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RESEARCH

Some Remarks on Recent Formalist Responses to the Hole Argument

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Abstract
In a recent article, Halvorson and Manchak (Br J Philos Sci, Forthcoming) claim that there is no basis for the Hole Argument, because (in a certain sense) hole isometries are unique. This raises two important questions: (a) does their argument succeed?; (b) how does this formalist response to the Hole Argument relate to other recent responses to the Hole Argument in the same tradition—in particular, that of Weatherall (Br J Philos Sci 69(2):329–350, 2018)? In this article, *ad* (a), we argue that Halvorson and Manchak's claim does not go through; *ad* (b), we argue that although one *prima facie* plausible reading would see Halvorson and Manchak as filling an important hole (no pun intended) in Weatherall's argument, in fact this reading is implausible; there is no need to supplement Weatherall's work with Halvorson and Manchak's results.

Keywords General relativity · Hole argument · Isometries · Mathematical structuralism

1 Introduction

In a recent article, Halvorson and Manchak [10] argue that there is no mathematical basis for the Hole Argument. They schematise the Hole Argument thus:

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1. Substantivalism.
2. Some mathematical facts.
C. Pernicious indeterminism.

Substantivalism is a metaphysical claim about the relationship between spacetime and matter: the former exists on (at least) the same ontological footing as the latter.[1] Right off the bat, we can identify a concern with the above schema: substantivalism is a claim about metaphysics, in the sense that it's a claim about the the constituents of possible worlds. In order for any mathematical claim to have traction in this discussion, it must be supplemented by a claim about how that mathematical fact affects the model-world relationship. So a slight refinement of the schema yields:

1. Substantivalism.
2a. Some mathematical facts.
2b. Some interpretational claim that the mathematical facts in (2a) affect the relationship between models and worlds.
C. Pernicious indeterminism.

Halvorson and Manchak argue that there are two mathematical claims in (2a) on which the conclusion of the Hole Argument (C) might plausibly be taken to rely[2]:

> `Hole isomorphism`: There exist distinct models of general relativity that are related by hole isomorphisms.
> `Distinct isometries`: For any two isometric Lorentzian manifolds, there is more than one diffeomorphism relating those Lorentzian manifolds which witnesses their being isometric.

Halvorson and Manchak claim that `Hole isomorphism` is "trivially true" (when the isomorphism under consideration is an isometry), but concur with Weatherall [28] that this is not sufficient to generate the Hole Argument.[3] They also claim, however, that if `Distinct isometries` were true, then in principle the Hole Argument *could* be generated within the framework of general relativity—but in fact `Distinct isometries` is false, due to a theorem due to Geroch. Consequently, they argue that discussion of the Hole Argument should be closed.[4]

In a sense—at least as we interpret them (to be clear, this is our own reading of Halvorson and Manchak [10]; they don't make the following claims explicitly in their article)—what Halvorson and Manchak seek to achieve in their article is

[1] Of course, there is a variety of ways of spelling out this claim—see e.g. [1, 4, 5, 17, 18]—but for now, this characterisation is sufficient.

[2] The following is our attempt to regiment Halvorson and Manchak's discussion, and is thus our terminology, not theirs.

[3] For a critique of Weatherall's arguments with which we engage further below, see [21].

[4] To be fair to Halvorson and Manchak [10], in footnote 2 of their article, they write: "Granted, there may be yet another mathematical claim upon which the Hole Argument could be built. By eliminating two possible mathematical claims, we hope at least to clarify the structure of the argument."

to plug a hole left outstanding in Weatherall's analysis. Roughly speaking (though the structure of [28] is complex and requires careful unpacking: see [21] as well as our own discussion below), Weatherall claims that for any two hole diffeomorphic models of general relativity, those models must be compared using the map which witnesses the isometry between them (which is to say that any interpretation must be such that if two manifold elements are related by that isometry, then they represent the same physical spacetime point); this, however—Weatherall claims—is not sufficient to generate the Hole Argument. However, if there were multiple such maps relating the two models, then in principle the Hole Argument could be generated within the framework of general relativity. (We spell out the reasoning here in detail in Sect. 3.2.) By denying `Distinct isometries`, Halvorson and Manchak take themselves to have closed this (loop)hole.

We have two goals in this article. *First*: to demonstrate that Halvorson and Manchak's invocation of Geroch's theorem fails on its own to close the Hole Argument, because the theorem is a purely formal result; to have any impact on the Hole Argument it needs to be supplemented by some claim of the form described in (2b). When we canvass some options for this supplementary claim, we discover that Halvorson and Manchak's claim fails to close the Hole Argument. And *second*: to argue that even to read the central purpose of Halvorson and Manchak [10] as being to close the above-described hole (supposedly) left by Weatherall is implausible, for whatever one makes of Weatherall's arguments, it is not necessary for them to be supplemented with Halvorson and Manchak's results. The upshot is that, as far as we can see, Halvorson and Manchak's central results add little to recent formalist responses to the Hole Argument.

2 The Hole Argument

In this section, we discuss the inputs (1), (2a), (2b), and conclusion (C) of the Hole Argument, as schematised above. Although this will be well-known to many readers, it is very important for the purposes of later sections of this article to be precise and explicit about all relevant moving parts.

2.1 Substantivalism

Spacetime substantivalism is generally understood to be the claim that spatiotemporal structure is on (at least) the same ontological footing as matter. Consider now a model of general relativity, $\langle M, g_{ab}, T_{ab}\rangle$.[5] The literature discusses two options for the substantivalist:

[5] Where M is a smooth 4-dimensional manifold, g_{ab} is a Lorentzian metric tensor, and T_{ab} is a symmetric tensor representing the stress energy of matter.

Manifold substantivalism: M represents spacetime, which is ontologically at least on par with the matter content whose stress-energy is represented by T_{ab}.
Metric manifold substantivalism: $\langle M, g_{ab} \rangle$ represents spacetime, which is ontologically at least on par with the matter content whose stress-energy is represented by T_{ab}.

(Authors who hold the first view regarding how to characterise substantivalism number among them Earman and Norton [7]; authors who hold the second view include Maudlin [14].) The Hole Argument purports to raise a radical indeterminism worry for both forms of substantivalism. For dialectical clarity, we discuss the Hole Argument in relation to Manifold substantivalism; all the arguments we discuss carry over to Metric manifold substantivalism.

The worry about indeterminism, however, does not arise solely because of the substantivalist's commitment to the ontological independence of spacetime and matter. It requires also an additional commitment, which Pooley [20] calls 'plurality':

Plurality: If W is a possible world according to the theory under consideration, then there is a plurality of possible worlds, W', W''..., that (i) involve the same pattern of spatiotemporal properties instantiated in W and contain the same material fields as W, but that (ii) differ from W solely over which spacetime points have which properties and serve as the locations of common material content.

As Hoefer [11] and Pooley [18] emphasise, substantivalism (of either of the above stripes) needn't entail the acceptance of Plurality. Furthermore, there are ways of setting up a substantivalist position which deny Plurality and are immune to the Hole Argument. So in order to set up the Hole Argument, the substantivalist position needs to be one that accepts Plurality; call this position Pluralist substantivalism. But even this on its own is not sufficient to set up the Hole Argument; to do so, we first need to introduce some mathematical facts.

2.2 Mathematical Facts

The mathematical fact about general relativity which is supposed to spell trouble for the pluralist substantivalist is the so-called 'general covariance' of the Einstein equation, the central dynamical equation of general relativity. To understand how this works, we begin by introducing the concept of a diffeomorphism and its drag-along. A diffeomorphism $d : M \to N$ is a smooth bijection from manifolds M to N whose inverse is also smooth. Insofar as d is a function, it simply associates (uniquely) with each element $p \in M$ some element $p' \in N$.

A coordinate system on a submanifold U of a (four-dimensional) manifold M is a map $x^\mu : U \to \mathbb{R}^4$. In practice, we restrict attention to only smooth coordinate systems. Relative to an interpretation linking models and worlds, we can now use these coordinate systems to make location claims about certain physical objects in regions of the world represented by U. For example, we can talk about the magnitude of some scalar field Φ in some region of W by looking at the value of some scalar function F, evaluated at the appropriate element $p \in U$ assigned the coordinate x^μ: $F(p) = f \circ x^\mu(p)$, where $f : \mathbb{R}^4 \to \mathbb{R}$. We can extend the action of a diffeomorphism to both the coordinate system and the representatives of physical fields as follows: (i) for a given coordinate system x^μ, we can define its 'drag-along' $d^* x^\mu(p) = x^\mu(d(p))$ for all $p \in d^{-1}(U)$; (ii) for a given magnitude $F(p)$, its drag-along defined by d is the object d^*F such that $d^*F(p') = F(p)$. Following the convention in the literature, we refer to models of the form $\mathcal{M} = \langle M, O_1, O_2, ...\rangle$ and $d^*\mathcal{M} = \langle d^*M = M, d^*O_1, d^*O_2, ...\rangle$, where $O_1, O_2, ...$ are tensorial objects, as *diffeomorphically-related* models.

We can now define general covariance (cf. [19]):

> `General covariance`: Let the models of a theory T be n-tuples of the form $\langle M, O_1, O_2, ...\rangle$. T is generally covariant iff: if $\langle M, O_1, O_2, ...\rangle$ is a structure of the relevant type and d is a diffeomorphism between M an N, then $\langle M, O_1, O_2, ...\rangle$ is a solution of T iff $\langle N, d^*O_1, d^*O_2, ...\rangle$ is also a solution of T.

Standardly, we associate solutions with physically possible worlds. (Precisely how this representation is to be achieved, and in particular how redundantly it may be achieved, is a subtle issue whose discussion is beyond the scope of this paper, but see e.g. [8, 21, 23] for recent discussion.[6]) An important point to note is that the objects in the tuples needn't be independent. Indeed, Earman and Norton's proof of the general covariance of general relativity relies on incorporating dynamical equations into the very structure of the models. For them, a model of general relativity is a tuple of the form $\langle M, g_{ab}, O_2\rangle$, where the object $O_2 = G_{ab} - T_{ab}$, (where G_{ab} is the Einstein tensor, built out of g_{ab} and its derivatives) is constrained dynamically to vanish. The vanishing of that object is equivalent to the satisfaction of the Einstein equation; the fact that the equation uses only tensors means that it is a fact that if O_2 vanishes in one model, then it vanishes in all diffeomorphically-related models. (NB: this isn't to say that all such diffeomorphically-related models are in fact kinematically possible, if one stipulates that some of the tensors in question be what Pooley calls 'fixed fields': see [18]. We return to this below.)[7]

[6] Note that there are also good reasons to question whether this is indeed the most appropriate way of understanding what is represented by models of physical theories—see [26, p. 3]—but we'll set these concerns aside here.

[7] An important point to highlight is that our discussion of equivalence should take place at the level of the full models, including all material tensorial dynamical content, as represented by stress-energy tensors. Often, as a matter of convenience, the standard of equivalence is taken to be the isometry of Lorentzian manifolds (we too will do this shortly). But it is important to remember that it is not merely the preservation of metrical structure that determines the equivalence of models; it is the further preservation of dynamical structure, namely the fact that certain models contain tensorial objects that satisfy dynamical equations like the Einstein equation or the Einstein-Maxwell equations. This is important because, as we'll see, the real physical content of general covariance is contained in the fact that solutionhood is preserved under diffeomorphisms, not merely that some diffeomorphisms are isometries.

Halvorson and Manchak read Earman and Norton's gauge theorem as follows: "[t]he idea here is that ϕ [the hole diffeomorphism] establishes an isomorphism between $[\langle M, \phi^* g_{ab}\rangle]$ and $[\langle M, g_{ab}\rangle]$ and since the latter is a model of the theory so is the former" [10, p. 14]. This is a mistake. First, isometry does not guarantee preservation of solutionhood: consider the two isometric models $\langle M, g_{ab}, T_{ab}\rangle$ and $\langle M, d^* g_{ab}, T_{ab}\rangle$. In the second model, we have not dragged along the stress-energy tensor; generically this will mean that the second model fails to satisfy the Einstein equation, and so is not a solution. Second, it is a property of the dynamical equations (namely their tensorial nature), *plus* stipulations as to which models represent kinematical possibilities (cf. [19]) that ensures the preservation of solutionhood. It is therefore a contingent claim that underwrites the general covariance of general relativity, not a mathematical truism.

This setup allows us to highlight three implicit mathematical commitments in the setup of the Hole Argument:

> `Tensorial dynamics`: Our theory's dynamical equations involve only tensorial objects.
> `Drag-along`: When the manifolds in two models are related by a diffeomorphism, the tensorial content of one model is related by the drag-along construction to the tensorial content of the other.[8]
> `No fixed fields`: Having fixed the mathematical objects and dynamics of the models of one's theory, one cannot further stipulate that certain models represent kinematical possibilities of one's theories while others do not. (Doing so would spoil general covariance, and is accordingly disallowed.)[9]

The satisfaction by a theory T of `Tensorial dynamics`, `Drag-along` and `No fixed fields` ensures that T is generally covariant. So, in particular, general relativity is a generally covariant theory. This is to be understood as the claim that under diffeomorphisms (and their associated drag-alongs which act on the tensorial constituents of the models) solutions are mapped to solutions. Of course, this is a substantive physical hypothesis (special relativity in its standard formulation is not generally covariant in this sense, for example—see [19]). But we will not quibble over nomenclature, and will continue to call this a mathematical fact (after all, a fact being mathematical does not preclude its having physical significance).

We are not yet in a position to level a charge of indeterminism to the proponent of `Pluralist substantivalism` in general relativity, even though general relativity is generally covariant. We require a further premise according to which the inferences we make about diffeomorphically-related models have non-trivial consequences for the worlds deemed possible according to the theory whose models are under consideration. After all, diffeomorphisms (and their drag-alongs) can only

8 To stress, `Drag-Along` doesn't hold as a matter of mathematics alone, for the reasons given in [10, §3]. It is to be taken as a restriction on the models which one is countenancing when one considers the Hole Argument.

9 For more on the definition of fixed fields, see [22].

be defined on models and not on possible worlds. Again following Pooley [20], we introduce an interpretative commitment:

> `Models`: If $\mathcal{M} = \langle M, g_{ab}, T_{ab} \rangle$ can be chosen to represent a possible world W then, relative to that choice, there is a permissible and natural interpretation of the diffeomorophically-related model $d^*\mathcal{M} = \langle M, d^*g_{ab}, d^*T_{ab} \rangle$ according to which $d^*\mathcal{M}$ represents a distinct possible world $W' \neq W$.

Call a pluralist substantivalist who accepts `Models` an *acid-test substantivalist*. But this still does not lead to an indeterminism worry; it does, however, lead to an underdetermination worry. Two diffeomorphically-related models $\mathcal{M}$ and $\mathcal{M}'$ which (according to `Models`) can be taken to represent respective worlds W and W' will *prima facie* correspond to two distinct possible worlds (according to `Plurality`). Pooley and Read identify the undeterdetermination worry:

> [S]uppose that according to $\mathcal{M}$, the observer at the salient stage of their trajectory is located at (the spacetime point represented by) p and that d maps p to a distinct point q. According to $\mathcal{M}'$, therefore, the relevant stage of the observer's trajectory is located at q. It follows that no measurement that the observer might perform at that point along their trajectory can determine whether they are located at (the point represented by) p or at (the point represented by) q, for the outcomes of any measurements are the same according to $\mathcal{M}$ and $\mathcal{M}'$. [21, p. 3]

Whether this underdetermination worry in fact goes through is a delicate matter—see [3] and [21, fn. 8] for discussion. But in any case, in general relativity, there arises also an indeterminism concern, when one takes the diffeomorphism relating the models under consideration to act non-trivially only to the future of some spacelike hypersurface. Let's explore this in more detail. As is common in the literature, in what follows, we focus on vacuum spacetimes, so that the only physical field being dragged along the relevant diffemomorphisms is the metric field. In other words, our standard of model isomorphism is now isometry.

The standard 'angle bracket' notation for models of general relativity gives rise to an ambiguity regarding how to understand isometries that needs to be ironed out before we proceed further. Consider the following definition of an isometry from Wald:

> If $d : M \rightarrow M$ is a diffeomorphism and T is a tensor field on M, we can compare T with d^*T. If $d^*T = T$, then even though we have "moved T" via d, it has "stayed the same". In other words, d is a *symmetry transformation* for the tensor field T. In the case of the metric g_{ab}, a symmetry transformation—i.e., a diffeomorphism d such that $(d^*g)_{ab} = g_{ab}$—is called an *isometry*. ([25, p. 438])

Contrast this definition with the following characterisation of an isometry from Weatherall (with notation adapted for consistency):

> Fix a model of a relativity theory, which is a relativistic spacetime, that is, a Lorentzian manifold $\left(M, g_{ab}\right)$ ... We define a relativistic spacetime $\left(M, \tilde{g}_{ab}\right)$, whose underlying manifold is once again M, and whose metric is defined by $\tilde{g}_{ab} = d^* g_{ab}$, where d^* is the [drag-along] map determined by d. One can easily confirm that $\left(M, g_{ab}\right)$ and $\left(M, \tilde{g}_{ab}\right)$ are isometric spacetimes, with the isometry realized by d. [28, p. 335]

The above two quotes invite, respectively, two distinct notions of isometry:

> $\texttt{Isometry}_1$: For all elements $p \in M$, $d^* g_{ab}(p) = g_{ab}(p)$.
> $\texttt{Isometry}_2$: For all elements $p, q \in M$, if $d(p) = q$, then, $d^* g_{ab}(q) = g_{ab}(p)$.

The fact that the angle bracket notation does not specify which elements are assigned which values of g_{ab} means that the standard notation of $\langle M, g_{ab} \rangle = \langle M, d^* g_{ab} \rangle$ is ambiguous between the above two readings of isometry. Note that $\texttt{Isometry}_1$ is a much more stringent requirement than $\texttt{Isometry}_2$. $\texttt{Isometry}_1$ requires the existence of Killing vector fields, whereas $\texttt{Isometry}_2$ can very straightforwardly be defined for any generic Lorentzian manifold: every diffeomorphism of M can be used to generate an $\texttt{Isometry}_2$, whereas only specific diffeomorphisms will generate an $\texttt{Isometry}_1$. For example, a Minkowski manifold $\langle M, \eta_{ab} \rangle$ is isometric$_2$ to $\langle M, d^* \eta_{ab} \rangle$ for any $d \in \mathrm{Diff}(M)$, but is isometric$_1$ to $\langle M, d^* \eta_{ab} \rangle$ only if $d \in SO(1,3) \ltimes \mathbb{R}^4$.[10]

Having disambiguated the two types of isometry relevant to diffeomorphisms from M to itself, let us divide our manifold M into two regions: (i) $H \subset M$, which is a compact open subset of M, and (ii) $M \setminus H$, which is the complement of H in M. Next, consider a diffeomorphism which we will call a *hole isometry*: $\tilde{\psi} : M \to M$ such that $\tilde{\psi}$ is the identity in $M \setminus H$ but differs from the identity in H (and the two regions join up smoothly). Call this region a 'hole'. If we drag the tensorial content of the model along the hole diffeomorphism—in particular, if we drag along the metric—then we construct a hole isometry. Note that in constructing a hole isometry, we do not rely on the metric manifold having any non-trivial Killing vector fields; all we are doing is smoothly changing which manifold elements are associated with particular tensorial magnitudes, without changing the lawlike relations between those tensorial magnitudes. Thus, to the extent that `General covariance` is what guarantees the existence of a hole diffeomorphism, as it is according to the standard understanding of the Hole Argument, every hole isometry is an $\texttt{Isometry}_2$, but not necessarily an $\texttt{Isometry}_1$.

We can now articulate another premise of the Hole Argument:

[10] Note that the notion of isometry deployed at [13, p. 85] is yet more general than either $\texttt{Isometry}_1$ or $\texttt{Isometry}_2$—our thanks to J. B. Manchak for pointing this out to us. ($\texttt{Isometry}_1$ and $\texttt{Isometry}_2$ can thus be regarded as special cases of Malament's definition.)

Hole isometry: Given a metric manifold $\langle M, g_{ab}\rangle$, there exists a distinct metric manifold $\langle M, g'_{ab}\rangle$ such that a (non-trivial) hole isometry (in the sense of $\texttt{Isometry}_2$) exists between them.[11]

Thus, given a metric manifold $\langle M, g_{ab}\rangle$, Hole isometry insists that one can construct a distinct metric manifold $\langle M, \psi^* g_{ab}\rangle$, isometric to the first, where the metrical content of the latter is dragged along with respect to that of the former using the diffeomorphism ψ, which is (defined to be) the map which *witnesses* the isometry between the former and the latter (note that Weatherall [28] also uses this terminology). Moreover, said isometry is a hole isometry, in the sense presented above. Since isometry is the standard of isomorphism for Lorentzian manifolds, in accepting Hole isomorphism (with Weatherall [28]—recall again that this is the "trivial claim"), Halvorson and Manchak [10] thereby *accept* Hole isometry.

Two points are worth stressing at this point. (a) There is (at this point, at least) no prohibition on comparing any two isometric models of general relativity using diffeomorphisms which do not witnesses those models' being isometric—in the above case, assuming that ψ is non-trivial, one could for example compare those models using the identity map 1_M, which (as we'll return to below) does not witness their being isometric (recall that to compare two models using a map means to use said map as a standard of cross-model identity of what the points related by that map represent). (b) At this point at least, nothing guarantees that the diffeomorphism which witnesses the isometry between two metric manifolds be unique: as already mentioned, such a claim amounts to the denial of Distinct isometries, and will be discussed further below.

2.3 Pernicious Indeterminism

To set up the worry about indeterminism, we restrict our attention to globally hyperbolic manifolds: metric manifolds $\langle M, g_{ab}\rangle$ which possess a Cauchy surface, i.e. a closed achronal set Σ whose domain of dependence is the entire manifold M.[12] Now consider some globally hyperbolic vacuum solution $\langle M, g_{ab}\rangle$. Hit this solution with a hole isometry whose hole H is entirely to the future of some Σ_t. This new solution is identical to the first solution up to time t. The acid-test substantivalist has to accept that two worlds W and W', represented by these models and both possible according to the theory, are identical up to some time slice, and non-identical thereafter. They are thereby committed to (a pernicious form of) indeterminism.

Let us say a little more about determinism. Consider the following definition from Pooley [20]:

[11] Muller [15] provides an explicit construction of such a hole isometry. Ultimately, we take this work of Muller to establish the existence of hole isometries, and Halvorson and Manchak's result invoking Geroch's theorem to establish the uniqueness of hole isometries. (What we mean by this should be clear from the main text to follow.)

[12] For definitions of 'achronal' and 'domain of dependence', see [25, p. 201].

`Intrinsic determinism`: A theory T is intrinsically deterministic iff for any two worlds W_1 and W_2 possible according to a given interpretation of T, if the past of W up to some timeslice in W is intrinsically identical to the past of W' up to some timeslice in W', then W are W' are intrinsically identical.

Here, we understand two sub-worlds as being intrinsically identical just in case they agree not only on the pattern of instantiation of properties (and relations) across particulars, but also over which particulars instantiate those properties (and relations). We should note that Halvorson and Manchak [10] dispute the intelligibility of this talk of intrinsic properties; we discuss this worry in more detail in Sect. 4.1. Granting, for now, the intelligibility of talk of intrinsic properties (`Plurality`, for example, assumes that such talk is intelligible when restricted to spatiotemporal relations and material properties), we would like to focus on a different aspect of the definition: determinism is ascribed to theories, but only in virtue of the nature of the worlds possible according to those theories. This is significant, because it makes the ascription of determinism depend on the manner in which theories (i.e., collections of models) represent possible worlds. It is not sufficient merely to look at the formalism of a theory; we need in addition to attend to the representational conventions involved before we can make any claims regarding determinism or a lack thereof.

Putting things together, we can set up the Hole Argument as follows:

(1) Substantivalism: `Pluralist substantivalism`
(2a) Mathematical claim: `Tensorial dynamics` ∧ `Drag-along` ∧ `No fixed fields` ∧ `Hole isometry`
(2b) Interpretative claim: `Models`
(C) Pernicious indeterminism: ¬`Intrinsic determinism`

Responses to the Hole Argument can be classified by which (and how many) of the above premises they deny. How do Halvorson and Manchak [10] fit into this classification? To answer this question, begin with Weatherall [28], who (at least focussing on his appeal to 'mathematical structuralism'—see [21] for discussion) argues that, when faced with models of general relativity related by a hole diffeomorphism, one is mandated by the formalism and/or practice of general relativity to compare those models using a map which witnesses those models' being isometric, in which case general relativity *per se* does not generate a philosophical problem of indeterminism. In other words, Weatherall's appeal to mathematical structuralism underwrites his denial of `Models`.

Turn now to Halvorson and Manchak [10]. The theorem proved by these authors might be claimed to plug a hole in Weatherall's arguments, for even on Weatherall's own terms he requires (so the claim goes) that, for any two models of general relativity related by a hole diffeomorphism, there be a *unique* map witnessing those models' being isometric. Non-uniqueness (the claim continues) would imply a multiplicity of 'legitimate' ways of comparing two models related by a hole diffeomorphism, some of which might correspond to redistributing field values on manifold points, in which case the spectre of indeterminism might re-arise. This uniqueness of isometries is (the claim ends) *assumed* by Weatherall, but is only *proved* by Halvorson and Manchak. Thus (in our reconstruction), Halvorson and Manchak also deny `Models`, but take themselves only to be warranted in doing so having proved the results presented in their article: in this sense, they agree with Weatherall, but (as we understand them) take themselves to be affording him the mathematical results required to underwrite the claims made in his article.

In brief, then: Halvorson and Manchak deny `Distinct isometries`, and this is what allows them (in their view, given their commitment to 'mathematical structuralism' of the kind which Weatherall also endorses) to deny `Models` (which is premise (2b) above), thereby evading the Hole Argument. By embracing this line of reasoning, they can be situated (with Weatherall) within a broader tradition, exemplified by Leeds [12] and Mundy [16], of formalist responses to the Hole Argument.[13] To anticipate: our response to this is going to be that (a) the denial of `Distinct isometries` needn't implicate one in the denial of `Models`, and (b) Weatherall's denial of `Models` doesn't rely in any significant sense upon the results of Halvorson and Manchak in any case: in this regard, then, these latter authors do not add to prior work on formalist responses to the Hole Argument.

3 Halvorson and Manchak's Reading of the Hole Argument

Halvorson and Manchak subscribe to the Hole Argument schema presented above [10, pp. 2–3]. However, they deny `Distinct isometries`, a denial which (as we have seen) they take to be sufficient to deny `Models`. They base their denial of `Distinct isometries` upon a theorem proved by Rynasiewicz [9]. We begin this section in Sect. 3.1 by discussing Geroch's result and how Halvorson and Manchak attempt to co-opt said result for their claim that the Hole Argument can be avoided. In Sect. 3.2, we demonstrate that Geroch's result cannot be used to support Halvorson and Manchak's claim.

3.1 Geroch's Theorem and Hole Isometries

In a paper published in 1969, Geroch proved the following theorem [9, pp. 188–9]:

> `Geroch's uniqueness theorem`: Let M and M' be connected [metric manifolds], and let w be an orthonormal tetrad at a[n element] $p \in M$ and w' at $p' \in M'$. Then there is at most one isometry $d : M \to M'$ which takes w into w'.

[13] In this broad context, our response to Halvorson and Manchak elaborated below is in the anti-formalist tradition of Rynasiewicz [24], who highlights (and argues against) the implicit interpretational claims found in the formalists' discussions. For further discussion of formalist responses to the Hole Argument, see [2].

It is worth discussing the construction that Geroch invokes in order to prove this theorem (although we will not discuss the actual proof). Consider an n-tuple of tangent vectors $\{\xi_1^a, \dots, \xi_n^a\}$ at some element p. The 'affine geodesic' γ_1 is defined to be the one whose tangent vector at p is ξ_1^a. We can parallel transport the remaining $(n-1)$ vectors $\{\xi_2^a, ...\xi_n^a\}$ along γ_1 to $T_{p'}M$, where p' is the element at a unit affine parameter distance of from p along γ_1. We can now repeat this procedure until we run out of vectors at some element q. The path composed of subsets of the n geodesics $\gamma_1, \dots, \gamma_n$ is called a 'broken geodesic'. Since M is a connected manifold, every element $q \in M$ is accessible via some broken geodesic from p. We can therefore, in each tangent space, associate uniquely an n-tuple of vectors with each element $q \in M$. Let us call the n-tuple associated with a broken geodesic its *generating tuple*.

`Geroch's uniqueness theorem` now tells us that if we map p to $d(p)$, and drag along each w at p to some $w' = d^*(w)$ at p', then any other isometry ψ that does so will agree with d on the images of all other elements. In other words, if $d(p) = \psi(p)$ and $d^*w = w' = \psi^*w$, then $d(q) = \psi(q)$ for all $q \in M$. It is straightforward to see why this is the case: the tetrad is a basis of the tangent space, in terms of which each vector of the generating tuple can be expressed. If the tetrad is preserved, then so too is the generating tuple. Following Geroch, let us call theories of whose models the preceding property is satisfied *rigid*.

Considering now two models of general relativity related by a hole isometry, `Geroch's uniqueness theorem` states that there is a unique map which witnesses those models' being isometric. This is a special case of Halvorson and Manchak's Theorem 1:

> Theorem 1: Let $\langle M, g\rangle$ and $\langle M', g'\rangle$ be relativistic spacetimes. If ϕ and ψ are isometries from $\langle M, g\rangle$ to $\langle M', g'\rangle$ such that $\phi|_O = \psi|_O$ for some non-empty open subset O of M, then $\phi = \psi$. [10, p. 17]

Having shown this, Halvorson and Manchak have demonstrated that `Distinct isometries` is false.

Now, at this point one might be confused—for how (one might ask) can Halvorson and Manchak's commitment to `Hole isometry` be consistent with Corollary 2 (regarding the "Non-existence of Hole Isomorphism") to Theorem 1 as presented in their article? Here is that corollary:

> Corollary 2 (Non-existence of Hole Isomorphism): Let $\langle M, g\rangle$ be a relativistic spacetime, and let O be a subset of M such that $M \setminus O$ has non-empty interior. If $\phi : \langle M, g\rangle \to \langle M, g\rangle$ is an isometry that is the identity outside of O, then ϕ is also the identity inside O. [10, p. 18]

Despite its name, Corollary 2 is consistent with `Hole isometry`, because the corollary states that any isometry from $\langle M, g_{ab}\rangle$ *to itself* must be the identity everywhere, so that non-trivial isometries (including hole isometries) relating $\langle M, g_{ab}\rangle$ to itself cannot exist. However, `Hole isometry` states that there exist two *distinct* models $\langle M, g_{ab}\rangle$ and $\langle M, \psi^* g_{ab}\rangle$ where ψ is a non-trivial map which witnesses those models' being isometric—and this, of course, is perfectly consistent with Corollary 2. Since Corollary 2 regards maps from $\langle M, g_{ab}\rangle$ to itself, both it, and any claims regarding the non-existence/triviality of hole isomorphisms with which it is associated, are—we contend—irrelevant for discussions of the Hole Argument as standardly construed, since those discussions trade on there being distinct models $\langle M, g_{ab} \rangle$ and $\langle M, \psi^* g_{ab} \rangle$.

3.2 Reopening the Hole Argument

It is at this point that it becomes plausible that Halvorson and Manchak can be read as plugging a gap in Weatherall's argument about how the Hole Argument is closed. (We have already presented this reading above, but we now elaborate upon it in more detail.) Let us briefly recapitulate Weatherall's argument.

Weatherall [28] claims that the Hole Argument is blocked if one accepts the following commitment (which he argues is to be derived from mathematical practice):

> `Structuralism`: The standard of cross-model sameness of points represented by manifold elements in different isometric models is to be given by the map which witnesses those models' being isometric.

This is a core thesis underlying what Pooley and Read [21] refer to as Weatherall's 'argument from mathematical structuralism'. As we have seen, one might claim that Halvorson and Manchak's central contribution to the recent formalist discussions of the Hole Argument is that in addition to highlighting that `Structuralism` is by itself insufficient to block the Hole Argument, they identify (and prove) the additional claim which they take to be required here—*viz.*, the negation of `Distinct isometries`.

It is easy to see why one might think that `Structuralism` on its own is insufficient to block the Hole Argument: if there were (*per impossibile*) multiple distinct diffeomorphisms that witnessed the isometry between $\mathcal{M}_1$ and $\mathcal{M}_2$, then there would be multiple legitimate (by the standards of `Structuralism`) ways of associating elements between $\mathcal{M}_1$ and $\mathcal{M}_2$, including some that agree on sameness of points represented by elements before some time t but disagree thereafter. On this reading, Weatherall's argument could be rendered as the following conditional:

`Structuralism` $\wedge \neg$`Distinct isometries` $\rightarrow \neg$`Models`.

The claim which we are countenancing here maintains that Weatherall assumes without proof the falsity of `Distinct isometries`; Halvorson and Manchak wield Geroch's theorem in order to establish this.

Presenting matters in this way raises three pertinent questions:

1. Does Weatherall's argument indeed presuppose the negation of `Distinct isometries`?

2. Are the commitments which Halvorson and Manchak take to dissolve the Hole Argument (namely, `Structuralism` and the denial of `Distinct isometries`) warranted?
3. Even if we accept said commitments, is the Hole Argument indeed thereby closed?

Regarding (1): we in fact think that the above line—that Halvorson and Manchak close a hole in Weatherall's argument by proving the negation of `Distinct isometries`—concedes too much to Halvorson and Manchak, and not enough to Weatherall. For in fact, the negation of `Distinct isometries` is unnecessary for Weatherall's argument (i.e., what Pooley and Read [21] dub the 'argument from mathematical structuralism', as presented in [28]) to proceed as intended (of course, whether Weatherall's argument is ultimately successful is another matter, to which we turn below). For even if there were to exist multiple diffeomorphisms witnessing the isometry between models $\mathcal{M} = \langle M, g_{ab} \rangle$ and $\mathcal{M}' = \langle M, \psi^* g_{ab} \rangle$, these maps would differ at most by a transformation which leaves the metric invariant (i.e., an automorphism of the metric)—in which case, a multiplicity of such maps would still not imply indeterminism. To see this, suppose that there are two pull-backs of the metric which coincide: $\psi_1^* g_{ab}(p) = \psi_2^* g_{ab}(p)$. From this, it follows that $(\psi_1 \circ \psi_2^{-1})^* g_{ab}(p) = g_{ab}(p)$—so $\psi_1 \circ \psi_2^{-1}$ is an $\texttt{Isometry}_1$ of g_{ab}. For a generic metric, these isometries are just the identity, so $\psi_1 = \psi_2$. And in the case in which g_{ab} has non-trivial isometries (in the sense of $\texttt{Isometry}_1$), $\psi_1 \circ \psi_2^{-1}$ is *still* an automorphism of the metric, and so does not shift fields on the manifold in such a way as to lead to the possibility of the Hole Argument re-arising. Given this, the above reconstruction of the contribution of Halvorson and Manchak's results to Weatherall's argument does not seem compelling: Weatherall's arguments needed nothing like such results to begin with; the denial of `Distinct isometries` is not a crucial-but-implicit element of his reasoning.[14]

In any case, turn now to (2): are `Structuralism` and the denial of `Distinct isometries` warranted? Clearly, the denial of `Distinct isometries` is correct; after all it is a mathematical claim established by a theorem. So let us focus our attention on `Structuralism`. In this case, we do not agree with the strictures which `Structuralism` imposes: in our view, there is no prohibition on comparing two models of a theory using any map which one pleases (after all, all such maps are perfectly well-defined mathematically); moreover, one can articulate the Hole Argument of general relativity by comparing hole diffeomorphic models using exclusively the identity map 1_M—a point made at length by Pooley and Read [21] in their discussion of Weatherall's 'equivocation argument'. Thus, in our view, the case for `Structuralism` is not compelling.

Regarding (3): suppose that we meet Halvorson and Manchak on their own terms, and accept (if only for the sake of argument) both `Structuralism` and (less

[14] We are very grateful to Henrique Gomes for discussion on the contents of this paragraph.

controversially) the denial of `Distinct isometries`. Even then, we do not think that the Hole Argument is thereby closed. The reasons here are more delicate, but are essentially those elaborated upon in depth by Pooley and Read [21] in their discussion of Weatherall's 'argument from mathematical structuralism'. We refer the reader to that article for the details, but in brief the point is easy to state: indeterminism is a metaphysical issue; simply insisting upon `Structuralism`, or (with Weatherall [28]) stating that isometric models of general relativity have "the same representational capacities" does not *per se* address this metaphysical issue. Insofar as one thinks that these metaphysical matters are (at least to some degree) independent of the mathematical tools which we use to represent possibilities, then—transparently—denying `Distinct isometries` does not help the advocate of `Structuralism` to overcome these issues: if one has reason to think that there is a plurality of haecceitistically distinct possible worlds (which is, arguably, the historically default substantivalist position—one which, indeed, is invited very naturally by the formalism of general relativity: see [21, p. 22]), then supplementing `Structuralism` with the negation of `Distinct isometries` will not assuage one's worries regarding determinism generated by the Hole Argument.

In brief, then, and in sum: there is a *prima facie* appealing reading of Halvorson and Manchak [10] as attempting to fill a lacuna in Weatherall's formalist argument against the force of the Hole Argument; in fact, however, we have seen that such a reading is too charitable to Halvorson and Manchak and insufficiently so to Weatherall, for whom—we claim—no such results were needed to begin with. Given this, it is unclear what Halvorson and Manchak's results on this front add to this class of recent responses to the Hole Argument. But in any case, we also do not think that the combination of `Structuralism` (the central tenet of all such formalist responses) and the denial of `Distinct isometries` is sufficient to "close" the Hole Argument. In fact, closing the Hole Argument, in the structuralist manner that Halvorson and Manchak suggest, would, in addition, require either (i) a non-trivial argument for the truth of the above conditional, or (ii) a supplementary conjunct in the antecedent that renders the conditional true. Halvorson and Manchak provide neither.

4 Outstanding Issues

In this section, we tie up a few loose ends. We discuss three claims made by Halvorson and Manchak in support of their view with which we disagree: the first on determinism (Sect. 4.1), the second on the structure of the category of models for general relativity (Sect. 4.2), and the third on essentialism (Sect. 4.3).

4.1 Determinism

Although determinism is at base a *metaphysical* issue—do the laws plus the state of the universe at a particular time fix the state of the universe at some or even all other times?—the reasons underlying why determinism is considered problematic are principally *epistemic*: if we know the laws and the appropriate boundary conditions, but are nonetheless unable to predict all other conditions, then, in particular, knowledge of the future via knowledge of the laws and of the past is limited to at least some degree.[15] By invoking both formal as well as metaphysical considerations, `Intrinsic determinism` latches onto this sense in which indeterminism is worrisome. That being said, it is important to recognise that `Intrinsic determinism` is far from being the only notion of determinism on the table (for an extensive catalogue of plausible alternatives, see [6]).

One might legitimately prefer not to characterise determinism in terms of intrinsic properties, depending upon one's metaphysical views (Halvorson and Manchak [10] fall into this camp). But whatever the alternative characterisation at which one arrives, it is crucial that, in order to capture the genuine epistemic worry of indeterminism, it make explicit the importance of both formal as well as representational commitments (more on this below). After complaining that `Intrinsic determinism` is insufficiently precise, Halvorson and Manchak consider two alternative characterisations: (i) `MLE determinism`,.[16] and (ii) `Dynamical rigidity`:

> `MLE determinism`: A theory T is MLE deterministic iff for any two models M and N of T, if there is an initial segment U such that $M_{|U} = N_{|U}$, then $M = N$, where an 'initial segment' $U \subseteq I$ is a suborder of a linear order I such that U is nonempty and for any $j \in U$, if $i \in I$ and $i \leq j$, then $i \in U$.
> `Dynamical rigidity`: A theory T is dynamically rigid iff for any two models M and N of T, and any two isomorphisms $f, g : M \rightarrow N$, if $f_i = g_i$ for all i in some initial segment U, then $f = g$.

Given our demand that a good definition of determinism capture the epistemic worry described above, we can, with Halvorson and Manchak, disregard `MLE determinism`: it is a purely formal characterisation, and as such, suffers from fairly generic worries that stem from Putnam- and Goodman-style paradoxes of reference, according to which, broadly speaking, nothing in the structure of models fixes the 'semantic glue' between words and their referents. As Halvorson and Manchak observe, "the construction... is not very interesting: it just uses the fact that for any set M_i, there is an isomorphic but non-identical set N_i" [10, p. 21].

Consider, then, `Dynamical rigidity`. In the present context, where for dialectical clarity we focus on vacuum solutions so that the standard of isomorphism is isometry of Lorentzian manifolds, `Dynamical rigidity` is very closely related to `Geroch's uniqueness theorem`. Indeed, if the conditions of Geroch's theorem are satisfied, then general relativity turns out to be dynamically rigid. And this is precisely what Halvorson and Manchak assume. Unsurprising, then, that if this constitutes the standard of determinism, then general relativity is deterministic.

[15] Assuming that we do not have other means of ascertaining the global state of the universe—e.g., divine insight. It's an open question whether the resources of indexicals can be brought to bear in order to resolve these epistemological issues—see [3] for discussion.

[16] After Montague, Lewis, and Earman: see [10, p. 16].

Here, one might charge Halvorson and Manchak with putting their thumbs on the scale by choosing to define the notion in terms of the mathematical property of `Dynamical rigidity`, when in fact there remains a rich metaphysics literature on determinism—recall Sect. 2.3—for which the status of general relativity with respect to determinism is far from trivial.

Although Halvorson and Manchak do argue that the definition of determinism in terms of `Dynamical rigidity` is preferable to `MLE determinism`, this strikes us as insufficient evidence that theirs is the most appropriate definition *tout court*.

4.2 A Category Mistake?

Following Halvorson and Manchak [10], consider three categories, **Man**, $\mathbf{Man_g}$, and **Lor**. Objects of **Man** are differentiable manifolds; morphisms are diffeomorphisms. Objects of $\mathbf{Man_g}$ are pairs $\langle M, g_{ab} \rangle$ of differentiable manifolds M and Lorentzian metric fields g_{ab} on M; morphisms are again diffeomorphisms between manifolds (i.e., the metrical structure of the objects is ignored). Objects of **Lor** are are again pairs $\langle M, g_{ab} \rangle$, but now morphisms are *isometries* (of the $\texttt{Isometry}_2$ type) between these pairs. There is a forgetful functor relating **Lor** and $\mathbf{Man_g}$: although the objects in these categories are the same, the former has fewer morphisms than the latter.[17]

Halvorson and Manchak point out that diffeomorphisms between differentiable manifolds needn't preserve a great many affine or geometrical features—e.g., lengths of curves, the timelike/spacelike nature of vectors, or flatness [10, pp. 9–10]. Accordingly, they claim that—historically—philosophers writing on the Hole Argument have been confused insofar as they have focussed on diffeomorphisms: in their preferred language of category theory, the claim is that authors have mistakenly focused on **Man** or $\mathbf{Man_g}$, whereas instead authors should have recognised the correct category as being **Lor**, which has a more discerning notion of equivalence.

As a statement about the correct category to consider, we can grant that this is correct—although we find Halvorson and Manchak's historical claim not to be entirely fair. Indeed, as we mentioned in Sect. 2, the term 'diffeomorphically related' as it applies to models is really a shorthand for models whose tensorial contents are dragged along by a diffeomorphism; nobody ever suggested otherwise. In any case, though, the claim that one should use **Lor** is certainly endorsed implicitly by commentators such as Fletcher [8] and Weatherall [28]. But the real question is: does using the standard of equivalence of models afforded by **Lor** suffice to block the Hole Argument? For the reasons already discussed above, we think not: even if one thinks (as, we take it, with Weatherall, Fletcher, and Halvorson and Manchak) that using the standard of equivalence of models afforded by **Lor** implies `Structuralism`, we have already seen that this commitment by itself is insufficient to block the Hole Argument.

[17] For accessible background to the relevant category theory here, see [27].

4.3 The Essentialism Tension

Towards the end of their article ([10, §7]), Halvorson and Manchak draw a comparison between their own work and Maudlin's 'metric essentialist' response to the Hole Argument [14]. Recall that, on metric essentialism, spacetime points have their metrical properties essentially, so that in fact only one of the class of possible worlds represented by models of general relativity related by a hole diffeomorphism is a genuine metaphysical possibility. Halvorson and Manchak profess to being inspired by Maudlin's attendance to the metric as being important in resolutions of the Hole Argument—although, quite rightly, they acknowledge that the details of their response differ substantially from the details of Maudlin's (mathematical considerations versus heavy-duty metaphysics, respectively).

They then proceed to make a claim about Maudlin's views which we find problematic. Consider a theory T' (see [10, p. 26]), according to which (i) there are exactly two people, (ii) exactly one of those people has blond hair, and (iii) hair colour is an essential property. On T', Halvorson and Manchak write that

> if T' has a model where Alice has blond hair, then it cannot have a model where Bob has blond hair; because if Alice has blond hair in one model, then she has blond hair in all models. However, Tim is now in an awkward position: he does not know what the models of his theory are until he determines which person has blond hair. So, to the extent that knowing a theory is knowing what possibilities it permits, Tim does not even know his own theory. In contrast, I know exactly which possibilities my theory permits. [10, pp. 26-27]

The idea here seems to be that, if hair colour is an essential property, then Alice and Bob cannot be represented as having different hair colours in different models of T'. This strikes us as confusing mathematics and metaphysics: hair colour being an essential property is a *metaphysical* issue; T' can have models in which Alice has blonde hair and hair colour is an essential property, and models in which Alice has some other hair colour and hair colour is an essential property—it's simply that one of those models—if hair colour is indeed an essential property!—will not correspond to any possible world.[18] For this reason, Maudlin's metric essentialism is not confused in the way that Halvorson and Manchak suggest.[19]

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[18] Of course, one could still maintain that a theory's having models which correspond to no possible world is pathological. Agreed—but this is a different objection to that raised by Halvorson and Manchak.

[19] Importantly, and very likely contrary to Halvorson and Manchak, we see no reason why these facts about essential properties need be represented *explicitly* in the models of one's theory.

Declarations

Conflict of interest The authors declare no competing interests.

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