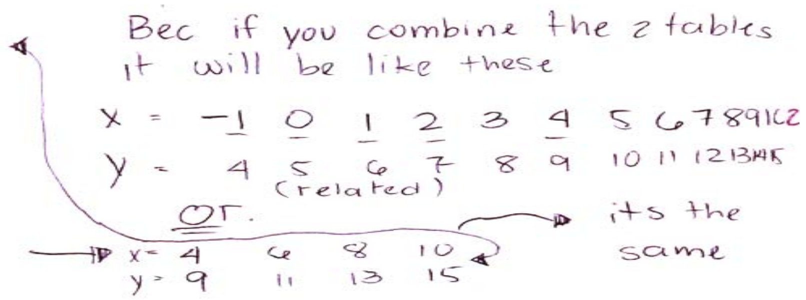
x	-1	0	1	2
y	4	5	6	7

Solution/Explanation:

Figure 73. Task 17 - Tables.

Students' strategies in Task 17 included point-plotting techniques either in the same grid or in separate grids, use of some properties or trends, comparing the corresponding equations, and use of invariant properties. Point-plotting techniques were not of course reliable. There were students who plotted the values correctly but came to the wrong conclusions.

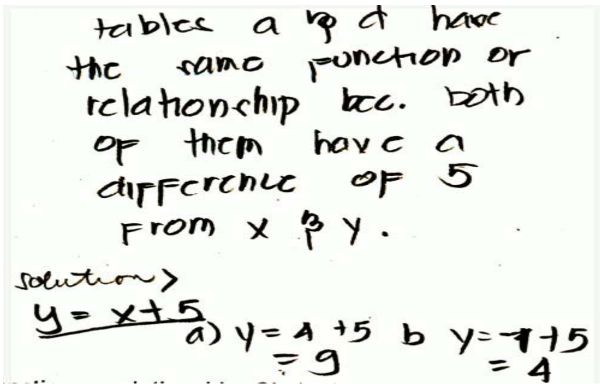
The coding of strategies for Task 17 was similar to that in Task 16. A correct answer using point-plotting techniques or comparing individual points was coded Strategy 1. A typical example is shown in Figure 74.



O2N26, Year 8

Figure 74. Task 17 - Strategy 1. Point-by-point analysis.

A solution was coded Strategy 2 if the student used trend, patterns or some properties, or showed that the tables had the same equation. Shown in Figure 75 is a typical students' solution coded Strategy 2.



B4B6, Year 10

Figure 75. Task 17 - Strategy 2. Common properties or rules.

A solution was coded Strategy 3 if it used invariant properties to justify the choice. This included showing that the two tables, *a* and *d*, have the same slope and intercept. However, no student was coded Strategy 3 for this task.

Table 36 shows the distribution of students coded for each strategy for Task 17. Data showed that the most frequently used solution involved the use of a rule or some properties apparent in the given tables. This may be because most of the learning experiences in the study of function involve finding the rule of a set of data given in tables.

That there were only few solutions which involved point-by-point analysis could be due to the fact that students need to make some interpolation or extrapolation in order to determine if the tables indeed share the same set of values. That is, they need to perform “actions” on the tables. This may be something they are not used to doing with tables.

Table 36
Distributions of Students at the Different Responses for Task 17

Responses	Year 8 (n = 149)		Year 9 (n = 152)		Year 10 (n = 143)	
	D1	D2	D1	D2	D1	D2
Strategy 1 – Point-wise	0	8	3	5	4	10
Strategy 2 –Common Properties/Rule	11	24	30	33	40	53
Strategy 3 - Invariant Properties	0	0	0	0	0	0
No answer	43	4	17	7	6	3
No explanation	1	3	2	1	1	0
Incorrect responses	94	110	100	106	92	46

A large majority of the correct responses involved finding the rule or equation for the tables, which is a familiar activity in algebra classes. These students may have succeeded in Task 17 because the relationship involved was a simple linear relationship. Nevertheless, using this solution shows that these students could conceive of the equivalent relationships, which share some properties (rule) rather than those using Strategy 1, which compares individual values.

No student was coded Strategy 3 or used invariant properties such as “same slope and same intercept” to justify their answers. I think this is not because they do not know how to calculate the slope of the function or identify the intercept. In fact, the most popular incorrect answers for all year levels was the pair, b and d. Some of the justifications were: because they have the same slope of 1 or both have a constant difference of 1 in x and y.

Results also showed that only a small percentage of students did not answer Task 17 or chose the correct answer without giving explanation. Thus, like Task 16, there were few instances where the basis for assessing students’ understanding was not very clear.

Overall, there was no evidence to suggest that students’ incorrect answers in Task 16 and Task 17 were due to misinterpretation of the tasks. It appeared to be due to their lack of understanding of the concepts needed to complete the task.

Task 18 (see Figure 76) is similar to Task 16 and Task 17 except that the representations involved were equations. This task also had an additional constraint: the variables used were different. Task 16 and Task 17 used the same letter symbols x and y.

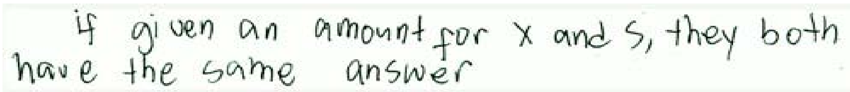
18. Select the equations that show the same function or relationship. Circle the letter corresponding to your choices. Please show or explain how you obtained your answers.

a. $y = 4x + 8$ b. $p = 4\ (s+2)$ c. $y = \frac{x-8}{4}$ d. $p = 4s + 8$

Solution or Explanation:

Figure 76. Task 18 - Equations.

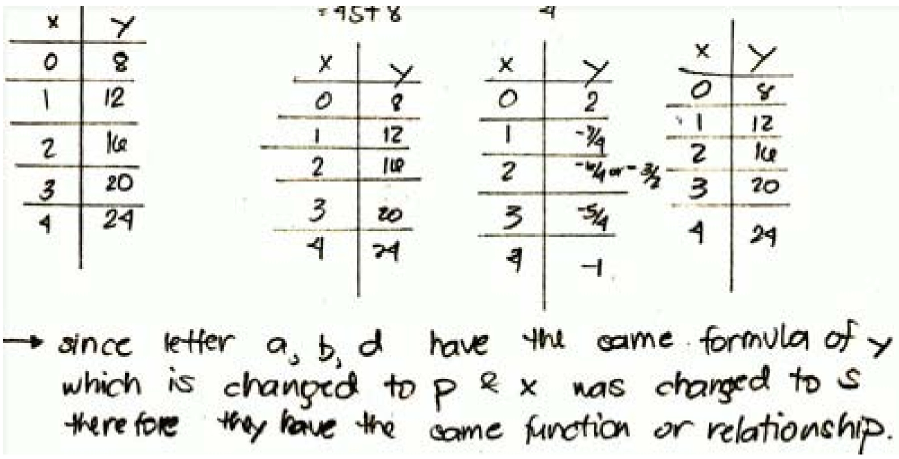
Students' solutions included evaluating the equations and the use of properties or structure of the equations. Many students chose only two equations. If they chose any two of *a*, *b* and *d* by substituting specific values in the equation, then the answer was coded Strategy 1. A sample of such a student solution is in Figure 77.



O2N5, Year 8

Figure 77. Task 18 - Strategy 1. Point by-point interpretations, letter symbols relevant.

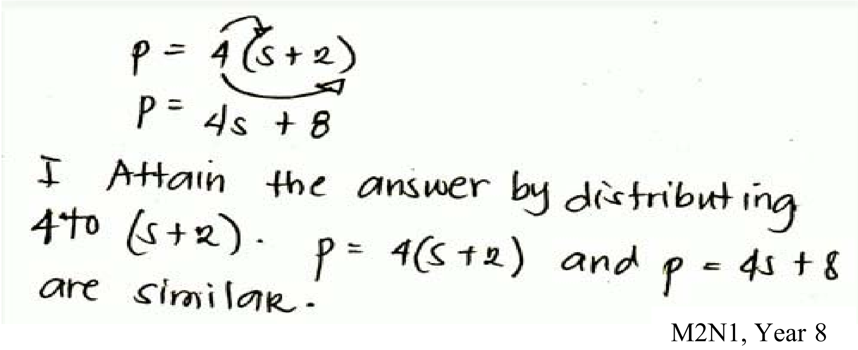
If students chose all the three correct answers, *a*, *b*, and *d* by substituting specific values in the equation, then the answer was coded Strategy 1.5. A student coded at this strategy understands the irrelevance of letter symbols used. A sample student solution is in Figure 78. In this solution, the student evaluated all three equations and compared their values.



B4B1, Year 10

Figure 78. Task 18 - Strategy 1.5. Point by-point interpretations with understanding of irrelevance of letter symbols used.

If students chose *b* and *d* after simplifying *b* or explaining that one is the factored form of another, then the answer was coded Strategy 2. A sample solution is in Figure 79.



M2N1, Year 8

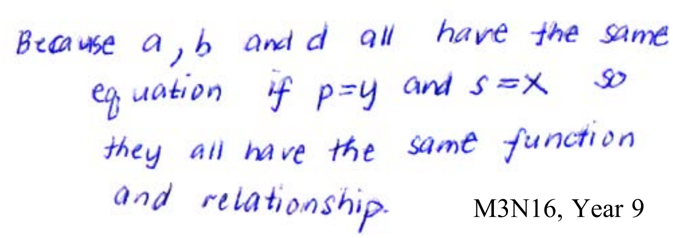
Figure 79. Task 18 - Strategy 2. Expressing into same form of equation but considered same letter symbols.

If the student's answer was *a* and *d* and they made a comment on the similarity of the structure of the equations or said that they had the same gradient and intercept or the same coefficient in *x* and same constant or other similar reasoning, then it was coded Strategy 3. A sample solution is in Figure 80.

B2D2, Year 10

Figure 80. Task 18 - Strategy 3. Form of the equations and letter symbols are irrelevant.

If *a*, *b* and *d* were chosen and then comment made on the similarity of the structure or irrelevance of the letter used or other similar reasoning, then the answer was coded Strategy 4. Shown in Figure 81 is typical student reasoning coded Strategy 4.



M3N16, Year 9

Figure 81. Task 18 - Strategy 4. Expressing into same form of equation with understanding of irrelevance of letter symbols used.

Table 37 shows the distribution of students coded the strategies in Task 18. Only responses coded Strategy 1.5 and 4 could be regarded as complete solutions to the task. The results indicates that there were very few students who understood that the function or relationship between variables is invariant with the change in letter symbols or change in the form of the equation. Furthermore, more students solved the task using Strategy 1.5 than Strategy 4 except for the Year 9 students in the second data collection period. This is again an indication of students' preference for point-by-point interpretations.

The majority of the students have incomplete solutions, that is, selecting only two of the three correct equations. The incomplete solutions are the ones coded Strategy 1, 2 and 3, depending on their justification. Among these three strategies, Strategy 2 has the highest frequency of students which is to be expected since all students need to do was to simplify the equation. Simplifying expressions and equations are familiar tasks. The parenthesis in choice *b* must have been the signal to the students to perform an operation on the equation.

Table 37
Distributions of Students at the Different Responses for Task 18

Responses	Year 8 (n = 149)		Year 9 (n = 152)		Year 10 (n = 143)	
	D1	D2	D1	D2	D1	D2
Strategy 1.5 – Point-wise, letter symbols irrelevant	3	10	14	23	22	31
Strategy 4 – Form and letter symbols irrelevant	0	10	5	26	8	29
Strategy 1 – Point-wise, letter symbols relevant	15	21	38	18	17	11
Strategy 2 – Form and letter symbols both relevant	23	70	35	53	59	48
Strategy 3 – Form relevant, letter symbols irrelevant	14	14	20	19	16	14
No answer	38	2	16	3	6	3
No explanation	9	4	2	1	0	2
Other incorrect responses	47	18	22	9	15	5

The incomplete solutions, Strategy 1, 2 and 3, were still considered in assessing students' growth points. This is explained in the next section, which describes the growth points, and how they were assessed.

There were three growth points identified under the Equivalent Relationships domain. Students must complete all the three tasks to be given the growth points. This means that students coded in any of the growth points could identify equivalent relationships in each of the three representational systems – graphs, tables and equations. They were coded at the growth points according to the strategies they used to complete the tasks. The growth points and the procedure for coding are described below.

Growth Point 1: Representations show equivalent relationships if they have the same set of values

Students at this growth point identified equivalent relationships through showing that the functions have the same set of values. Evidence of students' thinking at this level included solutions involving evaluating equations, plotting points or comparing ordered pairs. Students were coded at this growth point if they completed all three tasks and were coded Strategy 1 in at least one of the tasks.

Growth Point 2: Representations show equivalent relationships if they share some properties

Students at this growth point identified equivalent relationships through showing that the relationships shared some properties, including sharing the same equations for Task 16 (Graphs) and Task 17 (Tables). Thus, to be coded at this growth point, students must not have a code of Strategy 1 in any of the tasks and not qualify to be coded Growth Point 3.

Growth Point 3: Representations show equivalent relationships if they have the same invariant properties

Students at this growth point showed evidence of understanding the invariant properties of functions like irrelevance of letter symbols used, which was assessed by Task 18, and invariant properties of function such as intercepts and slope for linear relationships (Tasks 16 and 17). Student coded at this growth point were either coded Strategy 4 in Task 18, and coded Strategy 2 in the other two tasks or, they were coded Strategy 3 in Task 18 and coded Strategy 3 in Task 16 or Task 17. This means that the minimum achievement of students coded at Growth Point 3 was a solution involving the use of invariant properties of the relationships in at least one of the representations and use of patterns or some local properties in the other two representations.

Investigating for Typical Learning Trajectory

The preceding discussion described the growth points and the corresponding assessment tasks. This section presents the result of the investigation of the patterns in the order of the growth points.

The same tasks were used to assess each of the growth points. Thus, a student coded in any one of the growth points could not be coded in other growth points. Because of this, using the percentage of students coded at each growth point may not be appropriate to show the trend in the order of the growth points. The growth points with the highest frequency could not be interpreted as the first growth point that was first learned. It may be that during the time of the data collection, the majority of the students was already thinking at that level. Hence, in order to show a general trend in the order of the growth points, students' movements between the growth points in the two data collection periods were compared. Results of the cross-tabulation of students' achievement in the growth points in the two data collection periods confirmed the trend. The results for Year 8 are shown in Table 38 below. In the table, D1 refers to the first data collection and D2 refers to the second data collection.

Table 38
Cross-tabulation of Year 8 Students Achieving the Growth Points under Equivalent Relationships in Data Collections 1 and 2

		D2				
	Growth Points	GP 0	GP 1	GP 2	GP 3	Total
D1	GP 0	127	16	3	1	147
	GP 1	1	1	0	0	2
	Total	128	17	3	1	149

There were only two Year 8 students who were coded Growth Point 1 in the first data collection period. The rest were in Growth Point 0. This is expected since when the first data collection occurred, the Year 8 students were just starting to be introduced to the topic, and hence they could not yet be expected to understand the concept of equivalent relationships. In the second data collection, sixteen of the 147 Year 8 students coded GP 0 in the first data collection period, advanced to GP 1, three moved to GP 2 and only one moved to GP 3.

The decreasing number of students who were initially at GP 0 at the beginning of the year that advanced to GP 1, GP 2 and GP 3 indicated the order in which the growth points are likely to be acquired. This trend is also confirmed by data from Years 9 and 10 students shown in Tables 39 and 40 respectively.

Table 39
Cross-tabulation of Year 9 Students Achieving the Growth Points under Equivalent Relationships in Data Collections 1 and 2

		D2				
	Growth Points	GP 0	GP 1	GP 2	GP 3	Total
	GP 0	124	9	3	4	140
D1	GP 1	3	1	2	1	7
	GP 2	4	0	0	0	4
	GP 3	1	0	0	0	1
	Total	132	10	5	5	152

In Year 9, nine of the 140 students who were at GP 0 during the first data collection period moved to GP 1, three moved to GP 2 and four moved to GP 3. The trend was more pronounced in Year 10, where the number of students initially at GP 0 that moved to GP 1 to GP 2 to GP 3 was decreasing. The frequency of Year 10 students at each growth point was 18, 12 and 5. These decreasing numbers indicate that GP 1 is achieved first, followed by GP 2 then GP 3. Further confirming this trend is the movement of students who were at GP 1 during the first data collection period. In the second data collection period, two of the 15 students moved to GP 2 and none moved to GP 3. In Year 9, two of the seven students moved to GP 2 and only one moved to GP 3.

Table 40
Cross-tabulation of Year 10 Students Achieving the Growth Points under Equivalent Relationships in Data Collections 1 and 2

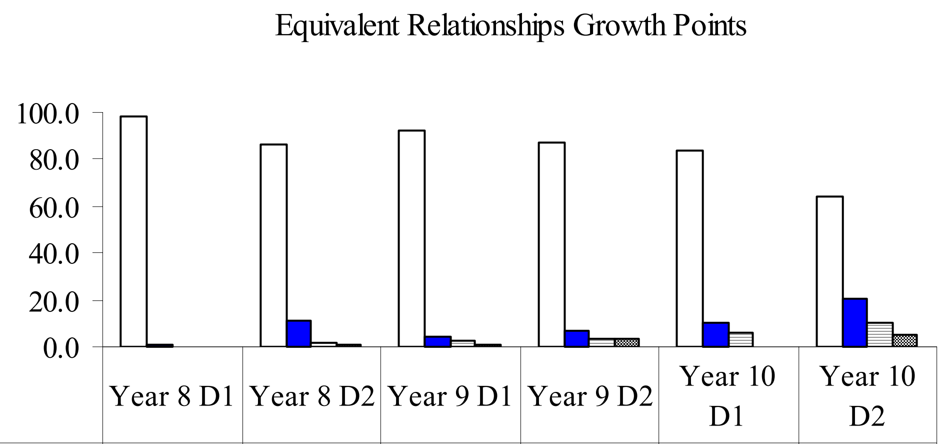
		D2				
	Growth Points	GP 0	GP 1	GP 2	GP 3	Total
	GP 0	85	18	12	5	120
D1	GP 1	6	7	2	0	15
	GP 2	1	4	1	2	8
	Total	92	29	15	7	143

Some students slid back to previous growth points. As expected, the proportion of students moving back to previous growth points was bigger in Year 9 than in Year 10. For example, among those who were initially at GP 2 during the first data collection period, all four of the Year 9 students slid back to GP 0, while four out of eight Year 10 students slid back to GP 1 and only one of the eight went back to GP 0. That is, the Year 10 students were still able to complete the tasks while the Year 9 failed to complete the tasks. This shows that

Year 10 students have a firmer grasp of the concept of equivalent relationships than do the Year 9 students. That half of the Year 10 students slid from GP 2 to GP 1 indicates that students are likely to revert to point-by-point interpretations when faced with unfamiliar situations.

Other Findings

Equivalent Relationships was a surprisingly difficult domain. Figure 82 shows the percentage of students coded at the growth points. The majority of the students were at GP 0.



	Year 8 D1	Year 8 D2	Year 9 D1	Year 9 D2	Year 10 D1	Year 10 D2
GP 0	98.7	85.9	92.1	86.8	83.9	64.3
GP 1	0.7	11.4	4.6	6.6	10.5	20.3
GP 2	0.0	2.0	2.6	3.3	5.6	10.5
GP 3	0.0	0.7	0.7	3.3	0.0	4.9

Figure 82. Percentages of students coded at the growth points under Equivalent Relationships.

Year 8 data showed that almost 99% of the 149 of the students were at GP 0 at the beginning of the year. This percentage decreased only slightly to 86% during the second data collection period. In Year 9, 92% of the 152 students were at GP 0 at the beginning of the school year. The percentage decreased only slightly to 87% during the second data collection period. The results also showed that the achievement of Year 8 and Year 9 were almost the same in the second data collection period indicating the concept of equivalent relationships is indeed difficult. It further indicates that this concept is not emphasised in the teaching of function to these students.

The percentage for Year 10 students who were in GP 0 in the second data collection period is small compared with Year 8 and 9 but 64% is still a large percentage considering their experiences with function. In fact, when the second data collection occurred, all the Year 10 classes that participated in the present study had completed the study of at least three families of function: the polynomial functions (linear and quadratic relationships); exponential functions; and, circular/ trigonometric functions.

Students' understanding of equivalent relationships in the different representations is compartmentalised. That is, they can identify equivalent relationship in graphical form, for example, but not in the other representations. Table 41 shows a summary, comparing their performance in the three tasks in the second data collection period.

Table 41
Cross-tabulation of the Number of Students Completing Correctly the Assessment Tasks in the Second Data Collection Period

	Task 17: Tables (n = 132)	Task 18: Equations (n = 62)
Task 16: Graphs (n = 172)	92	55
Task 17: Tables (n = 132)	--	46

Data in Table 41 show that of the 444 students, 172 were able to identify equivalent relationships in graphical form presented in Task 16. Of these 172, only 92 students were able to identify equivalent relationships in tabular form presented in Task 17, and only 55 were able to identify equivalent relationships in equation form presented in Task 18. Similarly, of the 132 students who completed Task 17, only 46 students completed Task 18. For Task 18, however, only those who selected all the three correct answers, that is, those coded Strategy 1.5 and 4 are counted and included in Table 41. This is why there were only 62 students. If the partially correct answers are to be included, which was considered in the coding for Growth Points 1 and 2, then it would be 402 students rather than just 62.

Equivalent Relationships versus Linking Representations

Equivalent Relationships and Linking Representations are related domains because both involved linking representational systems. The growth points identified in each of these domains are descriptive of strategies students used to work out the tasks. It would be interesting therefore to investigate whether students who could identify equivalent relationships in terms of a particular strategy are also more likely to use the same to link different representational systems.

Table 42 shows a summary of the comparison of the growth points in the two domains. Because almost all those who achieved at least GP 2 under Linking Representations also achieved GP 1 in this domain, the table started with GP 2. Growth Points 4 and 5 under Linking Representations were also combined since both involve interpretations of invariant properties. The data used in the table come from the second data collection period since there were more students coded the growth points in this period than in the first data collection period.

Table 42
Cross-tabulation of Students Coded the Growth Points under Equivalent Relationships and Linking Representations Domains

Equivalent Relationships	Linking Representations			Row Total (n = 204)
	GP 2: Point-wise (n = 157)	GP 3: Some Properties/ Trend (n = 28)	GP 4 & GP 5: Invariant Properties (n = 19)	
GP 1: Point-wise (n = 55)	23	2	3	28
GP 2: Some Properties (n = 23)	11	4	3	18
GP 3: Invariant Properties (n = 13)	7	0	0	7
Column Total (n = 91)	41	6	6	53

Out of the 91 students who could identify equivalent relationships in all three representations, only 53 could also link representations. Furthermore, the results showed that those who could identify equivalent relationships beyond point-wise interpretations did not use the same strategy to link representations. For example, of the 36 students in GP 2 and GP 3 under Equivalent Relationships, only seven (four at GP 3 and three at GP 4 and 5) used the same reasoning to link representations. Likewise, those who could link representations beyond point-wise techniques (28 at GP 3 and 19 at GP 4 or GP 5) did not use the same technique to identify equivalent relationships. This is an indication that Equivalent Relationships and Linking Representations are distinct domains. It also indicates that the students' understanding of function were still compartmentalised because they could not transfer knowledge in one domain to another domain, that is from Linking Representations to Equivalent Relationships and vice versa. Of course, the link between Linking Representations and Equivalent Relationships needed to be investigated further because the data in the study were too limited to make a strong conclusion.

Discussion and Summary

To assess students' developing understanding of function in terms of their understanding of equivalent relationships, students were given three tasks that required identifying equivalent relationships in graphs, equations and tables. Students' strategies in completing the tasks were classified and were used to describe growth points in understanding. Students were only coded at Growth Points 1, 2, or 3 if they were able to complete all three tasks.

The growth points identified under Equivalent Relationships domain were:

Growth Point 1: Representations show the same relationships if they share the same set of values.

Growth Point 2: Representations show the same relationships if they share some properties.

Growth Point 3: Representations show the same relationships if they have the same invariant properties.

The three growth points identified in this domain could also be located within the process-object and property-oriented paths, with Growth Point 1 nearest the process end and Growth Point 3 nearest the object end, reflecting an understanding of (linear) function as a permanent construct.

Students coded at any of the three growth points under this domain were able to identify equivalent relationships represented in all three representations – graphs, tables and equations – of linear functions. That there were just a few students coded at the Growth Points 1, 2, or 3 indicates that an understanding of equivalent linear relationship in all three representations is beyond the reach of the majority of the students, even those in Year 10. Of course, this may only be true of the participants of the study.

The students' difficulties in this domain lie in their understanding of properties that are invariant. With the tables (Task 17) for example, students needed to extend the tables (perform interpolation/ extrapolation) to make the tables look the same. Students who do this show an understanding that the relationship between the *x*'s and *y*'s in the table is not affected by additional values. The same could be said with the graphs in Task 16. Showing that graphs, which share the same set of ordered pairs, are equivalent, demonstrates an understanding that the relationship remains invariant with the change of scale in the axes. That students did not think of these solutions may also be due to their lack of experience in performing "action" with tables and graphs.

Students however, did not show lack of confidence in performing "action" on equations, which is to be expected since for many students equations signal action. For them, equations are there to be manipulated. The difficulty with Task 18 lies in the understanding that function would remain invariant in the equivalent forms of equation and in letter symbols used, something that may still be beyond the grasp of students with limited experience with functions.

Understanding of equivalent relationships in the three representations may not be parallel or may not go hand-in-hand, but in my opinion, for simple functions studied in secondary schools such as linear relationships, students, before leaving Year 10, should be expected to compare functions, at least those that are equivalent, within systems or between representational systems. Schwartz and Yerushalmy (1992) argued that the act of comparing functions is central to the discipline of algebra hence it is important that students be given these experiences at least with linear functions.

This chapter is the last chapter discussing separately the growth points for the four key domains of function considered in the framework. The next chapter consolidates the findings for the four previous chapters by presenting the final form of framework of growth points in students' understanding of function and discussion of key findings of the study.

CHAPTER 8

THE FRAMEWORK OF GROWTH POINTS

The findings and discussions presented in Chapters 4 to 7 are consolidated in this chapter. The chapter is organised into three sections. The first gives a brief summary of the research process, which led to the development of the final framework of growth points. The second section discusses the Framework of Growth Points that the study developed. The third section presents and discusses other key findings and outputs of the study.

Summary of the Research Process

The main objective of the study was to develop a framework of growth points in students' developing understanding of function. The aim of the framework was to provide teachers with a structure in monitoring and assessing students' developing understanding of this concept. The research started with the question:

> How might typical learning paths of secondary school students' developing understanding of function be described and assessed?

A review of related literature yielded an initial framework of growth points underpinned by process - object and property-oriented perspectives in the understanding of mathematical concepts. Thus, the present study described typical learning paths in secondary students' understanding of function by developing a framework of growth points in key domains of this concept that are typically included in secondary mathematics curricula.

The development of the framework was guided by the following specific questions:

1. What are the growth points in secondary school students' developing understanding of function?
2. What information on the students' understanding of function is revealed in the course of developing the framework of growth points that would be potentially useful for teachers?

The students' developing understanding of function was described in terms of growth points. Growth points as they came to be understood in the present study are "big ideas" in students' developing understanding of function. The descriptions of the growth points focus on students' knowledge, skills, strategies, and level of reasoning and abstraction involved and not merely on the completion of a particular task.

The study developed and enhanced the initial framework of growth points using both qualitative and quantitative approaches. The study was initially qualitative in organising the initial framework and identifying and describing the growth points. It then moved to a quantitative approach to identify trends in the order of the growth points, and finally moved back to a qualitative approach to refine the framework. The main data collection procedure used was a set of written-assessment tasks, but interviews using the tasks were also conducted to gain more insights on students' interpretations of the tasks and solutions.

The main data were collected from 149 Year 8 students, 152 Year 9 and 143 Year 10 students from three Regional Science High Schools in the Philippines. The data were collected twice, the first at the beginning of the school year and the second five months later, from the same students. Interviews were conducted with 18 students spread across schools and year levels. Four pilot studies were conducted prior to the main data collections.

Data analyses were based on students' strategies in working on the tasks and the nature of the tasks.

The Framework of Growth Points

The following discussion presents the final form of The Framework. The discussion is divided into three parts. The first part discusses the key domains of the function concept included in the framework, the second part discusses the growth points within these domains, and the third part compares growth points among the domains.

Key Domains

The domains in the initial framework were *Graphs*, *Tables*, *Equations* and *Modelling*. The pilot studies showed that students were more likely to do point-by-point analysis in almost all of the tasks including those in graphs and equations. As tables also lend themselves to this kind of analysis, it was decided not to make it as a separate domain, in order to simplify the framework. The aim was to encourage teachers to focus more on the function represented by tables and not on individual values within the table. Students' understanding of function in tabular form was therefore incorporated under a Linking Representations domain and an Equivalent Relationships domain.

Modelling was also deleted as a domain after the first two pilot studies because it was clear that the effect of context in students' developing understanding of function would require a separate study.

Linking Representations and Equivalent Relationships were not initially considered as major domains, but groups of tasks that would assess higher-level understanding of function under the domains Equations, Graphs and Tables. The concepts linking representations and equivalent functions require understanding of the connection between representations and of the ways in which they signify the same function or functional properties. However, the richness and range of students' strategies in the tasks showed that they should be considered as domains in their own right. Moreover, previous studies of students' difficulties have also shown persistent problems in linking information between representations (see, e.g., Leinhardt, Zaslavsky & Stein, 1990; Schwarz & Dreyfus, 1995). Hence, identifying growth points in students' developing understanding of the link between representations is a positive step towards addressing this difficulty. The inclusion, therefore, of Linking Representations and Equivalent Relationships as domains in their own right in the final framework of growth points enhanced the scope and depth of the framework in describing students' understanding of function.

If one would think of the process of understanding a mathematical concept as a growing network of conceptual nodes, the four key domains in the framework could be thought of as major conceptual nodes (see Figure 83). Of course, these are not the only key domains of the function concept. Function is a complex concept and it would not be possible for a single study to cover all key areas.

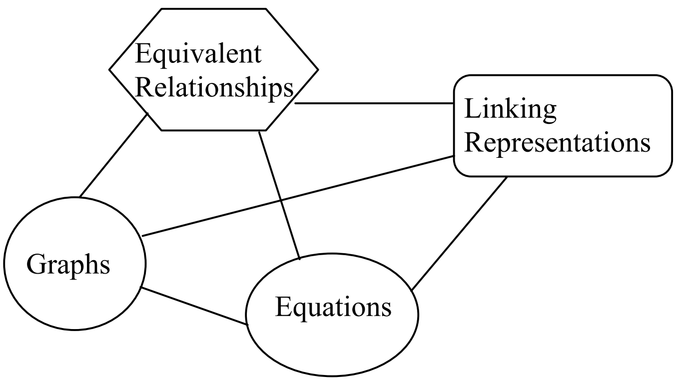


Figure 83. Major nodes in the network of students' understanding of function.

The representation of the concept is not the concept. That is, the representation (graph, table or equation) is not the function. Hence, although the major domains in the framework, except the Equivalent Relationships domain, refer to the representations of function, the descriptions of the growth points under the domains are not about the understanding of the representations but on the understanding of the concept of function. The growth points are presented below.

The Framework of Growth Points

The list of growth points in each domain is by no means exhaustive. Other growth points in the domain could be identified, but the focus here is on big ideas that are true to the data and have the potential to be useful to teacher-educators, researchers and teachers. A summary of the growth points in each domain is presented in a tabular framework shown in Table 43.

Table 43
The Framework of Growth Points in Students' Developing Understanding of Function

Equations	Graphs	Linking Representations	Equivalent Relationships
GP 1: Equations as procedures for generating values	GP 1: Interpretations based on individual points	GP 1: Linking equations and tables	GP 1: Representations show the same relationship if they have the same set of values
GP 2: Interpretations based on relationships	GP 2a: Interpretations based on rates	GP 2: Point-by-point linking of graphs with other representations	
GP 3: Interpretations based on local properties	GP 2b: Interpretations based on continuous property	GP 3: Linking representations by trends/patterns or some properties	GP 2: Representations show the same relationship if they share some properties
GP 4: Manipulation and transformation of equation seen as objects	GP 3: Interpretations based on point-wise and holistic analysis of relationships	GP 4: Linking by invariant properties	GP 3: Representations show the same relationship if they have the same invariant properties
		GP 5: Linking representations seen as objects	

It is acknowledge that teaching approaches, syllabus, textbooks and other instructional tools students have been exposed to might have had some effect on students' achievement of the growth points. The context of the problem and the conditions the students were at when the data were collected might also have had an effect on the result. But as far as the data gathered using the assessment tasks and the conditions present at the time of the data collection are concerned, the result of the present study holds.

The growth points in each domain in the Framework are arranged according to the likelihood that the majority of the students would acquire these levels of understanding. The order of the growth points were supported theoretically as well as empirically in the study as presented in Chapters 4 to 7.

The strength of linearity of the growth points within a domain, particularly that which involves two adjacent growth points, differs. Under the domain Equations, for example, during the second data collection period, the likelihood that those who achieved Growth Point 3 also achieved Growth Point 2 was 74%, while the likelihood that those who achieved Growth Point 4 also achieved Growth Point 3 was 89%. However, the order in which the growth points are acquired as presented in the framework held for the majority of the students.

Comparison between domains

In theory, all four domains should be connected as shown in Figure 83 since understanding is a function of the strength of connection between related concepts (Hiebert & Carpenter, 1992). However, the data did not show evidence that students' understanding had reached this level especially among the higher growth points in the four domains.

The results showed that the lower growth points under Graphs and Equations domains are acquired before the lower growth points under the other two domains. This is of course to be expected as these latter domains involve knowledge of the links among the representations. For example, except for five students out of 444 in the second data collection period, all those coded at least at Growth Point 1 under Linking Representations and under Equivalent Relationships were coded Growth Point 1 under Equations and Graphs. Those who could link equations with other representations (Growth Points 1 and 2 under Linking representations) could also make holistic interpretations of equations as representations of functional relationships (Growth Point 2 under Equations). The results also showed that 71% of 157 students from all the year levels coded at least a Growth Point 2 under Linking Representations were also coded Growth Point 2 under Equations as well. The same is true with Equivalent Relationships. Out of the 92 students who could identify equivalent relationships, 91% were coded at Growth Point 2 under Equations as well. However, there were 237 students coded at Growth Point 2 under Equations, which implies that this growth point is acquired earlier than Growth Point 2 under Linking Representations and the growth points under Equivalent Relationships.

While there is such a clear direction in the acquisition of the lower growth points, this could not be said among the upper level growth points among the domains. For example, of the 22 students coded Growth Point 4 under Equations and of the 18 students coded Growth Point 4 or 5 under Linking Representations, only two students were both coded at these growth points. Also of the 23 students at the higherst growth point under Graphs, only six students were coded Growth Point 4 under Linking Representations. Similarly, of the 13 students coded the highest growth point under Equivalent Relationships, only two were also coded the highest growth point under Equations and three under Graphs. These results are indications that the growth points under the domains Linking Representations and Equivalent Relationships involve more than just conceiving function under individual representations as objects; that in fact, concepts required under these two domains are different from those under Equations and Graphs domains.

There were students who could link graphs with other representations who were not able to identify equivalent relationships and vice versa. In addition, the students who could use a particular strategy to link representations are not likely to use the same strategy to identify equivalent relationships. This is an indication that the students who participated in this study were still not capable of transferring their knowledge and skill in Linking Representations to Equivalent Relationships domain even though both involve linking representations. This could also be construed as evidence that indeed these two domains represent distinct ideas hence they have to be considered major nodes or domains.

Levels of Understanding

The growth points included in the framework that the present study developed may fall into at least one of the following levels of understanding: Point-wise understanding; relationship-wise understanding; property-wise understanding; and, object-wise understanding. These levels of understanding more or less correspond to the stages identified by other authors but are named and described here in the present study in terms of the functions taught in secondary schools. As already mentioned, the study has incorporated understanding in terms of the properties.

The levels of understanding described below should not be viewed as discrete. They are only classified here for easy recall of teachers.

Point-wise understanding of function is evidenced by a point-by point interpretation of the representation. Growth Point 1 in each of the four domains including Growth Point 2 under Linking Representations is located in this node. These growth points are the entry level of understanding in students' developing understanding of function in their respective domains. However, the growth points involving point-wise interpretations in all the domains do not belong at the same level of understanding. For example, the point-wise interpretations under Linking representations and Equivalent Relationships involve higher-level understanding of function, even though students operate in terms of individual values.

Working with function holistically marks *relationship-wise* understanding. Students working at this level could understand the relationship that is represented and that this relationship is true to all individual values as defined in the domain of x and not only for a single value or set of values in the domain of the function. The understanding however at this level is not enough to conceive the function as an object in itself. Achievement of Growth Point 2 under Equations and Growth Point 2 under Equivalent Relationships are also evidence of conceiving function at this level. Growth Point 1 and Growth Point 2 under Linking Representations and Growth Point 1 under Equivalent Relationships could also be classified under relationship-wise understanding. Students at this level may still work using point-wise interpretation, but they already understand the function or relationship represented by tables, graphs and equations.

Property-wise understanding is marked by interpretation and analysis using the global, local and invariant properties of the function. Growth points classified under property-wise understanding involving local properties include Growth Point 3 under Equations and Growth Points 2 and 3 under Graphs. Growth Point 3 under Linking Representations is reflective of understanding of function using its global features. Growth Point 4 under Linking Representations and Growth Point 3 under Equivalent Relationships involve understanding the invariant properties of the function.

In the framework, all growth points involving interpretations based on properties come after the Growth Point 1s, indicating that interpretations based on the properties of the function whether local, global or invariant require at least point-wise understanding.

Object-wise understanding is evidenced by interpretation or analysis in terms of invariant properties of the function and/or working with the function as objects that can be manipulated or transformed. All the highest growth points in each domain including Growth Point 4 under Linking Representations tasks are considered reflective of an understanding of function as a permanent construct or as a mathematical object at least within the universe of linear and quadratic relationships.

Other Findings of the Study

The consolidation of the findings of the study in relation to the first research question was presented in the previous discussion. This section presents a summary of key findings of the study in relation to the second question, *What information on the students' understanding of function is revealed in the course of developing the framework of growth points that would be potentially useful for teachers?*

Difficulty with Working with Function beyond Point-wise Analysis

The majority of the students were only at the first two growth points under Graphs and Equations. Many could link the representations and identify equivalent relationships through point-by-point interpretations only.

The theory of epistemological obstacle explains that "knowledge is dialectically constructed both from and against previous knowledge, [thus], as far as some piece of knowledge has turned out to be successful in a wide range of situations, it becomes resistant to change, even if it must be strongly modified in order to cope effectively with new problems" (Artigue 1992, p. 10). This probably explains students' preference for point-wise thinking even if the task makes applying this approach tedious. One way to overcome this obstacle is to use tasks where students are confronted with situations where point-wise analyis is too cumbersome. This would provide a context for introducing a more holistic solution to the tasks. However, this should be done with caution. The students could be exposed to the idea of working with function as object or analysing it in terms of it invariant properties but not required to master it. Requiring mastery might lead to instrumental understanding or misconceptions especially for students who are not ready for this leap. The idea of function as an object, "as a static 'thing', when introduced too early is doomed to remain beyond the comprehension of many students" (Sfard, 1992, p. 77).

Compartmentalisation of Students' Understanding in the Different Representations

One of the fundamental conditions of understanding functions is to understand that the representations represent the same general concept (Sierpinska, 1992). The results of the present study showed that for many of the students, the understanding of functions in the different representations as well as in the different domains are still compartmentalised. Previous investigators had also observed this persistent problem of compartmentalisation of students' understanding of function in the different representations (see, e.g., Leinhardt, Zaslavsky & Stein, 1990; Schwarz & Dreyfus, 1995). The study confirmed that this is true even with the simplest of function as indicated by the students' performance, especially in the Equivalent Relationships domain. The tasks assessing students' understanding in this domain involved only linear relationships yet the tasks proved difficult.

Translating versus Linking

The ability to translate from one representation to another does not necessarily mean students understand the link between the representations. In the linking representations tasks, the students who could plot the points correctly failed to identify the correct graph, which indicates that mere translation exercises between representations are inadequate for students to link information between the representations and between the functions represented. The latter require knowledge of the properties of the function as well as the representations. This further implies that tasks and activities in mathematics should be much more than asking students to translate between representations or asking students to show the function in different representations. Learning activities should be designed so that students would be able to understand the link between the information and properties of the function in each representation, as well as understand the corresponding effect of performing procedures in one representation to the other representations.

Difficulty with Rate

The students' performance involving interpretations of rate or growth property of function represented by equations and graphs showed that rate is not an easy concept to interpret from these representations. This is because the concept of rate requires much more than reading points in a graph or substituting values in an equation. It requires coordinating change in the x's with that of the y's. However, for the students who participated in the study who belong to the above average group, the low success rate should be a cause for alarm. It appears that these students were not familiar with tasks involving the interpretations of rate from equations and graphs. The mathematics syllabus, textbooks and teaching and learning experiences provided for these students need to be revisited.

Students' strategies in dealing with rate could also be classified into differing levels of abstraction like global (visual) versus analytical or point-wise versus interpretation of parameters for equations. Since the study only considered a small aspect of the concept of rate in functions it is suggested that a study be conducted that would identify and describe growth points in students' understanding of this most fundamental property of function. This would greatly enhance the present framework of growth points.

Difficulty with Interpreting Continuity of Function in Graphs

The results of the study showed that students had difficulty interpreting continuous intervals, especially the endpoints of intervals in the graph. Furthermore, the results showed that there is little difference between the percentages of students coded at this growth point between the year levels. It is hard to tell whether this result is unique to these participants or not. As explained earlier, the difficulty may be in the ambiguity of the wording of the task, in which case the result is not unique to this group of students, or it could be because this is not given emphasis in the teaching, which means the result is true for this particular group only. This therefore needs further investigation.

Other Outputs of the Study

Aside from the Framework of Growth Points, which is the main output of the study, the assessment tasks and the corresponding rubric are also significant outputs of the study. The assessment tasks and the rubrics were developed hand in hand with the framework. Teachers could use the assessment tasks and the rubrics to assess and monitor their students' understanding of function. The assessment tasks could also serve as model in developing other assessment tasks. They can use the diagram in Figure 84 below, which served as a framework for the study in developing and analysing the tasks and in identifying and describing the growth points, in developing other assessment tasks.

	Process	⟶	**Object**
Point of analysis	• Individual points	• Set of points, interval	• Whole representation, Relationship
Strategies (action performed)	• Perform series of same procedure (point-by-point)	• Combination of points and whole representation	• Perform general operation on the representation
Interpretations based on properties	• Local properties	• Trends and patterns and local properties	• Invariant properties

Figure 84. Guide for analysing and designing assessment tasks.

The general conclusions, implications and recommendations of the study are presented in the final chapter.

CHAPTER 9

IMPLICATIONS AND RECOMMENDATIONS

The purpose of the study was to describe secondary students' developing understanding of function, which is based on the consideration of understanding as a growing network of conceptual nodes that is continuously being constructed and reorganised and, of learning of mathematics as a dynamic, multilevel process, where "mathematics acted out on one level becomes mathematics observed on the next" (Freudenthal, 1978, p. 33). The students' understanding of function was described in terms of a framework of growth points.

Contribution to Knowledge

Research-based

Existing frameworks on students' understanding of function are theoretical frameworks. While theoretical frameworks are useful in researching and analysing students' understanding of function, they are usually too general to be useful for classroom teachers, especially in monitoring and assessing students' understanding of this concept (see, e.g., the frameworks described in Chapter 2 of this report). The framework of growth points that the present study developed is research-based. It is descriptive and normative. It shows typical learning trajectories in the understanding of function in terms of "big ideas", described in terms of the level of "action" and reasoning students do, so that teachers would be encouraged to focus on these processes and not just on the outcomes.

Emphasis on Links between Representations

The framework of growth points that the study developed is related to the theoretical frameworks developed for analysing students' understanding of function, which combine the representations of functions and the process-object perspectives of function. The framework of De Marois and Tall and that of Moschkovich, Schoenfeld and Arcavi described in Chapter 2 are examples of these. The present study considered the three representations: equations, graphs and tables, but highlighted the links between the different representations of functions and students' understanding of equivalent relationships in these three representations, by describing growth points in students' understanding in these domains. The framework of growth points de-emphasised the use of tabular representation in assessing students' understanding of function by not making it a major domain, but incorporated it with domains involving linking representations, so that the focus would be on the functional relationships represented by tables and not on individual values. As an assessment tool, the framework aims to communicate to its users what is valued (Clarke, 1989) in the understanding of function.

Combined Process-Object and Property-Oriented Perspectives

Existing frameworks for analysing students' understanding of function identified stages in students' understanding of function in terms of the process-object route. However, the process-object route does not include understanding involving the properties of function. The framework that the study developed also included understanding in terms of the properties of function, local, global and invariant, as a means towards conceiving the function as a permanent construct.

Emphasis towards Abstraction and Generality

In the literature review of this thesis, a three-dimensional image was used to show key components of the function concept that should be considered in analysing and describing students' understanding of the concept. In one dimension are the different representations of function. In the second dimension are the situations that make the concept meaningful, which include tasks such as linking representations, identifying equivalent relationships, as well as applying the concept of function real life situations (modelling situations). In the third dimension are the different conceptions of function: the function as a process, as a mathematical object and as a concept possessing its own properties. The Framework of Growth Points developed in this study highlighted the vertical dimension by identifying and describing growth points in terms of abstraction and generality involved in the students' thinking, to make teachers become more aware of the different conceptions of function as this path points towards having a more abstract notion of the concept.

Focus on High School Students' Understanding of Function

The process-object and property-oriented perspectives have been used elsewhere to analyse students' understanding of function at the college level. The present study used these perspectives to describe students' developing understanding of function in terms of function as studied in secondary schools, thereby bridging the gap between the focus of mathematics taught in high school and at the college level.

Focus on Relational Understanding

The growth points are described in terms of big picture ideas to focus on relational understanding (Skemp, 1986), rather than on the outcome-based competencies reflected in most curricula.

Implications for Teaching and Learning

The present study aimed to contribute to bridging the gap between research and practice. The research-based framework of growth points was intended to provide teachers with a concrete map of students' developing understanding of function. The results of the study have implications for the following aspects of the teaching and learning process of this concept.

Teaching Sequence

The results of the study point to a need to revisit the teaching sequence for function. Because students' understanding of function proceeds in general from point-wise understanding of function, a teaching sequence could be designed based on students' understanding of function from point-wise to object-wise understanding of function as well as the understanding of properties of function in increasing abstraction. That is, students in the lower years could be given more tasks requiring lower level growth points than higher-level growth points initially. As the students move through the year levels they could be given tasks progressively requiring interpretations beyond point-wise thinking. This implies that the teaching of function at each level need not be confined to one particular kind of function only or using one particular representation only. Tasks requiring point-wise interpretations of individual points and interpretations of trends and patterns (global properties), even for functions traditionally introduced after linear function such as quadratics and exponentials and other general curves, could be introduced earlier. This could be done with graphical representations since point-by-point interpretations with graphs, whatever the shape, require the same skill. The same could be said with tables. Of course, the story is very different with equations. Students who have not learned what x^2 means could not necessarily be expected to generate values from a quadratic equation. The practice therefore of teaching the different kinds of function exclusively whether in different year levels or within a year level is not in keeping with the findings of the study.

Teaching approaches

The result of the study showed that students had difficulty working beyond point-by-point analysis. They stuck to this strategy even if it was tedious to do so. This is an indication that they were not confident in analysing a function in terms of its properties or conceiving it as an object. It could be assumed that tasks and activities given in the classroom are a major influence of the kind of students' understanding of function. Hence, to address the problem of students' difficulty in working with function beyond point-wise analysis, tasks could be structured so that point-wise analysis would be too cumbersome. This would provide a situation where they would seek a more efficient way of working on them. Graphs for example, need not only be on grids. Gridless graphs would encourage holistic and property-wise analysis. Values in tables need not always be presented in an orderly way. Studies have found that ordering the values in a table constrains students' recognition of the generality of the relationship between the dependent and independent variables (see e.g., Ryan & Williams, 1998). Students should also be encouraged to talk about the equation as "the relationship" rather than just using it for solving and generating values.

The use of calculators and function software in the teaching of function also have the potential to lead to a more holistic understanding of this concept. These tools provide immediate feedback, their graphs are usually gridless and the properties of the function and the effect of parameters to the graphs could be investigated. But teachers should understand that these technologies are just tools and the use of these *per se* does not encourage holistic thinking. The problems or tasks should be well-designed so that these tools are not just used for checking results. Furthermore, there are also misconceptions and difficulties that arise from the use of graphing calculators and function software (Penglase & Arnold, 1996), and so teachers should be aware of these in designing instruction.

Curriculum / Syllabus

The 1998 Revised Secondary Education Curriculum of the Philippines on which the curriculum of the Regional Science High Schools who participated in this study was based, starts with the linear function in Year 8, quadratics in Year 9 and at least three other functions in Year 10. In this approach, students are required to master all aspects of the function studied in the particular year level, including those requiring object-wise understanding. This does not correspond to the students' typical learning trajectory outlined in the framework of growth points. The results of the study showed that students could work on tasks involving quadratics using point-wise analysis but they had difficulty working on tasks involving simple linear functions but requiring analysis in more general terms. The 2002 secondary mathematics curriculum, which put the study of all classes of functions into Year 10, is not an improvement on the 1998 curriculum in terms of a teaching sequence for function. As found in this research, the study of function needs to be distributed across the year levels, but not in terms of content or class of functions but in terms of cognitive requirements as reflected by the growth points. In this curriculum, students are made to wait until Year 10 to learn and apply functions in investigating real-life situations. Tasks requiring point-wise and relationship-wise understanding of function may be too late for these students. All these imply the need for a review of the study of function in high school mathematics in the Philippines.

More Tasks Requiring Holistic Interpretations

Results showed that the majority of the students even at Year 10 were working in terms of point-wise analysis. This seems to indicate that conceiving function at a more general, abstract level is not emphasised in the teaching. There is therefore a need to focus the teaching so that students are required to use analyses beyond point-wise techniques. But as mentioned earlier, this should be done with caution. Sfard (1992) warned that tasks requiring object conception, when given too early, might alienate many students. Hence it is recommended that tasks should be designed so that they could be solved point-wise as well as in more holistic form. Many of the assessment tasks designed in this study are of this nature. These tasks and the diagram in Figure 84 may be used as a basis for designing other tasks.

The Role of Definitions

In the light of the findings of the study, where point-wise understanding precedes holistic, object-wise understanding of function, starting the teaching of function with definitions, which sets up the concept as an object, is not in keeping with the way students naturally make sense of this concept.

Use of Technology

The results of the study have shown that students had difficulty with tasks involving linking representations especially when they involve graphs not on grids. They also had difficulty identifying equivalent relationships. The use of graphing calculators and function software has the potential in facilitating the understanding of the links between the different representations of function. Calculators make possible the investigations of equations and their corresponding graphs. Most function software also allows the study of function in multiple representations.

Implication for Pre-Service and In-Service Programs

It would be difficult for teachers to use the framework of growth points if they have not acquired personally at least the highest level of understanding of function the framework demands and do not appreciate the importance and necessity of guiding students into the higher level growth points. This has implications for the emphasis on the mathematics and the teaching of mathematics, particularly function, in pre-service and in-service programs.

Recommendations for Further Research

The following discussion outlines recommendations for further research.

Teachers' Feedback

The study set out to develop a framework of growth points describing typical learning paths in students' developing understanding of function that had the potential to provide teachers with a structure in assessing and monitoring students' growth in the understanding of the function concept. However, the researcher, in consultation with the research supervisors and other mathematics educators and in the light of the data collected made the decisions on the form of the framework that would potentially be useful for teachers. There is a need therefore to have teachers try the framework or organise a seminar so that teachers could comment on the framework as well as on the descriptions of the growth points, if indeed it makes sense to them. In addition, because the framework was designed to provide teachers with a structure and a tool for assessing and monitoring students' understanding of function, there is a need to conduct a study on the effectiveness of the framework as an assessment tool and its effect on teaching and learning.

Effects of Teaching Approaches and Content

Given that the classroom experiences of students involved in the study may have varied considerably, the study has not been able to "tease out" the contribution of varying teaching approaches and content to students' understanding from the "natural" development of understanding. Moreover, it would have been ideal if data were also collected in Melbourne using the final set of assessment tasks. The results could have been compared and thus a stronger case could be made that the framework is not necessarily culture-bound and holds even for students who were provided with quite different learning experiences.

The study used data from two countries. Given that students' contact with function tends to be limited to the mathematics classroom, there would likely be some curriculum impact on the results. Investigations in other countries are needed to investigate whether the framework is more universal.

The students who participated in the present study belong to high-performing schools. It would have been ideal if the same set of assessment tasks was given to a low performing group and the results compared. Again, a stronger case could be made that the framework holds true for groups with different experiences, levels of knowledge, and understanding.

Need for Interview

The majority of the students who participated in the study were articulate enough and seemed used to explaining their thinking and solutions. However, written assessment may not yield the same richness in students' responses if they were given to other students. As per results of the pilot studies, many of the students from the low-performing group left the tasks unanswered and managed to give correct answers only to a few straightforward tasks. Hence, an interview using the assessment tasks is strongly recommended in assessing students' understanding for groups who may have difficulty in communicating their thinking, especially for students where English is a second language and mathematics is taught in English, as in the Philippines.

Graphing Calculators and Function Softwares

The study did not use technology such as graphing calculators and function software in the assessment of the growth points or in the identification of students' understanding of function. It is possible that more growth points could have been identified or perhaps the distribution of students at the growth points may have been different in this situation. This would be a worthwhile follow-up study.

More Domains

The present framework of growth points in students' developing understanding of function is far from complete. Function is a complex mathematical concept and there are other important domains not covered by the framework. For example, as mentioned previously, there is a need to identify and describe growth points involving secondary students' understanding of function in terms of rate or in general, the properties of the function. Slavit's levels (see Chapter 2) and property-based growth points in the framework could be a starting point but they need reinterpreting in terms of the functions included in secondary mathematics courses. Modelling is an important aspect in the learning of function, and identifying growth points in students' understanding of function under Modelling would enhance greatly the present framework.

Conclusion

This study identified growth points in the network of students' understanding of the function concept. Growth points are "big picture ideas" described in terms of the knowledge, understanding, strategies and reasoning students use to work on function tasks. The growth points were set in a framework to describe students' typical learning paths leading towards generality and abstraction in students' thinking in key domains of the function concept. The framework of growth points and the related assessment tasks provide a structure for teachers, teacher educators, researchers and students to think about function, and a means by which students' developing understanding can be assessed, monitored and enhanced.

In the course of the development of the framework of growth points, the study also identified key difficulties that students had in the learning of function and which teachers should take account in their teaching.

The study is of particular use to the teachers in the Philippines as it has provided a clear picture of what students in the science high schools in the Philippines know and can do in the broad domain and sub-domains of function, and raises questions about teaching approaches used in the classroom and the content of the current curriculum.

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APPENDIX A

ASSESSMENT BOOKLETS

Name:_________________________Sex:_____Age:______ Code: ________

Teacher’s name:______________________School:____________Date: _________

These questions are part of a research project designed to find out
what students know and can do in mathematics.

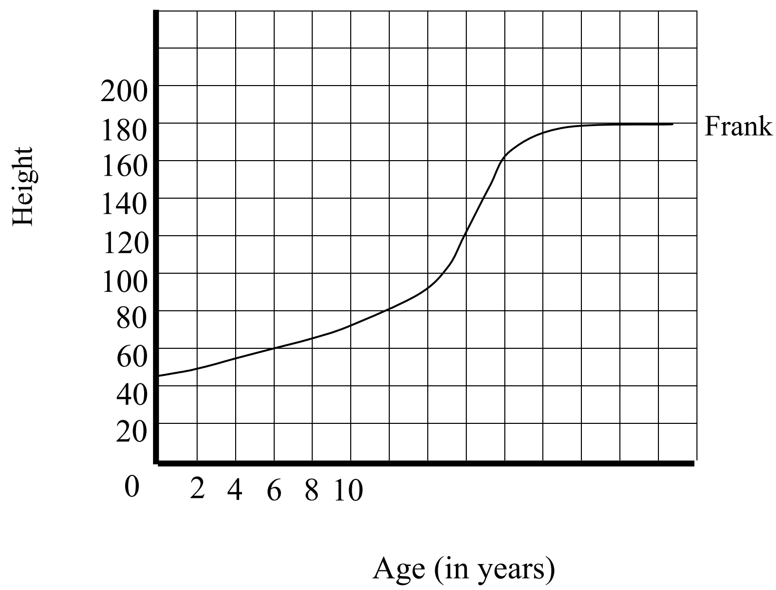
Booklet 1

I hope you find the questions interesting.

Erlina R. Ronda

Directions: Please answer the following questions. Write your answers on the space provided. Use the back page of each sheet if the space provided is not enough. You can use Filipino to explain your answers.

The graph below shows Frank's height and age.



Use this graph to answer questions 1-3.

1. When Frank was 140 cm, how old was he?

2. About how many centimetres did Frank grow between the age 6 and 12?

3. a. In which period was Frank's rate of growth the fastest? Circle the letter corresponding to your choice and explain why you chose this answer.

   a) Age 6 to 12 years b) Age 12 to 16 years c) Age 16 to 18 years d) Age 18 to 22 years e) Age 22 to 24 years

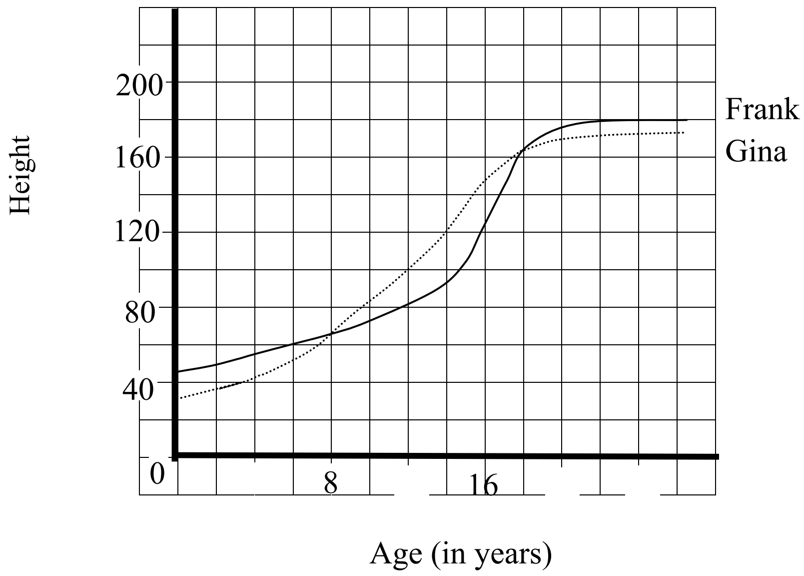
   Explanation:

   b. In which period was Frank rate of growth the slowest? Circle the letter corresponding to your choice and explain why you chose this answer.

   a) Age 6 to 12 years b) Age 12 to 16 years c) Age 16 to 18 years d) Age 18 to 22 years

   Explanation:

The second graph (in broken line) shows also the height of Frank's twin sister, Gina.

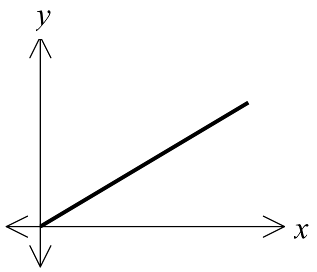


Use the graphs to answer questions 4 and 5.

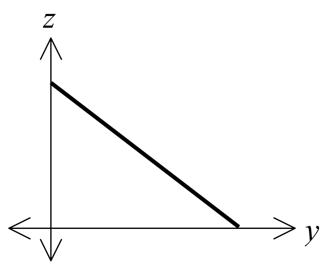
4. When did they both have the same height?

5. When was Frank taller than Gina?

6. Graph 1 shows how *x* is related to *y* and Graph 2 shows how *y* is related to *z*.

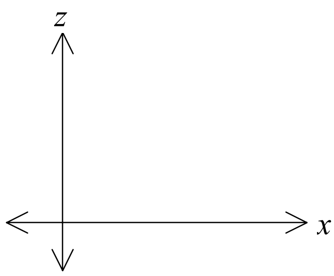


Graph 1



Graph 2

On the axes below, please draw the graph that shows how *x* is related to *z* based on the information from the two graphs above.



Please explain how you worked out your answer.

7. Imagine water flowing through a pipe into a container. The following equations show how the water level or height of the water (w) in the container was related to the number of minutes (t) when the pipe was opened for 10 minutes.

   $w = t + 8$ for the first four minutes ($t = 0$ to 4)
   $w = 3 \times t$ for the remaining six minutes ($t = 4$ to 10)
   where,
   w refers to the water level (height) in centimetres
   t refers to the number of minutes

   Please use the above information to answer the following questions.

   a. What was the height of the water in the container three minutes after the pipe was opened?

   b. From the given information, do you think the height of the water in the container is increasing at the same rate throughout the 10 minutes? Circle the letter corresponding to your answer.

      a) Yes, the water level increases at the same rate throughout the 10 minutes.

      b) No, the water level is <u>not</u> increasing at the same rate throughout the 10 minutes.

      Please show or explain how you obtained your answer.

   c. From the given information, do you think the container already contains water before the pipe was opened? Circle the letter corresponding to your answer.

      a) Yes, the container already contains water before the pipe was opened.
      b) No, the container does not contain water before the pipe was opened.

      Please state or show how you obtained your answer.

8. Which equation shows the fastest change in y when x takes values from 1 to 10? Please show/ explain how you worked out your answer.

   a. $x + y = 100$ b. $y = 6x - 3$ c. $4y = 8x$ d. $y = 75 + 5x$

   Solution or Explanation:

9. The relation of s with p is shown in the equation $s = 5p + 3$. The relation of p with n is shown in the equation $2p = 6n$. From this information, please write the equation that will show the relation of s with n. Please show your solution.

10. Examine the two tables shown below. The set of values in the table on the left shows specific values of $y = 2x^2 + 3$. Please write the equation whose values are shown in the table on the right. Please show or explain how you obtained your answer.

x	y
-1	5
0	3
1	5
2	11
3	21

$y = 2x^2 + 3$

x	y
-1	3
0	1
1	3
2	9
3	19

____

Solution or Explanation:

11. The relationship between x and y in Table 1 is $y = 2x + 1$. In Table 2, the values of x and y in Table 1 were swapped or interchanged. Please write the equation which shows the relationship between x and y in Table 2? Show how you obtained your answer.

Table 1

x	y
0	1
1	3
2	5
3	7
4	9

$y = 2x + 1$

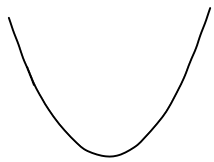
Table 2

x	y
1	0
3	1
5	2
7	3
9	4

____

Solution or Explanation:

12. Which table of values can be part of the graph on the right? Circle the letter corresponding to your choice and explain what you did to get your answer.



a.

y	0	8	12	10	4

b.

x	1	2	3	4	5
y	1	2	8	16	32

c.

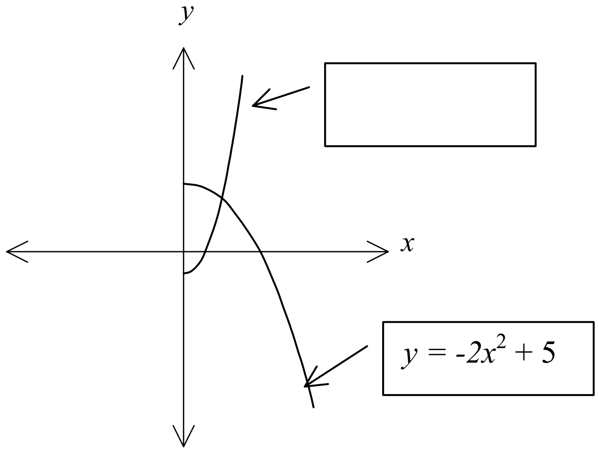
y	0	2	4	6	8

d.

x	1	2	3	4	5
y	8	5	3	2	2

Explanation:

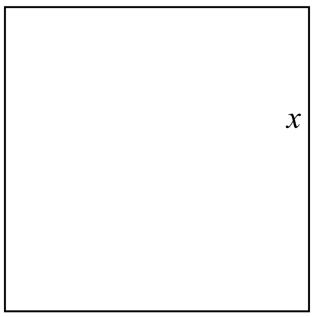
13. Part of the graph of $y = -2x^2 + 5$ is labeled in the figure below. What may be the equation of the other graph in the box? Please explain how you worked out your answer.



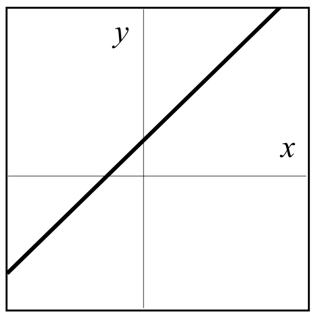
Explanation:

14. Which graph or graphs can be the graph of $y = 2x - 3$? Please explain how you obtained your answer(s).

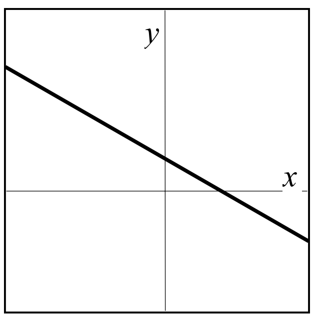
a.



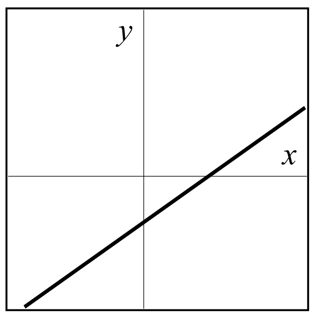
b.



c.



d.



Solution or Explanation:

Name:________________________Sex:_____Age:______ Code:________

Teacher's name:______________________School:____________Date: _________

These questions are part of a research project designed to find out
what students know and can do in mathematics.

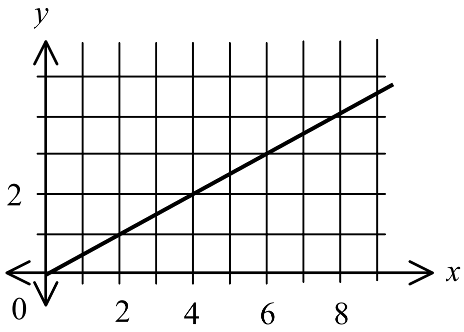
Booklet 2

I hope you find the questions interesting.

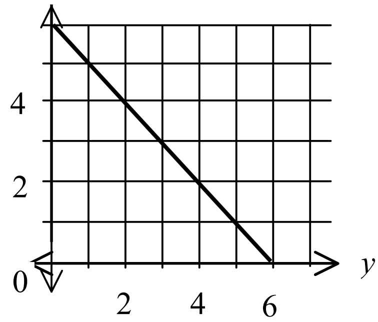
Erlina R. Ronda

Directions: Please answer the following questions. Write your answers on the space provided. Use the back page of each sheet if the space provided is not enough. You can use Filipino to explain your answers.

6.1 Graph 1 shows how *x* is related to *y* and Graph 2 shows how *y* is related to *z*.

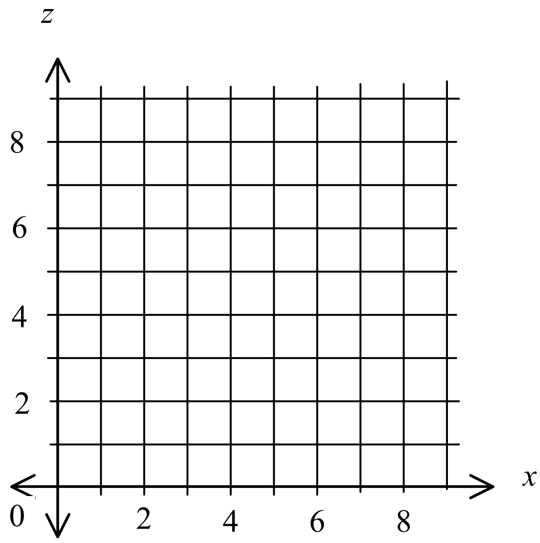


Graph 1



Graph 2

On the axes below, please draw the graph that shows how *x* is related to *z* based on the information from the two graphs above.



Please explain how you obtained your answer.

9.1 The relation of s with p is shown in the equation $s = 5p + 3$. The relation of p with n is shown in the equation $2p = 6n$. If n = 5, what is s? Please show your solution.

10.1 Examine the two equations shown below. The specific values of $y = x^2 + 3x + 3$ is shown in the table on the left. Fill in the table on the right with values of $y = x^2 + 3x$. Please explain/show how you obtained your answer.

$y = x^2 + 3x + 3$

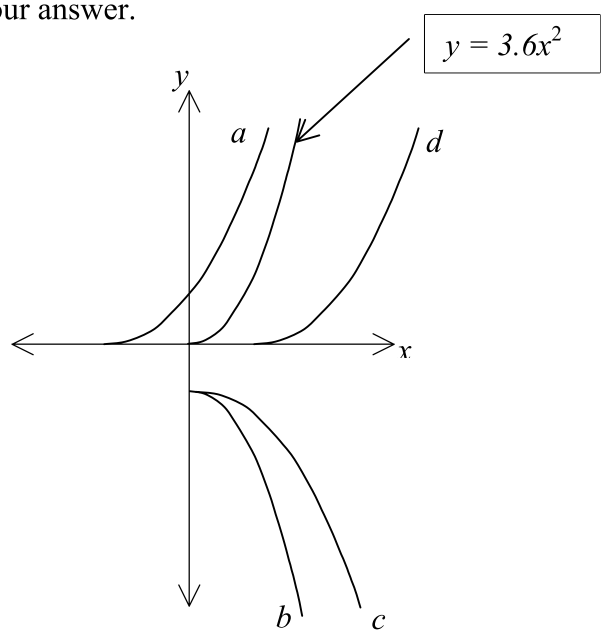
x	y
0	3
1	7
2	13
3	21
4	31

$y = x^2 + 3x$

x	y
0	
1	
2	
3	
4	

Explanation or solution:

13.1 Part of the graph of $y = 3.6x^2$ is labeled in the figure below. Which may be the graph of $y = -1.5x^2 - 4$? Circle the letter corresponding to the graph of your choice and explain how you determined your answer.

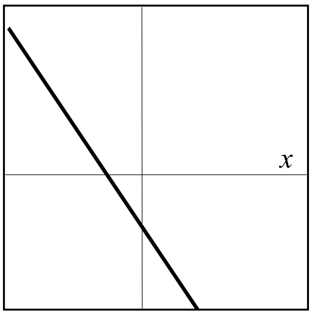


Explanation or Solution:

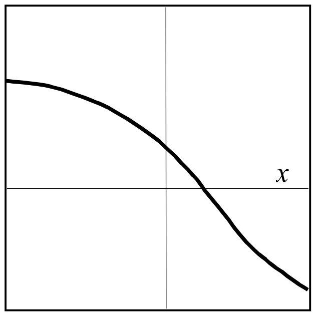
15. Which graph can match the set of values in the given table? Please show/explain what you did to determine your answer.

x	-4	-2	0	2	4
y	10	8	6	4	2

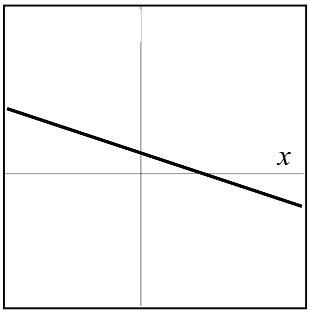
a.



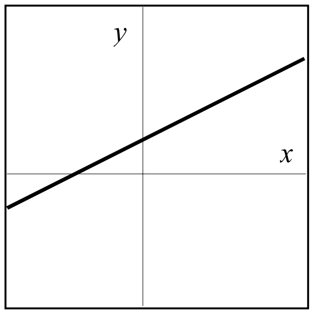
b.



c.



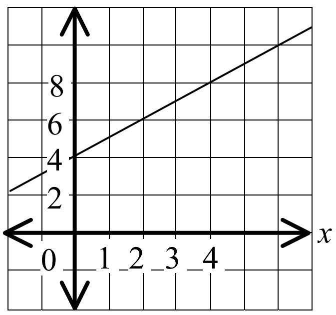
d.



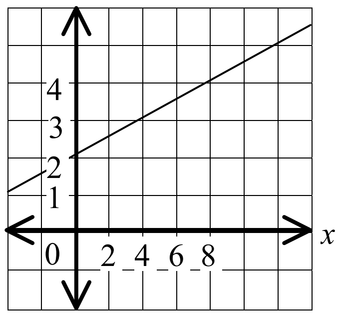
Solution or Explanation:

16. Select the graphs showing the same function or relationship. Please circle the letter corresponding to your choices. Please explain how you determined your answers.

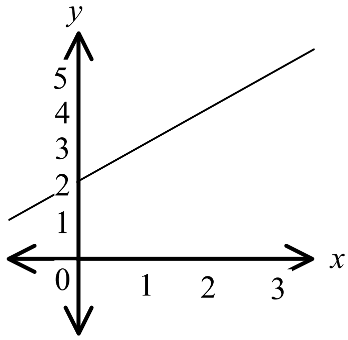
a.



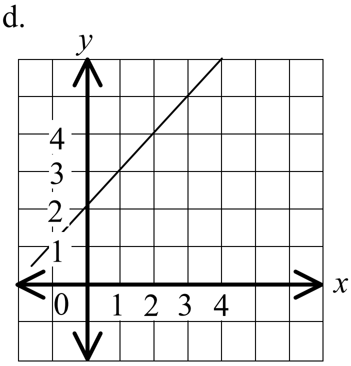
b.



c.



d.



Solution or Explanation:

17. Select the tables showing the same function or relationship. Circle the letter corresponding to your choices. Please show/explain how you obtained your answer.

Solution or Explanation:

a.

x	4	6	8	10
y	9	11	13	15

b.

x	2	3	4	5
y	9	10	11	12

c.

x	0	1	2	3
y	5	7	9	11

d.

x	-1	0	1	2
y	4	5	6	7

18. Select the equations that show the same function or relationship. Circle the letter corresponding to your choices. Please show or explain how you obtained your answers.

a. $y = 4x + 8$ b. $p = 4\ (s+2)$ c. $y = \frac{x-8}{4}$ d. $p = 4s + 8$

Solution or Explanation:

APPENDIX B

RECORD SHEET

Record Sheet

Name ________________________________Sex_____ Age______ Date: __________

Q1

0: incorrect:___; **1**: 17

Q2

0: incorrect:___; **1**: 20

Q3a

0: incorrect: ____; **1**: c

Q3b

0: incorrect: ____; **1**: a

Q4

0: incorrect: ___ ;

0.5: 8 or 18 **1**: 8 and 18

Q5

0: incorrect: ____;

1: some correct answers on the rigid intersections or enumerates values; 0, 1, … 7 & 19, 20, …

2: one interval only (e.g. 1-7 & 19-20, 1 - 8)

3: two intervals, but still incomplete (0 to 7 or 8 and from 18 or 18 to 24 or onwards)

4: 0 to 8 and from 18 or onwards (or any number beyond 24)

Q6

0: incorrect: ____ ; NA IR

0.5: NE **0.6**: INS

1:  using specific points

2:  reasoning in terms of relationship (words)

3:  using specific equation

4:  reasoning in terms of symbols

(y α x; y α 1/z; x α 1/z)

Q7a E

0: incorrect: ____; NA **1**: 11

Q7b E

0: incorrect:_____ ; NA IR NE

1: b: substituting values

2: b: using the invariant [coefficient of t]

Q7c

0: incorrect: ____; NA IR

1: a, correct reasoning

Q8

0: incorrect: ____; NA IR NE

1: b, by evaluating

2: b, using the invariant [highest coeff. of x]

Q9

0: incorrect ____ ; IR NE

1: $s = 15n{+}3$ or $n = \dfrac{s-3}{15}$, using numerical examples

1.5: s = 15n+3 or $n = \dfrac{s-3}{15}$, using proportional reasoning

2: $s = 15n{+}3$ or $n = \dfrac{s-3}{15}$ using composition

Q10

0: incorrect ____; IR NE

1: $y = 2x^2 + 1$, guess and check

1.5: $y = 2x^2 + 1$, used pattern, or the intercept or using algorithm for deriving quadratic equations

2: $y = 2x^2 + 1$, subtracts 2 from the given equation

Q11

0: incorrect: ______; IR NE

1: x = 2y + 1

2: $y = \dfrac{x-1}{2}$, point by point, algorithm for obtaining the equation of a linear function;

3: $y = \dfrac{x-1}{2}$, swaps x & y then solves for y

Q12

0: incorrect _____; IR NE

1: d, plotting points

1.5: d, using trend in the values of the table

3: d, using second difference as a property of quadratics

Q13 EG (Q)

0: incorrect: ____; IR NE

0.5: $y = 2x^2 - 5$

1: $y = kx^2 - 5$ using 1 value of $k > 2$

2: $y = kx^2 - 5$ using at least 2 values of $k > 2$

3: $y = kx^2 - 5$, $k > 2$

Q14
0: incorrect: ____; IR INR NE

1: d, plotting more than 2 points or evaluated 3 values

2: d, using pattern/trend or properties such as slope and intercept (sometimes insufficient or more than sufficient)

3: d, using invariant properties (slope and intercept, x and y intercepts)

Q6.1
0: incorrect ____; IR NE
[correct answer: connected pts along (0,6)&(8,2)]
1: plotting more than 2 points first then connect
2: plotting 2 points then connect (comment on the sufficiency of 2 points)

Q9.1
0: incorrect: ____;

1: 78, by evaluating the equations

Q10.1
0: incorrect ____; IR NE
1: correct entries, by evaluating
2: correct entries, subtracted 3 from the given table

Q13.1
0: incorrect: ___ NE
1: c, plotting points
2: incomplete reasoning but no misconception (at least correct interpretation of one of the parameters)
3: c, using the invariant properties

Q15
0: incorrect: ____; IR INR NE

1: c, plotting points

2: c, using pattern/trend or properties such as slope and intercept (reasoning may be insufficient or more than sufficient)

3: c, using invariant properties (slope and intercept)

Q16
0: incorrect ___ ; IR INR NE
1: cd, comparing more than 2 points; "same coord"
2: cd, using pattern/trend or properties such as slope and intercept (reasoning is insufficient or more than sufficient); compares equation
3: cd, using invariant properties (2 points only or slope and a point, slope and intercept)

Q17
0: incorrect: __ IR NE
1: ad, compares all entries in the table, plot the points
2: ad, uses pattern/trend or properties but ("Same slope", "same intercept", or "same difference between x and y") but reasoning is insufficient or more than sufficient; compares equation; same rule
3: ad, compares invariant properties

Q18
0: incorrect: ___ IR NE
1: any 2 of a, b & d, (evaluating, describing as procedure)
1.5: abd (evaluating)
2: bd, expanding/distributive
3: ad, same structure
4: abd, same structure, irrelevance of letter used

APPENDIX C

DISTRIBUTION OF INCORRECT RESPONSES

FOR SELECTED TASKS

Table 1
Distribution of Incorrect Responses in Task 15

Responses	Year 8		Year 9		Year 10		Total	
	(*n* = 149)		(*n* = 152)		(*n* = 143)		(*n* = 444)	
	D1	D2	D1	D2	D1	D2	D1	D2
a	23	13	15	11	5	4	43	28
b	31	29	32	36	28	11	91	76
d	31	32	29	8	7	1	67	41
bc	0	0	0	2	4	1	4	3
ad	0	1	0	0	0	0	0	1
cd	0	0	0	0	0	1	0	1
ac	1	0	0	0	0	0	1	0
Total	86	75	76	57	44	18	206	150

Table 2
Distribution of Incorrect Responses in Task 14

Responses	Year 8		Year 9		Year 10		Total	
	(*n* = 149)		(*n* = 152)		(*n* = 143)		(*n* = 444)	
	D1	D2	D1	D2	D1	D2	D1	D2
a	19	16	22	21	27	25	68	83
b	15	21	13	14	13	4	41	39
c	17	14	14	18	3	3	34	35
ad	5	8	11	14	4	12	20	34
ab	0	2	0	0	2	0	2	2
bd	1	5	2	4	3	2	6	11
cd	2	0	0	1	1	0	3	1
ac	1	0	1	0	1	0	3	0
bc	2	0	3	0	1	0	6	0
abd	0	0	1	3	0	0	1	3
abcd	0	2	0	1	0	0	0	3
bcd	0	0	0	0	1	0	1	0
acd	0	0	1	1	0	0	1	1
Total	62	68	68	77	56	46	186	212

Table 3

Distribution of Incorrect Responses in Task 12

Responses	Year 8 (*n* = 149)		Year 9 (*n* = 152)		Year 10 (*n* = 143)		Total (*n* = 444)	
	D1	D2	D1	D2	D1	D2	D1	D2
a	13	21	15	17	6	14	34	52
b	9	23	27	20	40	38	76	81
c	11	28	21	35	40	26	72	89
bc	0	0	0	2	1	1	1	3
bcd	0	0	0	0	1	1	1	1
Total	33	72	63	74	88	80	184	226

Table 4

Distribution of Incorrect Responses in Task 13.1

Responses	Year 8 (*n* = 149)		Year 9 (*n* = 152)		Year 10 (*n* = 143)		Total (*n* = 444)	
	D1	D2	D1	D2	D1	D2	D1	D2
a	41	31	41	17	20	14	102	62
b	34	72	47	60	64	56	145	188
d	15	3	5	1	7	6	27	10
ab	0	0	0	0	0	1	0	1
bc	0	0	2	0	0	0	2	
Total	90	106	95	78	91	77	276	261

Table 5

Distribution of Incorrect Responses in Task 16

Responses	Year 8 (*n* = 149)		Year 9 (*n* = 152)		Year 10 (*n* = 143)		Total (*n* = 444)	
	D1	D2	D1	D2	D1	D2	D1	D2
a, b	20	40	23	28	17	11	60	79
a, c	8	9	11	3	11	5	30	17
a, d	6	7	5	5	4	4	15	16
b, c	30	16	24	21	17	5	71	42
b, d	2	6	2	4	1	1	5	11
a, b, c	3	4	7	5	5	4	15	13
a, b, d	0	1	0	2	0	0	0	3
b, c, d	0	1	2	1	1	0	3	2
a, c, d	0	0	4	0	2	1	6	1
a, b, c, d	1	1	1	4	3	6	5	11
Single letter	45	6	21	5	11	3	77	14
Total	115	91	100	78	72	40	287	209

Table 6

Distribution of Incorrect Responses in Task 17

Responses	Year 8 (*n* = 149)		Year 9 (*n* = 152)		Year 10 (*n* = 143)		Total (*n* = 444)	
	D1	D2	D1	D2	D1	D2	D1	D2
a, b	18	38	28	21	32	8	78	67
a, c	10	6	9	11	4	4	23	21
c, d	0	9	1	8	4	2	5	19
b, c	6	7	5	0	3	1	14	8
b, d	21	25	23	28	20	26	64	79
a, b, c	0	1	0	4	0	2	0	7
a, b, d	1	4	5	15	3	3	9	22
b, c, d	0	0	0	2	0	0	0	2
a, c, d	0	2	0	1	1	1	1	4
a, b, c, d	1	0	0	9	4	20	5	29
a, c & b, d	0	0	2	2	4	6	6	8
Single letter	35	12	22	3	8	2	65	17
Total	92	104	95	104	83	75	270	283

APPENDIX D

CONSENT FORMS

TITLE OF RESEARCH PROJECT: A FRAMEWORK OF GROWTH POINTS IN SECONDARY STUDENTS' UNDERSTANDING OF FUNCTION

Research Supervisor: Associate Professor DOUG CLARKE
Student Researcher: ERLINA RONDA

22 January 2002

____
____
____
____

Dear ________:

I am seeking permission to conduct a research study in your school. The research is about developing a framework of growth points in secondary students' understanding of mathematical function. The study seeks to identify the growth points in students' understanding of this important mathematical domain and organise it into a form, which could be of direct use to teachers as basis for assessing and developing students' understanding of the concept.

I would be most grateful if you would nominate two mathematics classes at each of the year levels 2, 3 and 4 to participate in this research study. One of the four classes should represent the high ability group. A one-hour individual written test will be given to the students and follow-up interviews will be conducted with around two students per class, following the test. In the written test and interview, students will be asked to do tasks related to function and students' strategies and thinking processes will be noted. In this connection I would like to ask for your permission to use a small quiet room in your school where the interviews can be conducted.

Individual informed consent will be sought from parents. With permission, the interview will be audiotaped. The purpose of these recordings is for further analysis of data and will not be shown to anyone other than the research team.

Reports from the study will not enable anyone to identify the students or teachers. The data collected throughout this study may be aggregated and used in publications, used in teaching or shared with other researchers, but confidentiality of schools, teachers and students' identity will be retained at all times. Having given your consent you are free to withdraw your consent or discontinue your school's participation in the study at anytime, without giving reasons.

The work involved in this study has the written approval of the Australian Catholic University Human Research Ethics Committee. Any questions or clarifications about the research can be directed to the researcher at this number: (02) 9274276 or to my supervisor Doug Clarke at the following e-mail address: D.Clarke@patrick.acu.edu.au. Any complaints or query regarding the conduct of this work can be directed to:

Chair, Human Research Ethics Committee,
C/o Office of Research, Australian Catholic University
115 Victoria Pde. Fitzroy
VIC 3065
Tel: 613 9953 3157
Fax: 613 9953 3315

Any complaint made will be treated in confidence, investigated fully and the participant informed of the outcome. If you agree to participate in this study, I would be grateful if you could retain one signed copy of the Consent Form and return the other to me. Any questions regarding this study can be directed to me on telephone number (052) 838-0275, at my e-mail address: er.ronda@student.patrick.acu.edu.au or at postal address: Revolucion St. Guinobatan, Albay.

Yours sincerely,

Erlina Ronda

PRINCIPAL CONSENT FORM

TITLE OF RESEARCH PROJECT: A FRAMEWORK OF GROWTH POINTS IN SECONDARY STUDENTS' UNDERSTANDING OF FUNCTION

STUDENT RESEARCHER: ERLINA RONDA

I, ________________________________ agree for my school to participate in the Research Study on Growth Points in Secondary Students' Understanding of Function outlined in this letter. Any questions I have asked about the study have been answered to my satisfaction. I understand that the school and those who participate in the study will not be identified by name in any presentations or published reports regarding the study without my permission. I further understand that I can withdraw this consent at any time without giving reasons.

NAME OF SCHOOL: …………………………………………..

NAME OF PRINCIPAL: ……………………………………….

SIGNATURE.. DATE..................................

NAME OF RESEARCHER: ERLINA RONDA

SIGNATURE...
DATE......................................

INFORMATION LETTER TO PARENT/GUARDIAN

Title of Research Project: A FRAMEWORK OF GROWTH POINTS IN SECONDARY STUDENTS' UNDERSTANDING OF FUNCTION

Research Supervisor: Associate Professor DOUG CLARKE
Student Researcher: ERLINA RONDA

May 15, 2002

Dear __________________________:

A research study entitled *Growth Points in Secondary Students' Understanding of Function* is to be conducted in your child's school. The study seeks to identify the growth points in students' understanding of this important mathematical domain and organize it into a form, which could be of direct use to teachers as basis for assessing and developing students' understanding of the concept.

The study will involve consultations with teachers and a written test and interview with their students for not more than an hour for each interview. In the interview, students will be asked to do tasks related to function and their strategies and thinking processes will be noted.

I am writing to ask for consent for your child to be involved in this study. With permission, the interview will be audio taped. The purpose for these recordings is for further analysis of data. Reports from the study will not enable anyone to identify your child. The data collected throughout this study may be aggregated and used in publications, used in teaching or shared with other researchers, but schools, teachers and students' identity will be retained at all times. Having given your consent you are free to withdraw your consent or discontinue the participation of your child in the study at anytime, without giving reasons.

Any questions regarding this study can be directed to me at UP NISMED, Diliman, Q.C. at telephone number 02 9274276. In the event that you have any complaint or query regarding the conduct of this work that the researcher has not been able to satisfy, you can write to:

Chair, Human Research Ethics Committee,
C/o Office of Research, Australian Catholic University
115 Victoria Pde. Fitzroy
VIC 3065
Tel: 613 9953 3157, Fax: 613 9953 3315

Any complaint made will be treated in confidence, investigated fully and the participant informed of the outcome. If you agree to participate in this study, I would be grateful if you could retain one signed copy of the Consent Form and return the other to me.

Yours sincerely,

Erlina R. Ronda

PARENT CONSENT FORM

TITLE OF RESEARCH PROJECT: A FRAMEWORK OF GROWTH POINTS IN SECONDARY STUDENTS' UNDERSTANDING OF FUNCTION

STUDENT RESEARCHER: ERLINA RONDA

I, __________________________________ agree for my child to participate in the Research Study on Growth Points in Secondary Students' Understanding of Function outlined in this letter. Any questions I have asked about the study have been answered to my satisfaction. I understand that my child will not be identified by name in any presentations or published reports regarding the study without my permission. I further understand that I can withdraw this consent at any time without giving reasons.

NAME OF PARENT
…………………………………………………………………………………………..
(Block letters)
SIGNATURE.. DATE......................................

NAME OF CHILD
…………………………………………………………………………………………..
(Block letters)
SIGNATURE.. DATE......................................

NAME OF RESEARCHER: ERLINA RONDA

SIGNATURE...
DATE......................................

APPENDIX E

ETHICS APPROVAL

AUSTRALIAN CATHOLIC UNIVERSITY

Research Services

Human Research Ethics Committee

Ethics Clearance for a Research Project - Approval Form

Principal Investigator/s (if staff):	**1) A/Prof Doug Clarke**	**Campus: St Patrick's Campus**
Co Investigator:	**1)**	
Researcher(s) (if student/s)	**1) Ms Erlina Ronda**	

Ethics clearance has been provisionally approved for the following project: A Framework of growth points in secondary students' undertaking of function

for the period: 5.6.2001 - 31.12.2003 (subject to annual renewal)

Human Research Ethics Committee Register Number: V2000/01-68

subject to the following conditions as stipulated in the National Health and Medical Research Council (NHMRC) Statement on Human Experimentation and Supplementary Notes 1992:

(i) that principal investigators provide reports annually on the form supplied by the Institutional Ethics Committee, on matters including:
- security of records;
- compliance with approved consent procedures and documentation;
- compliance with special conditions, and

(ii) as a condition of approval of the research protocol, require that investigators report immediately anything which might affect ethical acceptance of the protocol, including:
- adverse effects on participants;
- proposed changes in the protocol, and/or
- unforeseen events that might affect continued ethical acceptability of the project.

and subject to clarification of the following to the Human Research Ethics Committee:

Information Letter to Participants
- Please amend the headings on the Information Letter to Participants to read "Information Letter to Parent/Guardian", "Information Letter to Teachers", "Information Letter to Principal"
- Please amend the title of the Committee to "Human Research Ethics Committee".
- In the heading section of the letters replace "Researcher" with "Student Researcher"
- In all the letters please amend the typographical mistake in the 1st line of the paragraph above the Committee address, "involve" should read "involved".
- In the last paragraph on page 2 of the letters, please amend sentence to "...retain one signed copy of the Consent Form and return the other to me."
- Following the address of the Chair, please add the statement "Any complaint made will be treated in confidence, investigated fully and the participant informed of the outcome".

Consent Form
- In the heading section of the Consent Forms replace "Researcher" with "Student Researcher"
- It was recommended that instead of a consent form for the Principal a separate letter should be requested from the Principal. If the consent form is to be used, the form needs to be amended to indicate "Name of Principal" and "Name of School", as the Principal is not a participant.
- The Consent Form to parents should include a section where children can indicate assent.
- Indicate on the Consent for whom it is for, for example, Parent Consent Form, Teacher Consent Forms.

Date of Approval 26/6/01

A Final Report Form will need to be completed and submitted to the HREC within one month of completion of the project
OR
An Annual Progress Report Form will need to be completed and submitted to the HREC within one month of the anniversary date of approval.

Please sign, date and return this form (with any additional information, or supporting documents to show completion of any amendments requested) to the Administrative Officer (Research) to whom you submitted your application. This is essential before final approval by the Human Research Ethics Committee is confirmed.

Signed: .... **Date:** 7.6.2001
Administrative Officer (Research)

(To be completed by the Principal Investigator, or Student and Supervisor, as appropriate.)

The date when I/we expect to commence contact with human participants or access their records is: 25-6-01

I/We hereby declare that I/We am/are aware of the conditions governing research involving human participants as set out in the Human Research Ethics Committee's *Guidelines and Instructions for Researchers/Students* and agree to the conditions stated above.

Signed: .... Date: 10-6-01
(Principal Investigator (if staff) or Supervisor, as appropriate)

Signed: .... Date: 12-6-01
(Researcher (if student))

OR/E30/974