A NUMBER SENSE APPROACH TO WRITTEN CALCULATION: EXPLORING THE EFFECTS IN THE MIDDLE YEARS OF SCHOOLING

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STATEMENT OF AUTHORSHIP AND SOURCES

This thesis contains no material published elsewhere or extracted in whole or in part from a thesis by which I have qualified for or been awarded another degree or diploma. No parts of this thesis have been submitted towards the award of any other degree or diploma in any other tertiary institution. No other person's work has been used without due acknowledgment in the main text of the thesis. All research procedures reported in the thesis received the approval of the relevant Ethics/Safety Committees.

RICK OWENS

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ABSTRACT

The purpose of this research was to investigate some of the effects on teachers and students of positioning written calculation within a commitment to building students' number sense.

The focus on number sense took shape initially through explicit teaching of a strategies approach to mental computation, followed by an exploration of approaches to written calculation which made use of effective mental computation strategies.

The impetus for this research came from the following observations of many classrooms and a review of the available literature:

- the dominant aspect of calculation in many schools in the primary and middle years of schooling (here deemed as up to Year 8 in schools in the Australian Capital Territory) is the teaching and using of formal written algorithms
- for many students this emphasis works against overall facility with calculation and the development of number sense.

This study investigated the following research question:

What are some of the effects on teachers and students within a junior high school setting, of aligning written calculation with a strategies approach to teaching and using mental computation?

Of specific interest, are there any effects related to:

- conceptual understanding of number and operations with numbers?
- facility with performing calculations?

This research used a mixed methods approach (Johnson & Onwuegbuzie, 2004) with the primary emphasis on case study methodology (Merriam, 1998). The bulk of the data involved descriptions of comments and behaviours that occurred during classroom lessons, planning and review sessions, and semi-structured interviews. This was augmented by a variety of student work samples, and student performance data on a range of calculation tasks, the analysis of which helped shape further elements of the case study. Initially the aim was to document fully three separate cases, where each case involved a teacher and his/her Year 8 class in a program of regular weekly intervention lessons over one school term. In the ACT students in Year 8 are typically 12 or 13 years

of age at the beginning of the school year. At the end of the data collection phase, one case in particular (Class C) offered a substantially more robust context for analysis and became the major focus of this study.

Each of the three teachers and their Year 8 classes worked for approximately the first half of the term with weekly sessions of explicit teaching around a strategies approach to mental computation using the four operations and whole numbers. Students were encouraged to explain, record and compare their strategies for calculating mentally. To varying degrees across the three cases, efficient and understood mental strategies were described visually and symbolically, with selected strategies extended to calculations which involved numbers just beyond the scope of reasonable mental computation. Completion of these tasks was supported with examples of pen and paper strategies for recording a series of calculations done mentally. This approach built on and extended earlier work by McIntosh (2002) which targeted students in Years 2 to 4 (aged between 6 and 9 years at the beginning of the school year) and the operations of addition and subtraction.

Data were collected on:

- teachers' stated beliefs about and practices in this part of the mathematics curriculum
- students' performance on mental and written computational tasks
- selected students' approaches to choosing and using the three forms of calculation: mental, written and electronic
- student and teacher behaviour during the weekly intervention sessions.

Instruments for collecting this data included:

- semi-structured and informal interviews with teachers and selected students
- written tasks identifying students' choice of computational approach and preferred method for doing a given calculation
- records of lesson observations
- written in-class tasks related to the focus of each weekly intervention.

Notes from the weekly intervention sessions with Class C, and the in-class tasks produced each week by the students from that class, became the primary sources of qualitative data used in this study.

There were three main areas for which the data from this study provided demonstrable support:

1. The importance of building conceptual understanding, illustrated most clearly in this study around understanding the operation of division.
2. The value of representing mathematical concepts in multiple and meaningful ways; typically (but not limited to) words, objects or pictures, and mathematical symbols.
3. The value of explicit teaching of mental computation for building number sense and improving performance with mental and written calculation.

This study has suggested that the teaching and use of number sense written methods, for a limited range of calculations with whole numbers, can contribute to strengthening conceptual understandings of place value and the four operations. It has also demonstrated the value of further discussion about and research into the relevance, purpose and scope of written calculation in the school mathematics curriculum.

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CHAPTER ONE: INTRODUCTION

Few would argue against the idea that understanding numbers and reliably calculating with them are important outcomes for all students, certainly by the time they exit the compulsory years of schooling. If not the full picture, then these things are at least core aspects of what we commonly understand as *being numerate* (Askew, 1999).

It is a continuing national priority that all students should be numerate by the time they leave school (Ministerial Council on Education, Employment, Training and Youth Affairs, 1999; 2008). Over ten years ago the *National Numeracy Benchmarks* (MCEETYA, 2000) were developed specifically to support schools in achieving this outcome.

Along with other selected aspects of the school mathematics curriculum these numeracy benchmarks describe key markers of the minimum acceptable performance under the heading *number sense* for students in Years 3, 5 and 7. As they move through their schooling all students in Australia are expected, among other things, to show progressive improvement in their understanding and application of numbers and the four operations of addition, subtraction, multiplication and division. They are also expected, from Year 3 onwards, to become increasingly competent with mental, written and electronic forms of calculation.

But how much classroom time and energy should be allocated to each form of calculation? And what might the teaching and use of each form of calculation look like in practice at different stages of schooling?

Number sense is not easily defined. Nonetheless developing students' number sense is becoming a priority in educational research and curriculum reform, and richer understandings of number sense have continued to evolve. Teaching that emphasises the meaning of, and the relationships between, mathematical concepts is being recommended (McIntosh, Reys, Reys, Bana, & Farrell, 1997; National Council of Teachers of Mathematics, 2000). There is a significant amount of literature which points to the value of improving students' mental computation as a tangible way of doing this (Cockcroft, 1982; McIntosh, 1990; McIntosh & Dole, 2004). There is also widespread acceptance that electronic forms of calculation are here to stay, with their use in schools and society almost certain to continue to increase (Cumming, 1999; Kalantzis & Cope, 2005).

Some literature has highlighted that the approach to written calculation which is typically taught in schools does not require the same level of understanding of number or the four operations as does effective mental or electronic computation (McIntosh, 2002; Plunkett, 1979). Others have levelled quite harsh criticism at the standard formal algorithms for written calculation commonly taught in schools, arguing that these formal written procedures actually *undermine* the development of number sense (Behrend, 2001; Kamii & Dominick, 1998).

Over the last 15 years in Australia the various national documents influencing the development of school mathematics curriculum have all made reference to, and accepted as valid, the notion of using alternate methods to formal algorithms for written calculation (Australian Education Council, 1990, 1994; MCEETYA, 2000). However there appears to have been little serious consideration of these frequently mentioned *informal* or *non-standard* written methods of computation as legitimate curriculum alternatives to formal algorithms (McIntosh, 2002). The documentation that does exist seems to focus on the early to middle years of primary school (Behrend, 2001; Kamii & Dominick, 1998; McIntosh, 2002).

THE RESEARCH PROBLEM

Plunkett (1979) gave one view of a balanced approach to calculation in schools:

> I think that the reasons for teaching the standard written algorithms are out of date, and that it is time we all took notice of this. I believe there is a place for mental algorithms, for the use of calculators, and for ad hoc, non-standard written methods. I think a large amount of time is at present wasted on attempts to teach and to learn the standard algorithms, and that the most common results are frustration, unhappiness and a deteriorating attitude to mathematics. (p.4)

Since then there has been considerable evidence collected that supports this view:

- mental computation is by far the most common approach to doing calculations in adult life (McIntosh & Northcote, 1999)

- mental computation and estimation are more important to success both in and beyond school than facility with formal pen and paper routines (Reys & Reys, 2004)
- using calculators in schools sensibly and effectively improves students' confidence with mathematics and their performance with mathematical tasks (Groves, 2004; Sparrow & Swan, 2004)
- formal written calculation still dominates classroom computation in the primary years of schooling (Sparrow & McIntosh, 2004).

Written methods of calculation which make use of number sense (referred to in this study as number sense written methods) may have several advantages for a mathematics curriculum that has developing students' number sense as a priority. These methods:

- are more like how most people calculate mentally
- make sense to the user, and so are easier to remember and review
- use and reinforce place value and the meanings of the different operations
- are more reliable, particularly for underperforming students (Carroll & Porter, 1998).

In considering a balanced approach to teaching and using calculation in the classroom, several factors draw the focus of the discussion to the place and nature of written calculation in schools. These factors include:

- an apparent misalignment in nature, purpose and effect between the traditional formal methods of written calculation common in school mathematics, and effective mental and electronic calculation
- in general a dominant emphasis on formal written computation in mathematics classrooms in the middle years of schooling[1].

[1] Deemed here to be years 5 to 8, with students typically between 9 and 13 years of age at the beginning of the school year

THE RESEARCH PURPOSE

The purpose of this research was to describe some of the effects, on teachers and students within a junior secondary setting, of positioning written calculation within a commitment to improving students' number sense.

This focus was likely to challenge a range of cultural norms regarding the place of written calculation in the mathematics curriculum. At the same time it was an opportunity to gain insights into ways of supporting teachers in describing and implementing a more balanced approach to this aspect of the curriculum.

THE RESEARCH QUESTION

This study investigated the following research question:

What are some of the effects on teachers and students within a junior high school setting, of aligning written calculation with a strategies approach to teaching and using mental computation?

Of specific interest, are there any effects related to:

- conceptual understanding of number and operations with numbers?
- facility with performing calculations?

SIGNIFICANCE OF THE RESEARCH

This research will contribute to understanding the issues around, and the impact of, aligning written methods of calculation with a focus on developing number sense through an emphasis on a strategies approach to mental computation. It is an attempt to explore aspects of the coherence and balance required of a contemporary approach to calculation in the later part of the middle years of schooling.

Here *coherence* means that the three approaches to calculation (mental, written and electronic) work together to reinforce key aspects of number sense, such as the relative size of numbers, part/whole relationships, and the nature and appropriate application of the four operations (McIntosh et al., 1997). *Balance* refers to an appropriate amount of curriculum emphasis and class time allocated to each of the three approaches to calculation. Ideally this balance would reflect what is now known about how people learn and use mathematics, and the demands and opportunities of the knowledge society within which schools currently exist. In this context balance does not necessarily imply equal amounts of class time and emphasis.

SCOPE AND DELIMITATIONS

There were several reasons for choosing Year 8, the second year of high school in the ACT, as the arena for this research:

- Much of the literature dealing with alternative approaches to written calculation has focused on the early to middle primary years (Behrend, 2001; Clarke, 2005; McIntosh, 2002).
- In the ACT, as in every Australian state and territory, Year 8 is bracketed by system assessment programs in Years 7 and 9. This established the potential for additional school-initiated research with this cohort, such as differences in improvement rates on number assessments between students involved in this research and their peers currently in parallel classes. It also removed any concerns from participating schools about altering the way this aspect of the mathematics curriculum is typically delivered.
- Access to teachers who were willing to participate in this research was available.

Chapter Two outlines a review of the available literature relevant to this research.

CHAPTER TWO: REVIEW OF THE LITERATURE

This literature review is focused on computation: calculating with numbers. The context for the review is the teaching and learning of mental, written and electronic approaches to computation in the middle years of schooling. This review provides the context for interpreting the research that was undertaken in this study into the effects on teachers and students of aligning approaches to written calculation with a focus on developing number sense through explicit teaching of a strategies approach to mental computation.

SETTING THE SCENE: THE COCKCROFT REPORT

Mathematics Counts, also known as the Cockcroft Report (Cockcroft, 1982), documented the findings of a comprehensive three-year inquiry into the teaching of mathematics in primary and secondary schools in England and Wales. The report addressed many things that were also relevant at that time to the teaching and learning of mathematics in other countries, including Australia (Stephens, 1984). Aspects of the inquiry included the mathematics required for trade and tertiary education, employment and adult life in general, and recommendations for improving the contribution of school mathematics to the mathematical development of society.

Particularly relevant to this literature review is the fact that the Cockcroft Report was written at a point in time that was a watershed in the use of electronic calculation, both in schools and in society generally. In this regard it provides a close-to-baseline description of issues around, and perspectives on, teaching calculation in the electronic age.

Key themes, findings and recommendations included the following:

- dealing effectively with the mathematical needs of adult life required “a feeling for number“ (p.10) that showed itself in facility with approximation, estimation and mental computation, and confidence in using whatever mathematical understandings and skills a person might have

- effective teaching and learning of mathematics involved understanding mathematical concepts and connections between them, and not just mechanical performance of mathematical processes
- mental computation was central to effective calculation
- in the workplace written calculation was rarely done the way it was taught in schools; longer processes that involved a series of short stages that could be used confidently were often preferred
- although electronic calculators were viewed positively in those workplaces in which there was a lot of calculation or data analysis, the inquiry found that calculators were totally absent from many primary classrooms, and many secondary mathematics teachers actively discouraged or would not allow their use.

Collectively these points raise some fundamental questions about the school mathematics curriculum. Firstly, what approaches to written computation should be taught in schools; and secondly, what might an appropriate balance between mental, electronic and written methods of computation look like in primary and secondary schools. The report was adamant that changes in the mathematics curriculum were necessary and inevitable. Suggested changes included:

- explicit and extended instruction in the sensible and effective use of calculators
- a reduction in the range of calculations expected to be performed by students with pen and paper
- less class time spent practising and using written methods of computation.

Much of what the Cockcroft Report said regarding the relative emphases on mental, written and electronic forms of calculation in the mathematics curriculum is still relevant. But what is painfully remarkable is how little impact the clear messages of that report appear to have had on the way calculation is taught and used in schools today (Sparrow & McIntosh, 2004).

NUMERACY AND THE MATHEMATICAL NEEDS OF ADULT LIFE

The concept of numeracy has continued as a focus in educational, social and political circles, but interpretations vary. Stephens (1984) criticised what he saw as the limited view of the mathematical needs of adult life in the Cockcroft Report (Cockcroft, 1982) which led to a description of numeracy as mostly about being comfortable with using number in day to day contexts, and understanding information in charts, tables and graphs. In reporting how the term numeracy was defined for the purposes of a specific research project, Askew (1999) propagated a similarly limited view by using numeracy to mean what others have described as elements of number sense (McIntosh et al., 1997).

However more people are using more mathematics in more aspects of life, and some current understandings of numeracy reflect this. In an attempt to capture a sense of the broad nature and expanding relevance of numeracy, both generally and with specific reference to schooling, the Australian Association of Mathematics Teachers defined numeracy this way:

> To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life.
> In school education, numeracy is a fundamental component of learning, discourse and critique across all areas of the curriculum. It involves the disposition to use, in context, a combination of:
> - underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical and algebraic);
> - mathematical thinking and strategies;
> - general thinking skills; and
> - grounded appreciation of context. (AAMT, 1998)

Related concepts include *mathematical literacy* or *quantitative literacy*. Steen (2003) drew attention to the relatively recent impact of technology in creating a data-rich global society, and examples of efforts around the world to improve the contribution of mathematics curriculums to developing numerate citizens. He quoted the *Program for International Student Assessment* (PISA) definition of mathematical literacy as

> the capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgments, and to engage in mathematics in ways that meet the needs of an individual's current and future life as a constructive, concerned and reflective citizen. (Organisation for Economic Co-operation and Development, as cited in Steen, 2003, p.211)

How this broader view of numeracy relates to the aims, content and delivery of the mathematics curriculum in schools is not straight-forward. Hogan (2004) reported on research into connections between students' performance on a range of mathematics assessments and their capacity to engage with the mathematical demands of other subjects across the curriculum. This two-year project involved approximately 1100 students in Years 5 and 7 across 18 schools in Western Australia. He described components of a numeracy framework that attempts to identify the dimensions required to make judgments about numerate behaviour. Such behaviour is seen as a mix of three types of *knowledge* (mathematical, contextual and strategic) and three *roles* (fluent operator, learner and critic). Descriptions of *numeracy moments* from across the curriculum illustrated the interconnections between these dimensions and suggest that numeracy is far more complex than just skill with arithmetic. Key findings included:

- the importance of understanding contextual factors in choosing and using mathematics appropriately
- the overriding significance of having the confidence to engage with mathematical tasks
- the importance of teachers across all subject areas taking responsibility for supporting their students' numeracy development (p.viii).

Richardson and McCallum (2003) discussed the place of quantitative literacy in the tertiary education sector, and as a corollary suggested two necessary emphases in the way mathematics as a subject should now be taught. The first is a focus on developing conceptual understanding. They described many students as "technically capable but unable to make reasonable decisions about which techniques to apply and how to apply them" and defined conceptual understanding as "an ability to recognize underlying (mathematical) concepts in a variety of different representations and applications" (p.101). The second emphasis is that the teaching of mathematics must take place within a meaningful context within which the mathematics is significant.

This view resonates with that of Cumming (1999) who stressed the importance of learning mathematics in ways similar to how mathematics is now used in our world. That is, critically applying mathematics with understanding, to firstly describe and shape problems and then solve them using whatever helpful technology is available. Some of the required shifts in classroom practice include more opportunities for oral language, a focus on mathematical modelling, and an expanding use of technology. These shifts align with the stated objectives for schooling in Australia in the Melbourne Declaration (MCEETYA, 1999).

The view of numeracy briefly outlined in the Cockcroft report (1982) drew attention to the importance of mathematics for effectively engaging with day to day life. More recent views of numeracy (AAMT, 1998; Hogan, 2004) reflect the increased complexity of day to day life, and the depth and breadth of mathematical thinking demanded by this complexity. Such conceptualisations of numeracy also raise questions about the contribution the school mathematics curriculum could and should make towards developing a numerate society.

TEACHING AND LEARNING MATHEMATICS

Many aspects of teaching and learning mathematics identified by the Cockcroft report (1982) are still significant. The following sections illustrate this, with particular emphasis on those aspects that appear directly relevant to developing a balanced approach to computation in the school curriculum. These include:

- an evolving understanding of the nature and importance of number sense
- the relationship between formal written calculation and number sense development
- student performance with applied calculation.

A move towards valuing conceptual understanding and number sense

There has been a strong shift in stated curriculum emphases towards teaching and learning mathematics with understanding. This focus on sense-making in mathematics is clearly articulated in the *Principles and standards for school mathematics* (2000) published by the National Council of Teachers of Mathematics in the US:

> Research has solidly established the importance of conceptual understanding in becoming proficient in a subject. When students understand mathematics, they are able to use their knowledge flexibly. They combine factual knowledge, procedural facility, and conceptual understanding in powerful ways.
> Learning the "basics" is important; however, students who memorize facts or procedures without understanding often are not sure when or how to use what they know. In contrast, conceptual understanding enables students to deal with novel problems and

> settings. They can solve problems that they have not encountered before. (NCTM, 2000, The Learning Principle, para. 1 & 2)

Contrasting recall of factual knowledge and facility with mathematical procedures on the one hand, with conceptual understanding on the other, brings to mind the seminal work of Richard Skemp (1976). Skemp has become associated with the terms *instrumental understanding* and *relational understanding,* which were coined by one of his colleagues in an attempt to distinguish between two quite different interpretations of the word 'understanding' as it was being used in the practice and discussion of mathematics education at the time:

> It was brought to my attention some years ago … that there are in current use two meanings of this word (understanding). These he distinguishes by calling them 'relational understanding' and 'instrumental understanding'. By the former is meant what I have always meant by understanding … knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as 'rules without reasons', without realising that for many pupils *and their teachers* the possession of such a rule, and ability to use it, was what they meant by 'understanding'. (Skemp, 1976, p.2)

In terms of teaching and learning about number, an emphasis on building conceptual understanding from Kindergarten to Year 12 would manifest itself in the curriculum as a foundational commitment to developing students' *number sense*:

> Students with number sense naturally decompose numbers, use particular numbers as referents, solve problems using the relationships among operations and knowledge about the base-ten system, estimate a reasonable result for a problem, and have a disposition to make sense of numbers, problems, and results. (NCTM, 2000, Number and Operations, para. 3)

The concept of number sense is often referred to in the literature (Markovits & Sowder, 1994; McIntosh, 1990). McIntosh, Reys, Reys, Bana and Farrell (1997) defined number sense as:

> a propensity for and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity (makes sense). (p.61)

They described a number sense framework with three core dimensions: knowledge of and facility with *numbers*, knowledge of and facility with *operations*, and applying these to *computational settings*. Aspects of these dimensions included:

- multiple representations of numbers, such as equivalences based on part/whole relationships
- understanding mathematical properties (such as associativity and commutativity) and the relationships between the basic operations
- interpreting a context to decide on levels of accuracy possible or required, and appropriate approaches to calculating an answer
- facility with using and interpreting the results of mental, written and electronic methods of calculating. (McIntosh et al., 1997)

Number sense and formal written calculation

There is evidence that number sense and facility with formal written calculation are not the same thing. McIntosh et al. (1997) reported on a project that involved three separate studies which attempted to assess the number sense of school students in Australia, Sweden, Taiwan and the United States. Two studies targeted student performance. One involved approximately 1100 students (in cohorts 8, 10, 12 and 14 years old) in eight schools from Australia and the US; the other, reported separately in Reys and Yang (1998) involved a total of 234 students in grades 6 and 8 in Taiwan. The number sense framework referred to earlier (McIntosh et al., 1997) was used to develop items for a written test of number sense at each age level. A mental computation test was also given to students participating in the US and Australian number sense study.

A key limitation of these studies was using pen and paper testing to assess students' number sense, and the project recommended further research into the value of assessing number sense this way. However a significant finding consistent across the age groups and countries represented was that core aspects of number sense did not develop as a result of facility with pen and paper calculation:

> The emphasis on developing standard written algorithms for dealing with whole numbers, decimals and fractions, which still pervades almost all schools at the ages tested, does not appear to bring with it a practical understanding of place value, an ability to estimate quantity or an instinctive or true feeling for the nature of decimals. (McIntosh et al., 1997, p.53)

This was most evident in the Taiwanese study (Reys & Yang, 1998) where there was consistently higher performance on tasks involving written computation than on parallel tasks that approached the same context from a number sense perspective. Detailed interviews were conducted with 17 of these students, with the group almost equally divided between high and medium performance on the previous written tasks. All students attempted to use formal written calculation as their opening strategy for solving number sense tasks. Interestingly only the students with high performance on the written tasks had alternate solution strategies that involved aspects of number sense, but these were only used when prompted by the interviewer. This suggested that the emphasis in the curriculum in Taiwan strongly affected students' choice of computation method, leading to mechanical performance as a default, and in so doing perhaps limited the development of number sense for the majority of students.

Gillies (2004) compared two methods for teaching calculating drug dosages to students in their first year of their nursing degree. The author took the distinction between relational and instrumental understanding developed by Skemp (1976) and applied it to teaching calculation strategies for use in a professional setting. Performance data on calculation tasks as well as interview responses were collected from 19 students. Seven students were taught to calculate using a problem-solving approach, the remaining students were taught using formula-based methods. The aim was to compare students' performance with and attitudes towards these tasks immediately at the end of, and then again sometime after, the teaching intervention on this topic. The author recognised that the short duration of the teaching intervention (four weeks) and the small number of participants (n=19) limited the strength of any conclusions. The procedural, formula-based approach led to greater gains on short-term performance measures, but contrasts in affective measures (such as participants' confidence and capacity to apply their learning in unfamiliar situations) raise serious questions about the long-term efficacy of teaching with a procedural emphasis. Given the high-stakes nature of the workplace decisions that will be affected by nurses' facility with calculation under duress this certainly warrants further research (Gillies, 2004).

These studies suggest that although students might rely on formal written approaches to calculation (and in fact demonstrate reasonable short-term performance with them) these methods do not appear to contribute to developing students' number sense, central components of which are an understanding of numbers and of operations with them.

Skill without understanding

It is reasonable to question the relevance of conceptual understanding to the effective development and application of mathematical skills such as calculating with numbers. Under the heading *Over-explanation of processes to duller pupils,* Schonell (1937) described one perspective on this issue:

> … what they need to know is how to get the sum right, and when they have learned the method so thoroughly that the possibilities of getting the particular type of sum wrong are only the ordinary ones due to chance, then explanations might be attempted … why should we burden children with unnecessary explanations in arithmetic? (p.73, quoted in Anghileri, 2001)

Pesek and Kirshner (2002) also drew on the model of understanding described by Skemp (1976) when they investigated the relative effects of emphasising teaching for meaning (relational understanding) or for skill development (instrumental understanding). The study involved two separate groups, each with three classes of Year 5 students. One group received five days of instrumental instruction on finding the area and perimeter of selected plane shapes. Then both groups were given three days each of relational instruction on the same material. The authors used performance data on written tasks (pre-test, post-test and a retention test given two weeks after the completion of instruction) in tandem with student interview data. They found evidence that initial rote learning of a concept can actually interfere with later meaningful learning and performance. It is worth considering the possibility of finding similar interference effects in other areas of the mathematics curriculum.

Behrend (2001) provided examples of several grade 3 students misapplying algorithmic procedures when attempting to solve addition and subtraction tasks. The rule-based strategies appeared to derail students' previously demonstrated number sense and generated answers that could not possibly be correct. A major concern was that these answers went unquestioned by the students. The author cautioned against teaching calculation through repetition of rules. She noted that such an emphasis interferes with students' ability to see relationships between mathematical facts and ideas, and is particularly disadvantageous for students with learning disabilities.

Kamii and Dominick (1998) were even more outspoken in their criticism of focusing on written calculation in grades 1 to 4. They stated that formal written algorithms actively undermine number sense by suspending place value and by promoting an unthinking approach to doing and learning mathematics. This refers to the way that formal algorithms make use of the *column value* of the digits that compose a number, and not the *quantity value* more common to estimation and mental calculation (Thompson, 2004). For example the number 237 is made up of the *digits* 2, 3 and 7. The *column value* of each digit is given by their position: 2 hundreds, 3 tens, and 7 ones. However the *quantity value* of each digit is 200, 30 and 7 respectively.

The examples above offer some support for the points made in the Cockcroft report (Cockcroft, 1982) on teaching and learning mathematics. In particular they suggest how important understanding is to the reliable and effective performance on computational tasks. They also illustrate the all too common disjunction between students' facility with procedures and their understanding of mathematical concepts. However, Star (2005) warned against the vilification of procedural understanding, as some have done in attempting to align themselves with Skemp's work on relational and instrumental understanding (Pimm, 2002; Skemp, 1976). He suggested there has not been sufficient focus on developing an appropriate concept of procedural knowledge, and that this is worthy of more attention.

As this debate goes back and forth, schools, teachers and students will continue to engage in various ways with teaching, learning and using three forms of calculation: mental, electronic, and written (AEC, 1990; MCEETYA, 2000). It is within this context that tangible steps forward towards the goal of developing students' number sense are most likely to occur.

Mental computation and estimation

McIntosh and Dole (2004) defined mental computation as calculating exact answers without using calculating devices or any recording of results. Effective mental computation requires a good understanding of numbers and operations, and the ways numbers can be connected through those operations; in essence, competence with mental calculation requires good number sense.

Other reasons why mental computation is important include the following:

- it is the most common form of calculation
- it is used when working out estimates (including when checking the reasonableness of answers produced by electronic calculation)
- it is often the simplest and easiest way to do a given calculation
- it involves active problem-solving
- it helps develop number sense (McIntosh, 2005).

Research by McIntosh and Northcote (1999) involved documenting the number and nature of calculations done in the day to day lives of a broad cross-section of 200 people in a 24-hour period. The authors found that almost 85% of all calculations were processed mentally, and that almost 60% of all calculations only demanded an estimate as opposed to an exact answer.

Mental computation has been an explicit focus in national and local curriculum documents since the national statement, as have applications of mental computation to calculating approximations and estimates (AEC, 1990; AEC, 1994; MCEETYA, 2000). Sowder (1994) stated these three things (mental computation, approximation and estimation) “have the power to increase number sense when students are encouraged to seek out alternative strategies and reflect on them” (p.141). There was evidence of this in the Australian and US study by McIntosh et al. (1997) where they found a connection between students’ performance on written number sense tasks and measures of their ability with mental computation, particularly with students over 12 years of age. They concluded that mental computation should be part of any assessment of students’ mathematical ability (a point directed largely at the national testing agenda topical at that time) and that improving mental computation would contribute significantly to developing number sense.

To this end in 2001 to 2003, education departments in Tasmania and the ACT collaborated with the University of Tasmania in a project to develop a framework for describing and assessing students’ abilities with mental computation, along with resources for teachers to support the coordinated and explicit teaching of a strategies approach to mental computation (McIntosh & Dole, 2004). This produced the first teaching resource that comprehensively addressed a strategies approach to teaching and assessing mental computation in schools.

Sowder (1994) described computational estimation as mental computation with approximations, with the ability to round numbers to reasonable approximations indicative of a good understanding of place value. She drew attention to the interconnections and mutual reinforcement of aspects of number sense in the estimation process. These included confidence in structuring and processing a calculation, and making sense of the result of a calculation.

Reys and Reys (2004) stated that despite curriculum references to the importance of estimation it is a much neglected focus in classrooms. One reason given was that classroom culture typically values exact answers; this is particularly true in Asian countries (McIntosh et al., 1997). The authors made clear connections between estimation and aspects of number sense, in particular the need to understand the relative magnitude of numbers and the nature of operations. They identified the importance of providing improved opportunities for valuing and developing skill in estimation, and framed doing this as a critical challenge for contemporary curriculum:

> As use of technology such as calculators and spreadsheets continues to proliferate at home, in school and in the workplace, the need to estimate and recognise the reasonableness of results will continue to grow. If the mathematics curriculum is to reflect these needs, then significant improvement in how we help students develop estimation is needed. (Reys & Reys, 2004, p. 110)

Calculating estimates mentally, using workable approximations to exact values, both applies and exercises number sense. This process reflects contemporary social practices, in particular the role of electronic calculation.

Electronic calculators

Electronic calculation is a widespread social reality. It represents a distinct stage in the constant evolution and application of technology to the task of processing calculations, and has reduced the amount of time and energy that people have to put into this procedural aspect of doing mathematical tasks (Brinkworth, 1985). Previous, more primitive technologies include the soroban, the abacus, and pen and paper (McIntosh, 1990). Change in the sophistication and capacity of electronic calculation appears to be happening at an ever increasing rate; spreadsheets such as *Excel* which now typically come with any home computer are more powerful than software used by professional accountants only two decades ago (Hogan, 2004).

The sophistication of electronic technology for calculation in the classroom has also increased over the past twenty years. Students now have access to simple four-function calculators that observe the conventions for order of operations, through to graphics calculators and computer packages that can display their results on interactive whiteboards. Cumming (1999) asserted that the "(u)se of computer technology allows a refocus of the emphasis in mathematics education, away from repetitious practice of the mundane, to application with understanding and eventually a stronger development of domain knowledge" (p.28).

This sentiment has been reflected repeatedly in curriculum documents in Australia. In Western Australia, the section of the curriculum framework for mathematics (Curriculum Council of Western Australia, 1998) around calculating with numbers reads very much like the national statement:

> (Students) use written approaches as a back-up for calculations they cannot store completely 'in the head' … They understand that calculators or computers are the sensible choice for repetitive, complex or lengthy calculations and use them efficiently, correctly interpreting calculator displays. (p.187)

The most recent curriculum framework for ACT schools, *Every chance to learn* stated:

> Using electronic calculators is an essential aspect of understanding and applying number. In all bands of development, the sensible and effective use of calculators can develop students' understanding of the meaning of numbers and operations and enable them to calculate efficiently when solving problems with real data. (ACT Department of Education and Training, 2007)

However, there seems to be a continuing reluctance to embrace the potential of electronic calculators to support the development of foundational concepts in number and to facilitate the use of real data in problem solving, at least in middle years' classrooms. White (1998, cited in Swan, 2007) referred to research in New South Wales into the beliefs of teachers in Years 5 and 6 towards students' use of calculators which revealed significant negative attitudes towards their use. On one item, 15% of surveyed teachers stated they believed that "using calculators in class would promote laziness and dependence" (p.79). On another item, 27% of respondents stated they believed that "using a calculator would result in students just accepting answers and not thinking" (p.79). Such stated beliefs stand in stark contrast to substantial evidence to the contrary (Groves, 2004). Ellington (2003) reported on a meta-analysis of 54 research studies that targeted the effects of calculators on students' attitudes and achievement. Key findings included that when calculators were integrated into instruction and testing, students' operational and problem solving skills improved, and students who used calculators had better attitudes to mathematics than students who did not use calculators.

A further example of similar positive effects from the use of calculators is reported by Ruthven (1998) who looked at the calculation strategies of a total of 56 year 6 students. One group of students had completed their primary schooling in schools that had participated in a several-year project where calculators were integral to learning and doing mathematics (post-project schools). In these schools, students had been supported in developing a range of effective mental computation strategies, and were expected to complement their mental computation with the responsible use of calculators. In the other group (non-project schools) students were in schools with a more traditional approach to teaching number. This showed itself in a focus on written methods for calculation, and little self-regulated use of calculators by students. Overall students in post-project schools made more use of mental calculation, which was hardly surprising. But they also made *less use of calculators* than students from non-project schools. The author suggested this may relate to the way mental computation in post-project schools was seen as a necessary foundation for sensible and effective calculator use.

There has been a huge increase in the availability, capacity and use of electronic calculation in society since the Cockcroft report (Cockcroft, 1982) was written. Yet it appears that there has not been a commensurate change in the use of calculators in the classroom. Reasons for this may relate to the traditional place of written calculation in the curriculum.

Written calculation

Written calculation has long had an explicit focus in the mathematics curriculum, and this focus has centred on the use of formal written algorithms.

Algorithms in mathematics

One definition of an algorithm is "a step-by-step method for calculating a result" (Bana, Marshall, & Swan, 2005, p.2). A fuller definition has been given by Maurer (1998):

> An algorithm is a precise, systematic method for solving a class of problems. An algorithm takes *input*, follows a *determinate* set of rules, and in a *finite* number of steps gives *output* that provides a *conclusive* answer. *Determinate* means that for each allowed input, the first action is precisely specified and, more generally, that after each action in the sequence the next action is precisely specified. *Conclusive* usually means that the output correctly solves the problem, but it can mean that the algorithm either solves the problem correctly or announces that it cannot solve it. (p.21)

Algorithmic processes are a significant aspect of the broader discipline of mathematics, and are not limited to calculating with the four basic operations. Applications in discrete mathematics include graph theory, game theory, combinatorics and recursion (Hart, 1998). Several examples are still visible in the senior secondary mathematics curriculum. These include differential and integral calculus (developed in the latter part of the seventeenth century) and the fields of logic and computability, which underpin the expanding application of computers in society (Struik, 1987).

But when applied inappropriately, the positive aspects of algorithms, such as their reliability and the written record of the specific step-by-step process that was used, can be lost. Usiskin (1998) illustrated this with direct reference to the formal written methods commonly taught in schools for calculating with whole numbers. Two dangers that are particularly relevant are that answers tend to be accepted uncritically (even if they do not make sense), and that written algorithms can become a default for calculations that are more easily and efficiently done mentally.

Algorithms in school mathematics

In Australia over the past twenty years, written calculation has continued to be mentioned in national mathematics curriculum documents (AEC, 1990; MCEETYA, 2000). Suggested approaches to pen and paper calculation in these documents have included “informal written methods” (AEC, 1990, p.109; McIntosh, 2002), but in practice the curriculum has been dominated by an emphasis on the teaching and learning of standard written algorithms, particularly in primary schools (Clarke, 2005; McIntosh, 2005).

Plunkett (1976) identified features of the formal written algorithms still commonly found in the school mathematics curriculum, which made them suited to the cultural demands and available technology of the pre-electronic age. These algorithms use symbol manipulation with relatively few steps and do not require context or an understanding of the process that is actually happening; they work for all numbers with no further choices required once the method is decided upon; and they provide a written record that can be easily checked and corrected. These features of formal written algorithms resonate with aspects of the traditional approach to curriculum, which has been shaped by the cultural characteristics of early industrial societies: specified content, often committed to memory; a focus on transmission, with the teacher as the centre of classroom discourse; and standardised delivery and testing (Kalantzis & Cope, 2005). Issues around what is easy to test in large-scale assessments still influence the emphases placed on different aspects of the mathematics curriculum in Australia (Sparrow & McIntosh, 2004).

The typical formal or standard written algorithms for calculating with whole numbers have several disadvantages which are not unrelated to those identified in the Cockcroft report (Cockcroft, 1982). These include the fact that formal written algorithms do not reflect the way people typically think about numbers, nor are they used very often beyond the classroom (Clarke, 2005; Plunkett, 1979). The latter point increases in its significance as more and more calculation in all areas of society is done electronically (Brinkworth, 1985). That the written methods for calculation commonly used in schools also tolerate a suspension of understanding about numbers and operations suggests their place in the curriculum might be in conflict with the contemporary curriculum imperative of developing students’ number sense (NCTM, 2000). The inflexibility of these formal approaches to written calculation clashes with the nature and demands of our increasingly technological society, where effective calculation across a range of situations typically requires understanding and strategic thinking in crafting solutions appropriate to the context (Anghileri, 2006; Sparrow & Swan, 2004).

The issue with the formal written algorithms commonly taught in schools is not that they are algorithms. The issue is that these written methods may not be the best choice for a 21st century mathematics curriculum which aims to support all students in developing number sense and applying it effectively beyond the mathematics classroom.

Two perspectives: the process of long division

The standard pen and paper algorithm for processing long division calculations has been used as evidence of the correctness of quite contrasting points of view.

Klein and Milgram (2000) described and analysed the written algorithm for long division with whole numbers and stated that it was "an essential tool for understanding what a real number is" (p. 4). They proposed an even stronger claim to curriculum relevance for all, saying that "the (long division) algorithm itself contains the initial exposure of topics which become crucial in the core applications of mathematics in our society today" (p. 1). The authors acknowledged the call from many for electronic calculators to be embedded in the Kindergarten to Year12 curriculum, but argued against any dismissal of formal written calculation. They claimed to represent the views of, among others, a growing number of mathematicians and scientists concerned about a perceived decline in mathematical standards in schools. But despite the link to school mathematics in the title, the bulk of their paper addressed topics more in line with studying mathematics at university level, such as Laplace transforms and eigenvectors, with no discussion of the many and complex issues surrounding the mathematical development of *all* school students, not just those who will engage with mathematics at the tertiary level.

Anghileri (2006) investigated the impact of five years' implementation of the *National Numeracy Strategy* in England by analysing students' performance on ten division tasks. The *National Numeracy Strategy* was implemented in 1998. One of its aims had been to 'open up' the primary mathematics curriculum to value a broader range of calculation strategies, including mental computation and more flexible approaches to written calculation. Data on 275 Year 5 students from 1998 was compared with data on substantially parallel division tasks from 308 Year 5 students in 2003. The purpose was to judge the degree to which the principles promoted by the *National Numeracy Strategy* were visible in students' solutions, and to identify any changes in students' solutions over the five years of the strategy's implementation.

In terms of performance there was some improvement on the eight directly comparable tasks: in 1998 the mean = 3.14, standard deviation = 2.1; in 2003 mean = 3.85, standard deviation = 2.2. However a further focus of this study was the documentation and analysis of the division strategies used by the students. Categories for written strategies for division included low level inefficient strategies, informal methods recorded in an unstructured way, and the traditional formal algorithm and the chunking algorithm. The chunking strategy for division was explicitly listed in the support materials offered with the English *National Numeracy Strategy* where it was classified as an informal written method for division. Although labelled 'informal' it is a standardized algorithm based on repeated subtraction and the quantity value of flexible multiples of the divisor. It is an efficient but less used written form of calculation for division that builds on more intuitive notions of that operation and the quantity value of component calculations.

This aspect of the study showed that there had been a shift away from the dominance of the traditional algorithm that was evident in 1998 towards a mix of traditional and informal methods, but with only marginal improvements in overall performance. On its own this might appear to suggest that an emphasis on informal methods leads to no better performance than the traditional division algorithm. However the 'chunking' strategy for division is the standard method for pen and paper division taught in the Netherlands. At the time of the original 1998 English study, comparable data was gathered on 256 Dutch students from ten schools, with results significantly superior to those of the English students. Anghileri (2001) suggested this difference was related to the nature of the written process used in each country. The formal written algorithm commonly used in England requires the greatest multiple of the divisor to be identified for each step. An advantage of the 'chunking' strategy is that it tolerates a range of choices of multiples of the divisor. This allows students over time to make more sophisticated choices and so calculate more efficiently (by taking greater multiples of the divisor and therefore fewer written steps) without changing the structure of the recording process (Angheliri, 2001).

Treffers and Beishuizen (1999) placed the use of the ‘chunking’ algorithm in Dutch schools in context. They described the approach to teaching mathematics in the Netherlands which is called *Realistic Mathematics Education* or RME. The seeds of this approach go back to the 1970s and the work of Hans Freudenthal. Central tenets of RME include starting with students’ informal strategies, structuring class activities that support “progressive mathematisation” (p. 33) where students move towards more sophisticated solution strategies but maintain understanding, and a strong emphasis on mental computation. A significant advantage of this approach to developing written computation is its inclusivity:

> There are, of course, some pupils who will not progress beyond intermediate levels (of abbreviated written strategies). However, every pupil learns to solve the long division problems at her/his own level, depending on personal abilities like knowledge of times tables and task-span capacity. (p.28)

The authors also reported that although there was evidence that RME supports improved performance for more students, a large number of teachers still use RME materials, such as specially designed texts, in a more traditional manner. This disjuncture between the teacher’s underlying beliefs and progressive aspects of their classroom practice may explain the limited gains in performance on division tasks of the samples of English students between 1998 and 2003 that were reported by Anghileri (2006).

The differing perspectives on long division described above give shape to two distinct orientations towards the teaching and learning of mathematics. One focuses on the perceived analytical coherence of the subject matter. This is something typically perceived by people who are already to some degree expert in the field. The other focuses on how to support all people in their learning of mathematics. Both orientations take the subject of mathematics seriously. But not all students in the compulsory years of schooling think the same way as professors of mathematics. It is important to find ways of constructing the curriculum that maximise the learning of *all* students.

Connectionist teachers: the importance of making links

Askew (1999) reported on a study into factors driving effective teaching of numeracy in primary schools as measured by demonstrable gains in student learning. In this study, *numeracy* focused on aspects of understanding and using number and was similar to the notion of *number sense* as described by McIntosh et al. (1997). Effective teaching in this context appeared more strongly connected to the way teaching practices related to and enacted teachers' beliefs about teaching, learning and mathematics, than to any specific classroom practices.

There were three interviews and three observation sessions with each of the 18 case study teachers, with two interviews with and observations of each of the 15 teachers in the validation group. Analysis of observational and interview data suggested three distinct but not mutually exclusive orientations towards teaching numeracy which were labelled:

- *transmissionist*, which focused on teaching and the procedural aspects of mathematics
- *discovery*, which focused on learning and students' invention of their own understandings of mathematics
- *connectionist*, where students' methods for doing mathematics were valued, as were connections within and beyond the mathematics.

The connectionist orientation was more consistently linked with greater gains in student learning. Teachers with this orientation stressed connections in the mathematics they taught such as between equivalent forms of numbers (for example fractions and percentages) and alternate representations of mathematical concepts (in symbols, words, pictures or concrete objects). They also connected with their students' approaches to doing mathematics. These teachers promoted efficient and effective use of a range of strategies, with the choice of strategy in a particular situation being influenced by both the operation and the numbers involved. The Dutch approach to teaching long division (Treffers & Beishuizen, 1999) is one example of developing an effective method for written calculation within a connectionist orientation.

Several aspects of the transmissionist/discovery/connectionist model make it relevant to the research conducted in this study, which looks at some of the effects of using written methods for calculation which mirror or make use of effective mental computation strategies. These written methods are labelled in the description and discussion of this research as number sense written methods. Fundamental to both mental computation and number sense written methods is the exploitation of connections within and between numbers, and between operations. The findings reported by Askew (1999) suggest that connecting different forms of calculation, specifically here mental and number-sense written forms (which themselves are about connections between numbers and operations) might lead to improved student performance in written calculation with whole numbers.

Adding weight to this view are corollaries involving the relatively less effective teaching orientations. Transmissionist approaches frequently involved an over-emphasis on formal written algorithms, while discovery approaches may unreasonably subjugate the guided development of meaningful and efficient methods for calculation to an over-emphasis on working everything out for one's self (Sparrow & McIntosh, 2004).

CALCULATION: LINKING WRITTEN TO MENTAL

McIntosh (2002) reported on a Tasmanian project that focused on linking students' development of written computation to competence with a strategies approach to mental calculation. The rationale for the project was to explore teaching written calculation in a way that might promote number sense. The project involved nine schools and 34 teachers over 18 months. It was limited to students in Years 2 to 4 and the operations of addition and subtraction (McIntosh, 2005).

To move students from mental computation through to understood written calculation the study progressed through planned stages of focused activity. In sequence these stages:

1. Strengthened students' mental calculation for addition and subtraction with two-digit numbers
2. Encouraged students to explain in writing their method for calculating (but only after students were comfortable and fluent in explaining their strategies orally; explanations needed to relate to the steps in the calculation, be intelligible without speculation to an outsider, and complete)

3. Compared, discussed and refined these written explanations (the written explanation of the mental method was shaped into a method for which pen and paper became necessary, for those numbers that were 'too much' for calculating mentally; the range of these written methods was limited, and the challenge was to draw out and build on the child's thinking while helping to refine the recording format through suggestions or models)
4. Used further calculations of similar difficulty to consolidate this method (students exercised their method over the full scope of calculations within the nominated range of complexity)
5. Extended the method to more difficult calculations
6. Consolidated the method as an understood, secure written method (McIntosh, 2005).

Numerous student work samples highlighted the tension between student ownership of the calculation process and the practical value of their approach as a method of calculation. For example, the student who attempted to calculate 53 times 24 by first drawing 53 circles, then putting 24 tally marks in each circle, appeared to understand the task, but used an approach that was quite rightly considered to be inappropriate (McIntosh, 2005).

The project used criteria outlined in Campbell, Rowan and Suarez (1998) to make judgments about which student explanations were able to be shaped into an understood, secure and practical written method for calculation. The three criteria were:

- *efficiency* (is the method reasonably concise and time effective?)
- *mathematical validity* (does the method yield the correct answer for the right reason?)
- *generalisability* (could the method work across a wide range of calculations?).

McIntosh (2005) listed a selection of informal written approaches for addition and subtraction developed from students' explanations, all of which meet the three criteria listed above and reflected effective mental computation strategies. At the completion of the project, participating teachers were unanimous in supporting the teaching of a strategies approach to teaching mental computation in primary schools, and developing informal written calculation with children. Reasons given for this included:

- the way both mental and informal written calculation exposed students' mathematical thinking (and so their levels of understanding) of key number concepts
- students' improved confidence and willingness to take risks with learning
- students' improved understanding of place value and performance on arithmetic tasks.

However, all project teachers also stated that they believed formal written algorithms still needed to be taught from grade 4 onwards because of perceived expectations of later years of schooling (McIntosh, 2005).

This last point highlights the cultural inertia around the teaching of formal written calculation in schools. The Cockcroft Report (Cockcroft, 1982) drew attention to the contrast between informal methods for written calculation commonly used in the workplace with understanding and confidence, and the formal written methods taught in schools which were rarely used in the workplace. Despite a range of practical and educational concerns, formal written algorithms still occupy a substantial amount of class time in the primary and early secondary years of schooling.

The research reported in this study used a similar methodology to the study reported by McIntosh (2002) but extended the focus into Year 8 and across the four operations with whole numbers.

IMPLICATIONS FOR THE SCHOOL MATHEMATICS CURRICULUM

Times have changed

The implications for developing a balanced approach to teaching computation as outlined in Cockcroft (1982) had been articulated several years earlier:

> I think that the reasons for teaching the standard written algorithms are out of date, and that it is time we all took notice of this. I believe there is a place for mental algorithms, for the use of calculators, and for ad hoc, non-standard written methods.
> I think a large amount of time is at present wasted on attempts to teach and to learn the standard algorithms, and that the most common results are frustration, unhappiness and a deteriorating attitude to mathematics. (Plunkett, 1979, p.4)

These sentiments were echoed by McIntosh et al. (1997):

> If learning these methods (standard written algorithms) does indeed cause a devaluing of understanding and sense-making on the part of the learner and results in only mediocre

> levels of proficiency, one must question the use of instructional time for such limited and counterproductive results. (p.5)

Some researchers (Thompson, 2004) and teachers (McIntosh, 2005) have advocated what appears to be a moderate position, suggesting the teaching of formal written calculation should come only after a strong platform of understanding of number and operations has been developed. However others have suggested a more radical approach. Regarding the teaching of formal written algorithms, Van de Walle (2005) stated “we no longer can afford the time required for teaching, re-teaching, and remediation of outmoded skills” (p.7) and cited research that showed no drop in performance on standardized assessments by students in the US who have not been taught formal written calculation methods. Ralston (1999) suggested the extreme view that pencil and paper arithmetic should be completely dropped out of the school curriculum as it is not only redundant but actively unhelpful in a calculator age.

The development and practice of a balanced approach to computation in primary and lower secondary schooling is being inhibited by an over-emphasis on formal written calculation, typically visible through an excessive amount of class time and assessment space given to the standard algorithms for performing the operations of addition, subtraction, multiplication and division with whole numbers. To improve the balance, Sparrow and McIntosh (2004) suggested a strong focus on mental computation, a reframing of the nature and role of written calculation, and more emphasis on supporting students in making sensible computational choices.

Reframing written computation

Reframing the role of written calculation in the curriculum requires the drawing out of some deep-seated and largely invisible beliefs and values. The laudable criteria for ‘good’ written algorithms of *efficiency*, *validity* and *generalisability* (Campbell et al., 1998) need themselves to be framed within the current social, and specifically technological, context. Any method for calculation should be mathematically valid. But regarding generalisability, it is fair to ask the question *How much is enough?* And with efficiency, *How much is too much?*

As Klein and Milgram (2000) stated, the long division algorithm is a concise process for performing this calculation with massively large whole numbers. But who really needs this, now that electronic forms of calculation are so pervasive in school, employment and day to day life? If written computation is limited to those calculations that are just outside the scope of reasonable mental computation, what does it matter if it takes a bit more paper to work out an answer? People rarely use written calculation outside the classroom, and when they do confidence and understanding seem to matter much more than compactness or mathematical elegance (Cockcroft, 1982). Even if this were not the case, the diminished frequency with which written computation is applied outside of school raises the question whether, once they leave school, people get enough practice with formal algorithms to keep whatever level of skill they might have had. Some question the continuing investment of class time and teacher energy into the teaching of formal written calculation (Sparrow & McIntosh, 2004).

Number sense written methods is a label for those methods that require an understanding of the numbers and operations involved in a calculation. These methods might grow out of approaches to solving problems that are *invented* or *discovered* by the student (Baek, 1998; McIntosh, 2005), but this is not a necessary requirement. Methods for written calculation that require number sense are frequently *non-standard* or *alternate* approaches when compared to what is typically taught in schools (Carroll & Porter, 1998; Thompson, 1999). But despite being referred to by some as *informal methods* (Anghileri, 2006), number sense written methods typically have a recognizable and mathematically sound structure and are, therefore, in a different sense, quite 'formal' (Thompson, 1999; Treffers & Beishuizen, 1999).

As part of the evolution of the contemporary mathematics curriculum to better meet the needs of citizens in the 21st century, the standard or formal written algorithms could be replaced by number sense written methods which:

- emphasise the quantity value of the digits in the number
- maintain the sense of the operation being processed
- use pen and paper to record the results of calculations of partial quantity value that are done mentally

- more readily make sense to the user (Carroll & Porter, 1998; Plunkett, 1979; Thompson, 1999).

SUMMARY AND CONCLUSION

A workable reframing of written calculation for current times positions the use of number sense written methods for a limited range of calculations, namely those that are just outside the scope of reasonable mental computation, alongside a strong emphasis on mental computation and the sensible and effective use of electronic forms of calculation (Plunkett, 1979). Linking the three modes of computation (mental, electronic and written) by the common thread of number sense increases the connectedness that appears to underpin better performance for more students (Askew, 1999). The central argument of this chapter has been that using number sense written methods within a balanced approach to computation is likely to produce positive outcomes for more students by contributing more effectively to building and strengthening students' number sense.

A reframing of written computation along the lines suggested above would involve a major shift in many teachers', students' and parents' perspectives and values. The research project described in the next chapter has been an attempt to gather evidence from the middle years of schooling that such a shift is worthwhile. The research took a similar approach to that reported by McIntosh (2002) and investigated the following questions:

> What are some of the effects on teachers and students within a junior high school setting, of aligning written calculation with a strategies approach to teaching and using mental computation?

Of specific interest, are there any effects related to:

- conceptual understanding of number and operations with numbers?
- facility with performing calculations?

Chapter Three describes the methodology that was used in this study.

CHAPTER THREE: METHODOLOGY

INTRODUCTION

This chapter outlines the theoretical perspective behind the research and describes the methodology within which the collecting of data took place.

This study planned to use a mixed methods approach, with the dominant emphasis on the qualitative methodology of case study using multiple cases. Each case was to involve the teaching and learning activity within a Year 8 mathematics class (where typically students are 12 or 13 years old at the beginning of the school year) as they engaged with a series of weekly lessons that targeted mental and written calculation with whole numbers. The description of each of the three cases was to be augmented by comparisons of quantitative measures of selected students' performance on calculation tasks with whole numbers.

Data collection and analysis would focus on:

- records of observations of a series of weekly lessons with three Year 8 classes
- records of discussions with the teachers of the Year 8 classes
- student work samples produced in the observed lessons
- student performance and choice of calculation method on a set of 11 calculations that was administered with each class at the beginning and at the end of the series of weekly lessons
- semi-structured interviews with selected students.

THEORETICAL PERSPECTIVE

The choice of methodology used in this study was influenced by the discourse around the characteristics and philosophical platforms underpinning quantitative and qualitative orientations to research in the social sciences.

Traditional approaches to educational research

The discourse around educational research commonly draws on understandings of quantitative and qualitative research methods (Creswell, 2005). The differences in the historical orientations of these two approaches to research have led some to believe that quantitative and qualitative methods are incompatible (Howe, 1985; Johnson & Onwuegbuzie, 2004).

Quantitative methods purists tend to align social research with a particular construction of scientific research known as *positivism*. Characteristics of a positivist perspective include:

- a belief that science is objective in its descriptions and conclusions
- a focus on cause and effect and generalizations about these
- a separation of the observer and that which is being observed
- discourses involving notions of reliability and validity (Johnson & Onwuegbuzie, 2004).

With quantitative research in education the influence of this view of science shows itself through a strong emphasis on the collection and statistical analysis of numerical data (Creswell, 2005).

Qualitative research methods reflect different perspectives and priorities to those outlined above:

> The key philosophical assumption … upon which all types of qualitative research are based is the view that reality is constructed by individuals interacting with their social worlds. Qualitative researchers are interested in understanding the meaning people have constructed, that is, how they make sense of their world and the experiences they have in the world. (Merriam, 1998)

Purists within this research tradition accept value judgments as inherent parts of the research process and see reality as a situated construct, rejecting the idea of generalisations without context. They see the observer and observed as intimately connected, and strive for trustworthiness and usefulness in research findings. Qualitative research usually involves rich and detailed descriptive accounts (Johnson & Onwuegbuzie, 2004; Merriam, 1998; Robson, 2002).

Extreme expressions of both approaches to research have their critics. Though popular, the positivist construction of science is contestable. Chalmers (1982) noted the trend throughout the 20th century for proponents of various bodies of knowledge to describe themselves as ‘scientific’ in an attempt to gain some sense of elevated status. The motivation for doing this is related to how people commonly, but mistakenly, perceive the physical sciences as dealing solely with objective, proven knowledge, gained through rigorous experimental processes using only testable sensory perception (p.1). Chalmers refuted this common view of science, drawing attention to the *theory laden-ness of observation*, whereby the description of all observation inherently involves some theory. This imbues all observation with an essential subjectivity that the positivist view of science seeks to, but cannot possibly, avoid. Howe (1985) discussed the continuing impact of positivist values on the perceived superior status of quantitative methods in educational research. He argued that the implied sense of objectivity and value-free observation, present in some versions of quantitative research in education that allegedly leads to proven 'facts', is as untenable in that setting as it is in strict positivist views of science. He added this warning: "… employing the fact-value distinction to avoid value bias instead exacerbates the danger of bias by cloaking value judgments with names such as 'objectivity', 'truth' and 'science'" (p.10).

Extreme relativism has its own difficulties. Accepting that a person's beliefs, values and interpretations shape what that person takes to be real does not imply that their construction is an equally valid representation of the world as any other. It is hard to argue against the notion of a common base of existence which is independent of our perceptions and constructions of reality (Robson, 2002).

Mixed methods research and critical realism

Strict representations of qualitative and quantitative methodologies do not seem adequate as tools for representing the social and tangible worlds we deal with on a daily basis. Johnson and Onwuegbuzie (2004) questioned the theoretical perspectives that underpin this rigid dichotomisation of research methodologies. They suggested that combining those aspects from both traditions which appear likely to address specific contexts to best effect, facilitates better quality research than either quantitative or qualitative approaches on their own. They labelled this approach *mixed methods research*, and defined it as "the class of research where the researcher mixes or combines quantitative and qualitative research techniques, methods, approaches, concepts or language into a single study" (p.17).

This weaving together of quantitative and qualitative research contrasts with the false opposition between the two traditions that was described and rejected by Howe

(1985), who claimed the two paradigms have more in common in practice than their theoretical positions might suggest. However, this philosophical antagonism continues to influence the focus and approach of contemporary educational research. There still exists a sense, in some influential quarters, that quantitative methods are superior to other research approaches. This situation is lamented by Meier (2007):

> The (US) federal government defines research only as the experimental design … (But) what makes for an educated and successful citizen is not always easily quantifiable, and definitions vary. Therefore, the narrow type of research the government allows also restricts what types of questions even get asked by the research. (p.3)

Robson (2002) provided an alternate construction of scientific inquiry that diffuses the traditional criticisms of qualitative research as being ‘unscientific’. He described a scientific approach to conducting social research as research that is done “systematically, sceptically and ethically” (p.18). This reframing of what makes research scientific underpins what Robson described as *real world research*, for which he argued a *realist* perspective provides an adequate philosophical basis for meaningful and productive research (p.29). In a realist explanation of social phenomena, the outcome of an action is produced by a mechanism which is nested within a context. The work of a researcher is to craft opportunities to activate, engage with and describe such mechanisms, and identify and reduce the effect of factors that interfere with the operation of the mechanism being studied. This requires the researcher to be familiar with the context within which the research takes place, and that he or she cautiously but actively engages with the context and the operation of the mechanism.

Case study methodology

Merriam (1998) stated that “(a) case study design is employed to gain an in-depth understanding of the situation and meaning for those involved” (p.19). When considering what constitutes a case for the purposes of research she suggested that it is essential to be able to define boundaries that effectively demarcate the case to be studied:

> I can “fence in” what I am going to study. The case then, could be a person such as a student, a teacher, a principal; a program; a group such as a class, a school, a community; a specific policy; and so on. (p.27)

Creswell (2005) identifies several types of case study:

- the *intrinsic case* is something that is interesting in its own right, perhaps because it is unusual
- the *instrumental case* is an instance that may serve to illustrate a particular issue
- a *collective case study* involves the description of several cases, aspects of which can then be compared.

Regardless of the category of case study it is likely that a range of data sources will be used to provide a rich picture of the case(s) in question. The researcher acts as the primary medium for data collection, either in the selection or the recording of the various aspects of the descriptions that aim to capture the specifics of the case(s). These descriptions are distilled into themes from which the researcher makes interpretations about what was learned from the case(s). The researcher is necessarily embedded in the research to a greater or lesser degree, which presents both challenges and opportunities (Creswell, 2005; Merriam, 1998; Robson, 2002).

Validity, reliability and the trustworthiness of results

Robson (2002) stated that validity in research relates to conceptualisations of being accurate, correct or true (p.170) , while reliability in research relates to the degree to which "the [research] tool or instrument produces consistent results" (p.176). He acknowledged that notions of validity and reliability have often been interpreted and expressed differently in quantitative and qualitative methodologies. This situation is explained, at least in part, by the disparity in the fundamental ontological assumptions in each camp:

> One of the assumptions underlying qualitative research is that reality is holistic, multidimensional, and ever-changing; it is not a single, fixed, objective phenomenon waiting to be discovered, observed and measured as in quantitative research. Assessing the isomorphism between data collected and the 'reality' from which they were derived is thus an inappropriate determinate of validity. (Merriam, 1998; p.202)

However, Robson urged qualitative researchers not to replace these terms, which are well-established in the culture and discourse around research methodology, with others like 'credibility' or 'dependability':

> The problem is not so much with the apple-pie desirability of doing reliable and valid research, but the fact that these terms have been operationalised so rigidly in fixed design

> quantitative research. An answer is to find alternative ways of operationalising them that are appropriate to the conditions and circumstances of flexible, qualitative enquiry. (p.170)

Robson (2002) described some of the characteristics of rigorous research conducted by case study:

- the specific case that is the focus of the research is clearly defined
- it involves empirical research, often using multiple sources of evidence with an emphasis on qualitative data
- an understanding of the context within which the case is situated is seen as vital.

Role of the researcher

In this study the role of the researcher was intended to be somewhere between an observer/participant and a participant/observer. Merriam (1998) described the nature of the first of these roles as "(t)he researcher's observer activities are known to the group; participation in the group is definitely secondary to the role of information gatherer" (p.101). This description is a reasonable fit for the researcher's relationship to the students in this study, whereas his relationship to the teachers included additional elements of direction and coaching that were in line with the researcher's professional context within which the research took place.

Mixed methods research and this study

The use of a mixed methods design for this study was prompted by the seminal discussion by Skemp (1976) who popularised the terms *instrumental understanding* and *relational understanding* in mathematics education. He paraphrased instrumental understanding as "rules without reasons", which involves learning what to do without necessarily knowing what is happening or why it works. This may at times appear to support acceptable performance on mathematical tasks. However, Skemp suggested that it is vital, for the genuine learning and effective application of mathematics, to also include an adequate emphasis on the conceptual basis of the mathematics being learned or used (relational understanding).

The mix of qualitative and quantitative approaches used in this study was crafted to provide, in a manageable way, a basis for understanding the different layers of the classroom context that related to the research question. These aspects were deemed to include the teachers' and students' levels of conceptual understanding of whole numbers and operations with numbers, which were taken as indicators of participants' relational understanding, and students' facility with calculations done mentally and with pen and paper. Reliably performing calculations correctly does matter, but levels of conceptual understanding may not be directly evident in the correctness or otherwise of the answers to computational tasks. The correct answer to a calculation may only indicate a satisfactory level of instrumental understanding. Some indication of a student's conceptual development is more likely to be evident in a description of the process he or she used to calculate a response (McIntosh, 2005). Targeting descriptions of how calculations were performed was considered to be an appropriate orientation for exploring any effects on learning and performance of promoting curriculum approaches that required greater amounts of relational understanding than are usually evident in this topic (Clarke, 2005).

DATA COLLECTION: STRATEGIES AND PURPOSES

It was planned that multiple sources of data would be explored. For teachers, the planned sources of data included:

- semi-structured interviews
- records of interactions with the researcher
- records of classroom observations.

For students, the planned sources of data included:

- an initial and repeat aural assessment of mental computation performance
- an initial and repeat written task
- semi-structured interviews with up to six selected students (two from each class)
- records of classroom observations
- selected samples of students' work.

It was anticipated that the repeat assessments for the mental computation and written tasks would use the same instrument as for the initial assessment in each case. This was to facilitate direct comparison of the results of the respective initial and repeat assessments.

PURPOSE AND NATURE OF THE WEEKLY INTERVENTION SESSIONS

Overview of the intervention

The initial intention around the purpose and nature of the program of weekly interventions was that teachers would collaborate with the researcher to:

- use the data from the initial mental computation assessment to identify strengths and likely gaps in students' understandings of number
- identify priorities for explicit teaching of mental computation strategies to build on these strengths and target identified gaps
- spend at least 30 minutes class time per week for eight weeks on explicitly sharing and developing mental computation strategies with their students
- in weeks five to eight of this mental computation focus, explore number-sense written methods for extending or applying the mental computation strategies that had been used in class.

Number sense written methods

In this context, *number sense written methods* were written methods for calculation which included the following characteristics:

- the digits within a multi-digit number maintained their *quantity value* (Thompson, 1999)
- pen (or pencil) and paper were used to keep track of the stages of multi-staged applications of effective and efficient mental computation strategies (as described in McIntosh & Dole, 2004)
- the person using the method could explain what was happening at each stage of the calculation in terms of the original numbers and operation.

All students would be explicitly taught some number-sense written methods. Alternate effective and efficient written approaches that might be suggested by students would be shared within the class and promoted as valid and worthwhile examples of this approach to written calculation.

Coverage and depth of treatment

The focus of the intervention was limited to the four operations (addition, subtraction, multiplication and division) with whole numbers. The degree of computational complexity that was to be targeted and some suggested strategies are outlined in Appendix A. Teachers would be free to extend these methods to numbers with more digits than described in the tables.

Mental computation assessment instrument

The instrument that was chosen to collect data on students' performance with mental computation tasks is in Appendix B. This instrument had been developed by the researcher several years earlier in collaboration with several teachers at School C. At the time of this study, the instrument had been in use for three years as a way of grouping students in Years 7 and 8 for targeted number sense sessions each week, and was deemed useful for that purpose.

Developing the items for the student written task

Plunkett (1979) described one view of an appropriate and reasonable set of expectations for calculating with the four basic operations [addition, subtraction, multiplication and division] at a time when electronic calculators had started to become readily available.

Table 3.1 is adapted from Plunkett's "spectrum of calculations" (Plunkett, 1979) and represents the platform from which the items used in the student written task in this study were developed.

Table 3.1. "Spectrum of calculations" adapted from Plunkett (1979).

Red	***Orange***	***Yellow***	***Green***	***Blue***
Basic addition and multiplication facts and inverses, up to 10 + 10 and 10 x 10	One-step mental processing, using basic facts combined with powers of 10	Two step mental processing; aim is for all students to have this facility by exit Year 10	Not expected to be done mentally but do-able by extending mental computation strategies with some, perhaps idiosyncratic, written support	In the past usually done with standard written algorithms; now most people would use some form of electronic calculation
Basic number facts	*Mental computation (1 step)*	*Mental computation (2 step)*	*Written or electronic calculation*	*Electronic calculation*
5 + 9 13 – 8 4 x 7 35 ÷ 5	135 + 100 85 – 20 5 x 30 90 ÷ 3	139 + 28 83 – 26 17 x 3 72 ÷ 4	592 + 276 592 – 276 931 x 8 693 ÷ 7	3964 + 7123 + 4918 + 5960 931 x 768 8391 ÷ 57

Table 3.2 contains the 20 items that were in the initial draft of the written task. Some items had been selected from the mental computation assessment tool (Appendix B). This was to allow for the possibility of comparisons in performance on the same item between encountering it as a mental calculation only (as on the mental computation assessment) and when written support was available (as on the written task in this study). The column headed *MC link* gives the reference of the item on the mental computation assessment tool.

All items were indicative of either the yellow or green columns in Table 3.1.

Table 3.2. Items in the initial draft of the written task.

item	*answer*	*MC link*	*spectrum*
27 + 25	52	B.1./D4	yellow
92 – 34	58	B.1./D3	yellow
105 – 26	79	B.1./D5	yellow
107 + 51	158		yellow
264 – 99	165	B.1./D6	yellow
68 + 44	112		yellow
700 + 283	983		yellow
26 x 7	182	B.2./D4	yellow
43 x 5	215		yellow
92 ÷ 4	23	B.2./D5	yellow
256 + 68	324		green
585 + 337	922		green
631 – 54	577		green
620 – 466	154		green
143 x 6	858		green
4 x 729	2916		green
34 x 31	1054		green
53 x 28	1484		green
378 ÷ 7	54		green
1062 ÷ 9	118		green

Several repeat items were considered. These items had a similar structure but with different numbers (such as 26 x 7 and 43 x 5) and were designed to provide opportunities for confirmatory results within an individual student's response. However having a total of 20 items was eventually considered unwieldy and likely to work against getting a reliable attempt from students on all items. Table 3.3 gives the final 11 items that were selected from the initial draft.

Table 3.3. Items in the final draft of the written task.

number	*item*	*answer*	*MC link*	*spectrum*
1	68 + 44	112		yellow
2	92 – 34	58	B.1/D3	yellow
3	264 – 99	165	B.1/D6	yellow
4	700 + 283	983		yellow
5	26 x 7	182	B.2/D4	yellow
6	92 ÷ 4	23	B.2/D5	yellow
7	256 + 68	324		green
8	631 – 54	577		green
9	143 x 6	858		green
10	1062 ÷ 9	118		green
11	53 x 28	1484		green

The student written task, which contained the same items and format in the initial and repeat administrations of the instrument, is in Appendix C.

On the written task students would be asked to indicate several things:

- their answer to the calculation
- how they worked out their answer
- whether they calculated an answer mentally, using pen (or pencil) and paper, or if they would need to use a calculator.

An ‘explanation’ was some written description or indication of the strategy or method that the student used to calculate the answer. For example, when mentally adding 68 and 44, an explanation might be given:

- as if explaining the process verbally:

  “I knew that 60 + 40 was 100 so then I just had to add 4 and 8 which is 12 and added both answers together”

- in arithmetic notation:

  60 + 40 = 100

  8 + 4 = 12

  100 + 12 = 112

In order to familiarise students with what was being asked of them on the 11-item written task, students in all classes were given a brief task with three items (different to any of those on the 11-item task) and talked through the response process by the researcher. Once the students in each class had clarified the expectations around the task they were given the 11-item written task.

CONTEXT FOR THE RESEARCH

The schools

The three schools involved in this study are all government high schools in the ACT. In the ACT system 'high school' typically caters for Years 7 through to 10, with Year 11 and 12 students attending separate senior colleges. Some ACT government high schools have nominal connections to a neighbouring primary school and together see themselves as Kindergarten to Year 10 (or Preschool to Year 10) schools.

School A had been classified as a P to 10 school for several years, with separate primary and high school campuses which were separated physically by a school oval. At the time of this study the student population in Years 7 to 10 was 374. The high school was structured around a middle years program for Years 7 and 8, and a separate senior school program for Years 9 and 10.

There were separate staff rooms for the bulk of the teachers in the respective programs. Teachers typically had a subject-specific connection to their classes; that is a teacher was seen as a mathematics teacher, or an English teacher and so on. Because of shortages in the availability of trained and qualified teaching personnel it had become increasingly the case that teachers at this school taught classes in both the middle years and senior programs, and at times classes outside of their area of formal training.

At the time of this study School B was in the process of becoming a P to 10 school with a combined population of 718 students. The high school component formally amalgamated with the adjacent primary school campus. It was anticipated that the Year 6 cohort would in future be housed on the high school campus as part of a middle years program involving Years 6 to 8. This would be a major shift in the culture on the high school campus which had previously been structured around four discrete sub-schools, one for each year group in Years 7 to 10. On this sub-school model a group of teachers stayed with a given cohort as those students progressed through their four years at high school. These teachers had the continuing responsibility for providing the academic and social programs for their year group, and for meeting their students' welfare and behaviour management needs.

School C was a 7 to 10 high school with a student population of 626. The junior program (Years 7 and 8) and senior program (Years 9 and 10) ran on separate timetables, such that breaks for recess and lunch were staggered. Teachers were typically attached to one of three curriculum banks, each of which had responsibility for several learning areas. The curriculum bank responsible for the teaching of mathematics was also responsible for the teaching of science and health/physical education.

There were three staff rooms for teachers, one for each of the three curriculum banks. These staff rooms housed the teachers from that curriculum bank for both the junior and senior programs.

The teaching of mathematics

The three schools all had a history of poor performance on the numeracy instruments used for collecting population performance data in Years 7 and 9. Culturally 'poor performance' is seen as having numeracy results that are significantly below the system mean on the ACT Assessment Program (ACTAP). ACTAP was superseded by the National Assessment Program in Literacy and Numeracy (NAPLAN) in 2008.

All schools had few if any teachers who had mathematics as the focus of their teacher training. Of these some teachers were in the first few years of their teaching careers. It was not uncommon for the teaching of mathematics to be assigned to teachers with training in science. However School C had a high proportion of teachers of mathematics who had physical education as the focus of their pre-service teacher education.

These two factors (chronic underperformance on population tests, and the need for building expertise in the teaching of mathematics in high schools) were the impetus for the initiative which established the context for the researcher's professional connection with the three schools. The researcher had been employed by a consortium of government high schools (which included Schools A, B and C) to work with teachers to improve the teaching and learning of mathematics in each school. This initiative was self-funded by the participating schools and had started two years before the research in this study.

At the time this study was conducted the ACT school system had just completed a curriculum renewal process and was in its first year of implementing the new curriculum framework *Every chance to learn* (ACT DET, 2008). In this document the development of number sense (through mental computation in particular) is listed as an explicit curriculum priority in the middle years of schooling. At that time the three schools in this study were at different points of engagement with this agenda around developing number sense.

Of the three schools, School C had the most cultural and curriculum capital targeting number sense development. It was into its third year of using the results of a mental computation assessment of all junior classes as the basis for flexible groupings of students for weekly targeted support to address gaps in foundational understandings in number. A strategies approach to mental computation had been embedded in the Year 7 and 8 curriculum and although it was a long way from being fully developed and totally consistent, it was a recognisable aspect of classroom practice in the middle years program. This model for supporting students with gaps in their understanding intentionally served a concurrent purpose. Through regular weekly collaborative planning and debriefing sessions it connected the more experienced support teachers with the many classroom teachers who had limited experience of teaching mathematics. These sessions provided a focused and on-going arena for expanding content-pedagogical knowledge in mathematics and improving general classroom practices.

School B had a commitment to an on-line program called *Mathletics*. This program provides a range of mathematical tasks with which students can engage, including calculation tasks. The program was used to varying degrees in mathematics classes throughout the school but not linked to any articulated program of explicit teaching of a strategies approach to mental computation.

School A had previously made some attempts to learn about and adapt the approach being developed at School C. However efforts were generally confined to a couple of teachers who had a stronger interest in this approach, and the implementation of an adapted model had been sporadic. Some teachers in School A had been involved in trailing *Mathletics* but appeared to do so somewhat independently of any classroom instruction on mental computation.

The classes

All classes were in Year 8 which is the second year of high school in ACT schools. Classes A (n = 29) and C (n = 25) had not been streamed; that is, allocated according to results or perceived ability. Both had roughly equal proportions of male and female students although there was some variation in the nominal composition of each class over the term. Class C had less session to session variation in student attendance than Class A. Class B (n = 15) was streamed and was comprised of students considered as typically underperforming in mathematics. It nominally had almost three times as many male students as female but attendance varied considerably from lesson to lesson. A high proportion of students appeared to have significant difficulties in maintaining compliant and on-task behaviour.

The teachers

Prior to this study the researcher had an established professional connection with each school around the teaching and learning of mathematics. The principals of the three schools saw the participation of one of their teachers in this research as useful professional development for those teachers. The involvement in the research was seen to align with the aims of the researcher's professional context as a curriculum deputy in each school.

Each of the three teachers in this study participated in the research voluntarily. They saw their involvement as both purposeful in terms of a teaching focus for their respective classes and valuable as a professional learning opportunity for themselves as teachers of mathematics. Teacher A was trained as a high school mathematics teacher. He was in his first year of teaching. Teacher B was an experienced teacher whose training had been in teaching science. She had been at the school for more than five years and had regularly taught mathematics classes during that time. Teacher C was trained in physical education and was in her third year of teaching.

The bulk of the data collection for the study took place during term two 2008, with most of the data relating to a structured weekly intervention of up to eight weeks with each class. The weekly intervention focused initially on strengthening students' mental computation, then on approaches to written calculation that are structurally parallel to, and make use of, efficient strategies for mental computation. Examples of these written strategies have been documented by Thompson (1999) and Carroll and Porter (1998). Workable written strategies that arose from students' explanations of their own approaches were valued as in the study by McIntosh (2002). The specific nature of the implementation of the intervention varied across the three schools. In each case it was planned collaboratively by the teacher and the researcher week to week, and reflected the opportunities available with each group of students.

IMPLEMENTATION OF THE INTERVENTION PROGRAM

Preliminary sessions

In Term One, 2008, the researcher had discussed with each teacher the purpose of the research, the nature and structure of the planned intervention sessions, and their likely levels of involvement. Each teacher had been given a brief outline of the intervention and some background material on number sense in general, and written calculation from a number sense perspective.

Planning sessions

The series of intervention sessions took place in Term Two, 2008. Each intervention session was preceded by a planning session involving the classroom teacher and the researcher. On a few occasions this was done through email, but mostly the planning sessions were face to face.

The duration of these planning sessions varied from ten to fifteen minutes to over an hour, depending on the time available to, and the interest of, the participating teacher. As well as planning related to the next classroom intervention session there was frequent discussion around related ideas and issues, such as other aspects of the mathematics curriculum, the role of conceptual understanding in learning mathematics in general, and managing student behaviour.

Classroom sessions

The duration of the weekly classroom intervention varied. At times the intervention was as brief as 15 minutes of effective time, due on occasion to interruptions from outside the classroom (such as visits by teachers or students to complete an administrative task) or as a result of chronic off-task behaviour from students within the class. Other intervention sessions occupied the bulk of the class time (up to 50 minutes) as the teacher ran with the students' engagement with the intervention material.

Differences between schools

The three schools varied in their depth of engagement with the intended intervention. Class C followed the intended sequence more closely than Classes A and B. However, all three teachers provided the researcher with valuable insights into their beliefs about teaching mathematics, and in particular their thinking about the practice and the teaching of calculation. Collectively the three teachers gave the researcher the range of professional backgrounds and dispositions towards being a teacher (and a learner) of mathematics that had been hoped for.

The characteristics of each class impacted on their capacity to engage with the intervention. Throughout the intervention the behaviour of students in Class A was typically unsettled. The regular disruptive behaviour of several students prevented extended presentations by the teacher. Some quieter passive students would regularly go through a lesson with what appeared to be little time on task. These students were not used to explaining their thinking around solving a mathematical task. They had limited experience with mental computation.

Class B was a group of students with high needs. Attendance, staying seated and maintaining time on classroom tasks were all a challenge for the majority of this class. This impacted negatively on the capacity of less disruptive students to engage with the intended learning experiences.

Class C was a very settled group of students. There were few disruptions from within the class. These students were used to explaining their thinking around solving a mathematical task. They had experience with mental computation in particular and were comfortable with explaining the strategies they used for processing calculations.

SUMMARY

In crafting the approaches to collecting data in this study it was believed that quantitative data on student performance with a range of computational tasks typical of the Year 8 mathematics curriculum would provide pivot points for discussions with participants, and serve as doorways into teachers' and selected students' thinking about this aspect of the curriculum. Complementary qualitative data sources which Burns (2000) identified with case study methodology included:

- teacher surveys and interviews
- student work samples which illustrated choice and explanation of computational approach
- audio recordings of interviews with selected students
- records of observations of teachers and students.

It was expected that together, these approaches would provide complementary vantage points from which to derive answers to the fundamental research questions for this study.

Chapter Four contains descriptions of aspects of the series of eight intervention sessions with Class C, based on the researcher's field notes, records of discussions with Teacher C, and the in-class tasks that were completed by students in these sessions.

CHAPTER FOUR: CLASSROOM OBSERVATIONS AND STUDENT WORK SAMPLES

This chapter contains descriptions of aspects of the series of eight intervention sessions with Class C. Field notes of observations from these eight sessions, together with records of the discussions with Teacher C and the in-class tasks that were completed by students in these sessions, all constituted important sources of data in this study.

These sources of data were always intended to be part of the overall data collection, but as this study progressed their position changed from being incidental to pivotal. This shift reflected several aspects of qualitative research, in particular case study methodology, described by Merriam (1998), and reflects the following:

- the design of a qualitative study is somewhat flexible; it can respond to changing or emerging conditions to best capture the relevant parts of a situation
- the sample in the study may well be small and its selection purposeful
- the object of the study, the case, is an involved but bounded system.

The research question for this study was:

What are some of the effects on teachers and students within a junior high school setting, of aligning written calculation with a strategies approach to teaching and using mental computation?

Of specific interest, are there any effects related to:

- conceptual understanding of number and operations with numbers?
- facility with performing calculations?

Extended discussions of selected elements of the descriptions of the intervention sessions which are relevant to addressing this research question occur in several places throughout this chapter.

WORKING WITH TEACHER C

Background

The intention in this study was to implement a program of eight weekly interventions, during which Teacher C would collaborate with the researcher to:

- identify priorities for the explicit teaching of mental computation strategies which built on students' strengths and targeted identified gaps
- spend at least 30 minutes class time per week for eight weeks on explicitly teaching, sharing and developing mental computation strategies with their students
- in weeks five to eight of this mental computation focus explore number-sense written methods for extending or applying the mental computation strategies that had been used in class.

Prior to the program of intervention sessions there had been some discussion between the researcher and Teacher C about the mental computation performance of students in this class, based on existing data generated by the school's use of the mental computation assessment instrument (Appendix B). The researcher also visited Class C in the term before the research intervention to observe the class informally and establish a connection with the group. Such class observations and discussions were a natural part of the researcher's professional context in that school.

Over the course of the classroom interventions several planning discussions with Teacher C drew on existing data on students' mental computation performance as determined by responses to the mental computation assessment instrument. Assessing students' performance with the items in this instrument was routine practice with all Year 7 and Year 8 classes at this school. The data were typically put to several purposes which included establishing flexible groupings for weekly targeted support sessions for students identified on this instrument as having gaps in their understandings of foundational aspects of number. A Learning Assistance (LA) teacher was assigned to each class to develop and implement 30 minute support sessions each week to address the needs of selected students as identified from the mental computation data. The LA teacher would gather further evidence of learning through selected work samples from, or recorded anecdotes about, students in their group. The constitution of the weekly groups varied according to the relative strengths and areas of concern in the various aspects of mental computation (such as addition and subtraction of whole numbers, or benchmark fractions) assessed by the instrument, and in response to the complementary sources of evidence of learning collected each week by the LA teacher.

A concurrent purpose of the weekly LA intervention was to support the many classroom teachers who were teaching mathematics without a strong background in the subject. At this school most junior mathematics classes (Years 7 and 8) had teachers whose major teaching area was not mathematics. Some teachers had undertaken no study of mathematics or mathematics teaching in their tertiary education.

Teacher C had experience as a LA teacher on this model. She was chosen for that role because she was committed to building students' number sense. Despite having only a couple of years of teaching experience and limited training in teaching mathematics, Teacher C was a relaxed, confident and effective classroom teacher who saw the value to students' learning of targeting mental computation as a curriculum area. One consistent priority in her mathematics classes was an expectation that, as well as giving the answer to a question, students would also explain how they arrived at their answer. This included but was certainly not limited to questions requiring calculating with numbers.

In-class tasks

Teacher C decided to develop and give students a one-page activity as the final phase of each intervention session. Typically this involved no more than four tasks, plus a request to illustrate and/or explain the strategy the student used to complete each task. These tasks were intended to consolidate the learning from that session, and provide Teacher C and the researcher with evidence of students' conceptual understanding of the material and their performance on tasks of the type provided. Teacher C and the researcher would usually review the samples of student work from the week before prior to meeting to plan the focus of the next intervention session.

Framework for developing multiplicative thinking

In the first two intervention sessions, Class C had shown facility with using efficient mental strategies for processing addition and subtraction tasks involving two two-digit numbers. The researcher prepared a framework for Teacher C around aspects and indicators of multiplicative thinking specifically geared towards the purposes of the intervention sessions. The aims of providing this material were to scaffold Teacher C's planning around teaching mental strategies for multiplication and division, and to build her content-pedagogical knowledge, and confidence, in this area of the curriculum.

The content of the framework drew on material common to programs such as *Counting On* (NSW DET, 2004) and resources such as *Mental computation: a strategies approach* (McIntosh & Dole, 2004). The focus of the material was firstly on how to establish strong conceptual understanding of multiplication and division as operations, and then on how to connect this understanding to number sense written calculation methods for tasks within the identified scope of difficulty.

The framework described a progression of key phases of development and a primary visual strategy to scaffold understanding and performance at each phase. The phases and primary strategies are listed in Table 4.1.

Table 4.1. **Contents of support materials for teaching multiplication and division.**

Phase	Focus	Main strategy
1	Establish multiplication as repeated equal groups	*rectangular arrays*
2	Explore visual representations of multiplicative part/whole relationships	*split arrays*
3	Consolidate symbolic representations of multiplicative part/whole relationships	*empty rectangle*
4	Develop language and notation for extended multiplication tasks	*empty rectangle*
5	Use an abridged version of the same sequence to establish meaningful connections between multiplication and division	*empty rectangle and reverse multiplication facts*
6	Introduce the chunking algorithm for division	*building with multiples of powers of ten*

Substantial detail was developed initially for phases one, two and three. This gave Teacher C enough support and direction for her lesson planning. Detail for phases four, five and six was developed during the planning of later interventions in the series and used in class, but was documented during the term following the intervention. The finished version of the framework was titled *Support notes for scaffolding number-sense written methods for multiplication and division*. This material is reproduced in Appendix E in a form similar to that in which it was developed.

DETAILED DESCRIPTIONS OF THE CLASSROOM INTERVENTIONS WITH CLASS C

Background

There were eight teaching sessions with Class C. These were bracketed by the two sessions during which the initial and repeat written tasks were administered. The ten sessions that made up the planned intervention that is the focus of this study all took place within the second school term of 2008.

The researcher was a passive observer for the majority of the teaching sessions, during which he took notes on a range of behaviours of Teacher C and the members of her class. On a few occasions the researcher was invited by Teacher C to contribute his point of view on an idea that had been raised, or a comment from a student, or to add something within an explanatory phase of a session. These actions were in keeping with the researcher's role in the school.

The text that makes up the remainder of this chapter is distilled from the researcher's field notes from observations of Class C and planning and debriefing sessions with Teacher C, and the researcher's own reflections as the data-collection phase of this study unfolded. In several cases particular points have been illustrated with samples of student responses to the in-class tasks that became a regular feature of the intervention sessions, and which have become a rich and significant source of data for this study.

A description of the mental computation strategies for addition and subtraction that are referred to in this chapter is in Appendix D.

Initial planning

Teacher C and the researcher initially reviewed students' results on the initial student written task (Appendix D). Observations included that out of the 23 initial responses to the written task:

- 21 students solved Item 1 (68 + 44) mentally and correctly but only 13 solved Item 2 (92 – 23) mentally and correctly; several students appeared to misapply the split strategy with the subtraction task
- relatively few students solved Item 6 (92 ÷ 4) correctly (mentally n =7; written n = 3).

As a result it was decided to start with explicit teaching on using the empty number line to visually represent strategies for mentally solving 2-digit from 2-digit subtraction tasks, and to represent each task as "take away" (starting at the larger number and *stepping back* a total of the lower number) and as "difference" (marking the lower number on the number line and *stepping up* to the larger number). The intention was to connect with some aspects of mental calculation with which students appeared reasonably familiar but also in need of further development (here strategies for addition and subtraction) and through this establish a level of comfort and familiarity with a strategies-based approach to calculating mentally that would support later work with multiplication and division tasks.

Session One

The focus for this session was to:

- use the empty number line (ENL) to represent addition and subtraction of two two-digit numbers
- provide explicit language around the jump strategy and around subtraction as both *take away* and as *difference.*

Students were already familiar with the jump, split and compensation strategies for the addition tasks. Some students applied the split strategy incorrectly to subtraction tasks that required regrouping such as 52 – 17. Several students processed this task by calculating 5 – 1 (40) and 7 – 2 (5), giving an answer of 45 (the *larger number minus the smaller number* error).

Teacher C explored this through questions such as: is 3 + 5 = 5 + 3? [yes] … is 3 – 5 = 5 – 3? [no] …

The in-class task required students to solve two subtraction calculations (63 – 18 and 74 – 26). Each calculation was shown on separate ENLs; once as take away and then as difference, along with a written explanation of the way the student processed each calculation.

There was some discussion about what constituted the answer to the subtraction task on each of the two models. With 63 – 18 for example:

- using the jump strategy to represent *take away,* students would mark 63 on an ENL, then move back in steps a total of 18 and mark the answer 45 as a point on the ENL
- using the jump strategy to represent *difference*, students would mark 18 on the left and 63 on the right of an ENL, and then move in steps from one to the other, with 45 being the total of these steps between the two given numbers and not a point on the ENL.

In representing the jump strategy on the ENL with both interpretations of subtraction students showed a range of part/whole strategies. These included:

- jumping in single tens: for 63 – 18 … 18, 28, 38, 48, 58, 63 (45)
- jumping in multiples of ten: for 74 – 26 … 26, 46, 66, 74 (48)
- compensation: for 63 – 18 … 63, 53, 43 then +2 (45).

Some misconceptions or points warranting discussion emerged. Examples of this included:

- sometimes placing the larger of the two numbers to the left on the ENL
- limited language for explanations in words
- representing steps on the ENL in numerals: for 74 – 26 a student wrote the following string on top of an ENL … 70 – 20 = 50 + 4 – 6 – 2 = 48.

Discussion

The choice to start with representing addition and subtraction on the ENL proved to be appropriate for several purposes. Students were mostly competent with addition tasks, which gave them a foothold with connecting the processing of symbolic calculations with their visual representation on the ENL and oral language for describing in words the processes they used. This requirement for multiple representations also helped expose several conceptual weaknesses in some students. These included the relatively minor issue of placing larger numbers to the right on an ENL, and the major concern of using a *larger number minus the smaller number* strategy for processing multi-digit subtraction tasks. The use of this flawed strategy was noted by Behrend (2001) as present in her group of Year 3 students, and it is likely that the students in Class C who demonstrated this misconception in Session One had similarly adopted this strategy much earlier in their schooling, as their way of making sense of what was being asked of them to do. That this misconception had remained such that it was demonstrable in Year 8 was a concern.

Insisting that students represented each subtraction task as both *take away* and as *difference* meant that the learning demands on students were greater than what was needed to simply process each calculation to get a numerical answer. Students were scaffolded and pushed to understand what the three numbers in each task meant on each of the two interpretations of the operation of subtraction, and how to represent these visually and with language. Attacking the issue of improved performance on arithmetic tasks through improving students' understandings of foundational concepts (in this case a deep understanding of the operation of subtraction) was an attempt to promote *relational understanding* as described in Skemp (1976) where students know what to do (in this case complete a subtraction task) and why (that they are finding what is left after taking one number away from another, or the difference between the two numbers).

Session Two

The focus of this session was to:

- consolidate representing subtraction tasks on the empty number line (ENL) as both *take away* and as *difference*

- use negative numbers with the split strategy on subtraction tasks.

Some home-grown terminology developed:

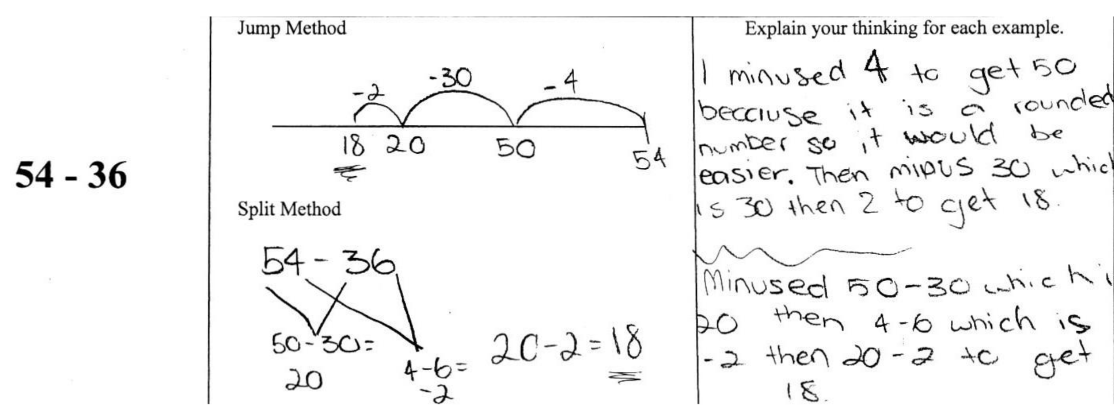
- difference was described as *how far apart?*
- take away was described as *what's left* (that is, *what amount or quantity is remaining*)*?*
- the split strategy was described as *break up/make up* (a phrase introduced the year before by another teacher at School C and remembered and used by Teacher C).

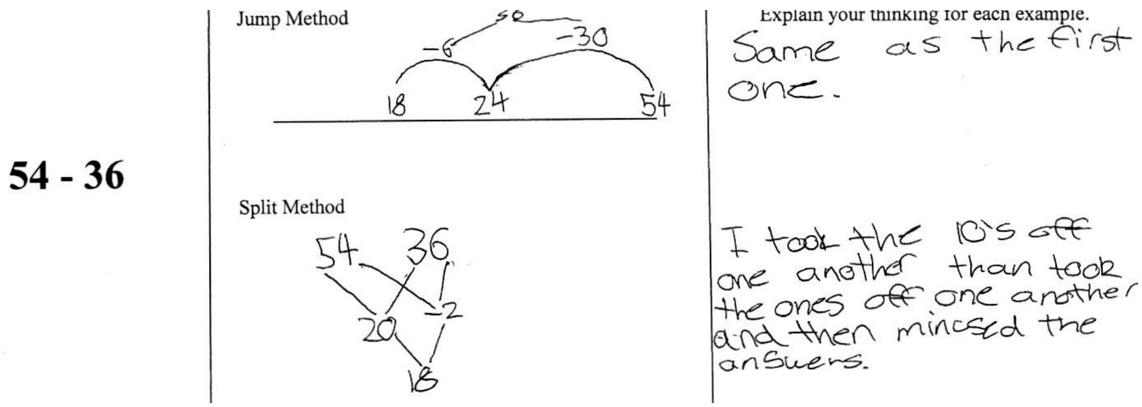
For example 74 – 38 was calculated using the split strategy by first doing 70 – 30 (40), then 4 – 8 (-4), then 40 – 4 (36). Although the symbolic representation of the calculations seemed to make sense to most students they struggled to find language for what was happening.

Teacher C used the ENL to represent the steps used with the jump strategy to process mentally a subtraction task. She introduced a shorthand notation for visually representing the steps in using the split strategy (labelled "V notation", taken from the shape of the two lines connecting pairs of tens and ones). Almost all students used this notation to correctly represent their strategies for solving the two subtraction tasks (73 – 28 and 54 – 36) using the split method.

The examples in Figure 4.1 are from responses to that session's in-class task.

Figure 4.1. Examples of students' written explanations of using the jump and split methods for calculating 54 – 36 mentally.





Subtraction can be represented as *take away* or as *difference*, but the first of these interpretations seemed more intuitively obvious to students. During discussions between Teacher C and the researcher it was decided that it was important for students' conceptual development of the operation of subtraction to understand both interpretations. It was agreed that both within and beyond their further learning of mathematics in school, students would be expected to interpret contexts and to deal with tasks requiring both interpretations, and that by working with both models of subtraction students were likely to develop a more robust concept of this operation. This emphasis on conceptual development was in line with the position taken by Richardson and McCallum (2003) on how the teaching and learning of mathematics can contribute to the development of quantitative literacy in university students. They suggested that improved performance with using mathematics in other contexts was linked to a greater emphasis on building understanding of the underlying mathematical concepts. They had found that too many students were technically proficient with a range of mathematical processes but were often unable recognise which process was relevant within a given context.

A similar rationale was behind exploring ways of representing the split strategy when applied to subtraction tasks that required regrouping. Students seemed quite comfortable with the concept of a negative value to a component calculation. What they needed was oral language and mathematical symbols to communicate and keep track of their thinking.

In Figure 4.1 the two examples of the jump strategy show its flexibility in representing different processing preferences, such as first subtracting a bit to get to a round number (multiple of ten), or first subtracting the multiple of tens. The V notation developed by Teacher C for representing the split method for processing these tasks seemed to more than adequately expose students' thinking, with the adjacent written text suggesting that these students clearly understood what was happening in the calculation.

Discussion

The empty number line and the V notation are not tools for processing calculations in the same way as the traditional formal written algorithms typically associated with addition and subtraction with two two-digit numbers. Rather they are ways of visually representing, recording and communicating effective and efficient mental strategies for performing these tasks. This visual representation, augmented by the oral or written language that is used to explain what is happening during a calculation, helps add meaning to the symbols that are used in our number system. Combining these three modes of representation of a mathematical concept (visual or concrete, verbal and symbolic) is a central tenet of some sense-making approaches to teaching mathematics (McIntosh & Dole, 2004; NCTM, 2000).

The empty number line is convenient for representing the jump strategy with addition and both interpretations of subtraction, and for representing compensation strategies. The V notation blends obvious applications to addition tasks with intuitive notions of negative numbers to provide a comprehensive approach to representing the split strategy for both addition and subtraction tasks. Taken together the empty number line and the V notation provided useful visual representations of the three main efficient mental strategies for calculating addition and subtraction with two two-digit numbers. Such visual representations align with similar approaches described by Trafton and Thiessen (2004) in their attempts to position learning about and developing facility with computation within a sense-making approach to learning and doing mathematics. Their work with student-invented computational strategies suggested, among other things, that students develop a stronger and more connected concept of place value by engaging with calculations in ways that make sense to them.

The empty number line and the V notation expose the quantity value of the digits making up the numbers in a calculation, maintain the sense of the operation being processed, and use pen and paper only to record the results of calculations of partial quantity value done mentally. This helps the user and others to make sense of what is happening, both in terms of quantities and operations with them, thereby contributing to building number sense. Students' responses to Item 7 (256 +68) and Item 8 (631 -54) in the repeat written task both showed greater use of these strategies and improved performance compared to the initial written task.

Session Six in this series introduced *procedural representations* of these strategies through the number-sense written methods used for addition and subtraction with larger numbers. The expectation was that the investment in developing students' capacity with using, explaining and communicating efficient strategies for mental calculation would have a visible return in facilitating their use of the planned number sense written methods for addition and subtraction.

Session Three

The focus of this session was to:

- introduce split arrays as a direct representation of quantity showing the total of a rectangular array as the sum of component products
- build visual/symbolic representations of multiplication tasks as splits using the empty rectangle (ER) and sums of component products.

The mental computation data for this class suggested that students were likely to cope with representing two-digit by one-digit whole number products on the empty rectangle. Also students were familiar with representing products as rectangular arrays. It was decided to orientate planning for this session around the material in section three of the framework for multiplication and division that the researcher had developed for Teacher C (Appendix E).

Students used a 7 x 9 array of dots (7 rows with 9 dots in each row) to represent 7 x 9 as the sum of component calculations. There were several instances of both the teacher and students clearly identifying the number of dots in the group being repeated, and the number of groups. One student used the column as the group when explaining his compensation strategy of 7 x 10 – 7 for calculating 7 x 9. This student's explanation of his approach suggested he was clear that the 7 being subtracted was because one more group of 7 than was actually required had been included in calculating 7 x 10.

Teacher C represented several calculations using a split ER as a visual organiser for keeping track of component products. Here pen and paper were used to record the totals of component calculations, each of which calculated mentally, and for which the sum was also calculated mentally. Several ways to split the numbers in the example products were suggested by students:

- one student split 15 x 7 into 10 x 7 (70) and 5 x 7 (35) and calculated these separately
- another student recognized 5 as half of 10 and halved the answer of 10 x 7 to get the answer of 5 x 7
- a third student recognized 15 as 5 x 3 and multiplied the product of 5 x 7 by 3.

Towards the end of the session Teacher C asked the class if they might use this approach to calculate 24 x 15. One student asked if it was 10 x 20 plus 5 x 4. Teacher C hinted that 5 x 24 is half of 10 x 24 so the original calculation (24 x 15) is probably possible to calculate mentally.

Discussion

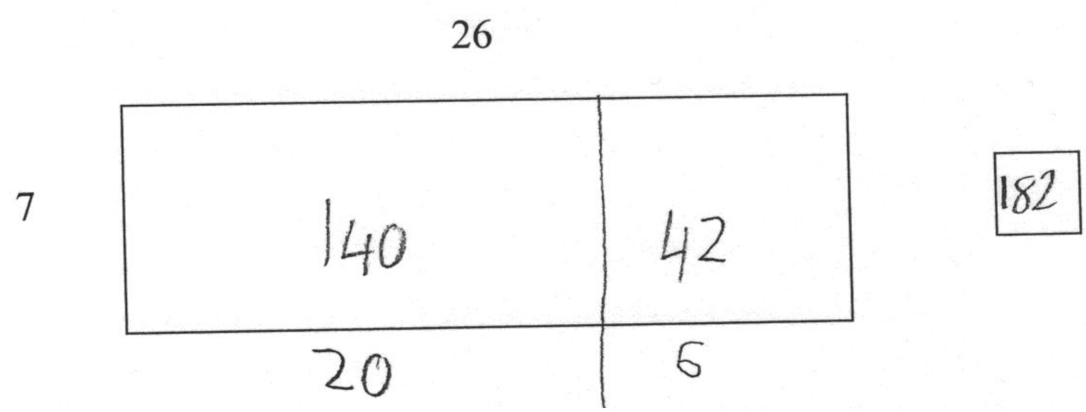
The three examples of how students reworked the calculation 15 x 7 described above illustrated the sort of facility with numbers and operations that indicated and consolidated number sense. In contrast, the splitting of 24 x 15 only into tens and ones suggests a naive generalisation of the split strategy for addition or subtraction into a multiplicative structure. Here number sense, or an understanding of how additive and multiplicative thinking are different, or both, appeared to be lacking. If this incorrect split were followed through, the value of the tens/ones only split is 220; just 10 lots of 24 is greater than this, and there are more than 10 lots of 24 in the full product.

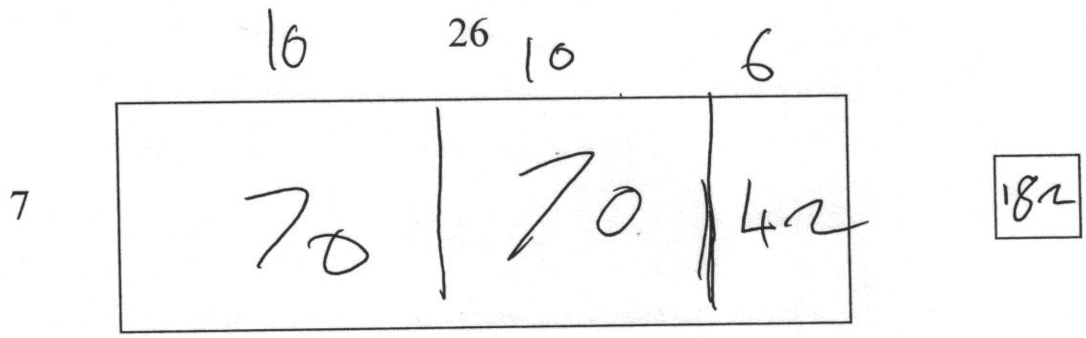
The capacity for accommodating variation in the way numbers might be split when using the ER is an advantage of this approach. Students are given a structure for an efficient process for splitting a larger multiplication task into smaller, more manageable pieces, and then keeping track of those pieces, but are free to make the connections between numbers as they choose or are able. The three responses above to the task 7 x 26 illustrate this. This goes some way to catering for individual differences when calculating, and prompts discussion of alternate and perhaps more efficient strategies when reviewed by the group, increasing students' exposure to a richer repertoire of strategies.

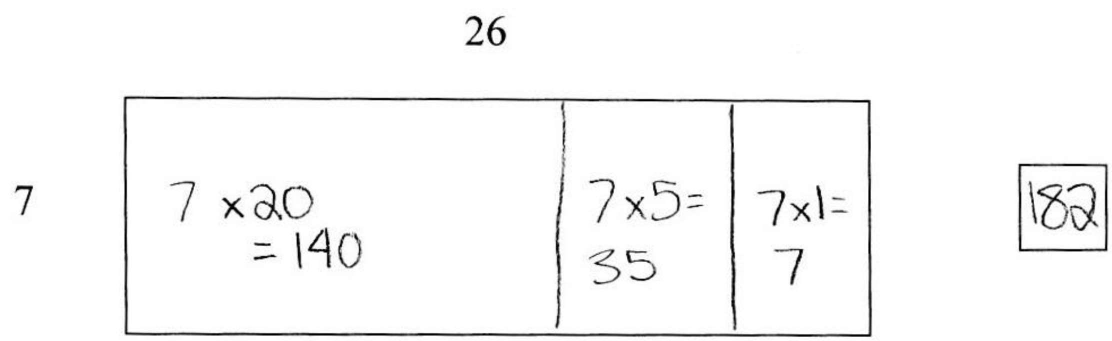
Figure 4.2 shows three examples that are representative of students' responses to the in-class task for this session. Points to note include:

- the variation in the ways numbers were partitioned or split (26 as 20 + 6, 10 + 10 + 6, and 20 + 5 + 1)
- the inclusion of full number sentences for partial products (such as *7 x 20 = 140*) inside the ER on one response but not others
- differences in the positioning of numbers in and around the ER.

Figure 4.2. Examples of students' use of the empty rectangle to describe strategies for calculating 26 x 7 mentally.







The way the various elements of a calculation are represented visually on the ER seemed important; also that each element carried meaning relative to the whole process. A minimalist representation with no record within the cells of the factors being multiplied perhaps prompts the calculation of these products mentally. In the first example in Figure 2 the top number (26) is split into manageable parts (20 and 6) which are written below the opposite long side of the ER, providing a visual connection between the whole factor and its parts. Here a vertical line separates the ER into two cells which contain their respective products, and is extended by this student to separate the lower long side of the ER into two intervals, representing the partitioning of the larger factor. Explicitly establishing this or a variation as a convention to be followed may provide some stability with using the ER structure through keeping the locations of the various parts of the visual organiser consistent.

Session Four

Division, like subtraction, can have two interpretations. Subtraction can be interpreted as *take away* and as *difference*, and division can be interpreted using either the *quotative* or *partitive* models. Consider the task 56 ÷ 4:

- on a quotative model the 4 represents the number in each group, while the answer to 56 ÷ 4 (14) represents how many groups of 4 are in 56
- on a partitive model the 4 represents the number of groups into which 56 is to be equally shared, and the answer (14) represents the number in each of the four equal shares.

The focus of this session was to use the ER to find the missing factor in additive splits of numbers as a way of answering the question: *how many groups (of a set size) are in (a given number)?* This is the quotative model of division, and to scaffold students' engagement with this approach to division the emphasis in this session was on framing division as the reverse of multiplication, and linking this to earlier work with modelling multiplication on the ER.

Warm-up activities involved multiplication tasks where a factor was split and component products calculated then combined. For example 6 x 14 was described as 6 x 10 (60) and 6 x 4 (24) to give 84. After a brief discussion about the meaning of the division sign (÷) within the quotative model, students were asked to calculate 39 ÷ 3. Responses included:

- split 39 into 30 and 9 and divide each by 3
- 12 x 3 = 36 as a known fact, and add on one more 3 to get 13 lots of 3
- 3 into 15 (5), another 5 (makes 30) then 3 more (5 + 5 + 3 = 13).

The ER was introduced as a way of representing visually what was happening when division calculations were done mentally. Known values were:

- the components of the additive split(s) of the number being divided (the dividend); these were written inside the cells of the ER
- the number being divided by (the divisor); this was written on the left just outside of the ER.

The quotients for each of the component division tasks were written on top of the ER, centred on the respective cells. The answer to the original division task was the sum of these numbers. This was written to the right of the ER. Teacher C emphasised that the way we were using the ER for division was the reverse of the multiplication tasks that were the focus of the previous week. Students were given two division calculations for their in-class task (64 ÷ 4 and 119 ÷ 7) and were asked to calculate the answers using the ER method they had been shown during this session.

Discussion

Many students in this session could give the correct answer to division tasks that were the reverse of familiar multiplication facts up to 10 x 10. However when pushed further most students seemed to struggle with dealing with division as an operation. The extent of this difficulty would become more apparent in the next session.

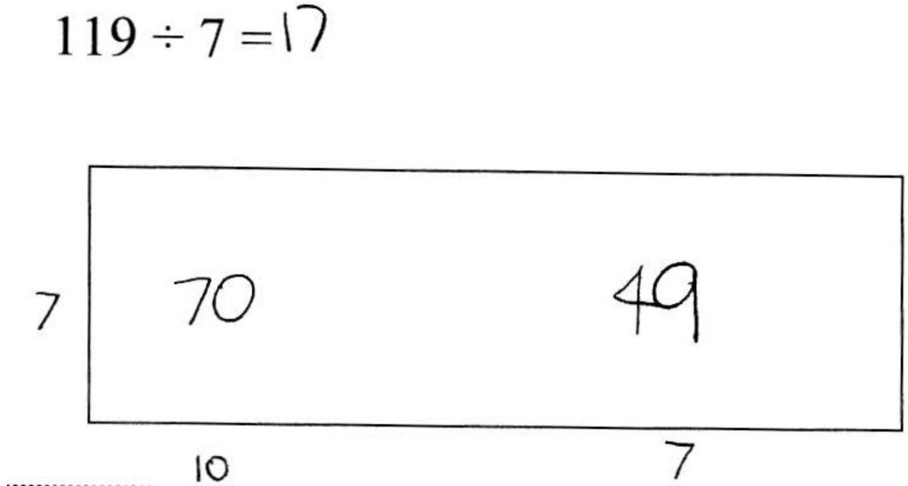
Of the 23 responses to the in-class task that were collected in Session Four:

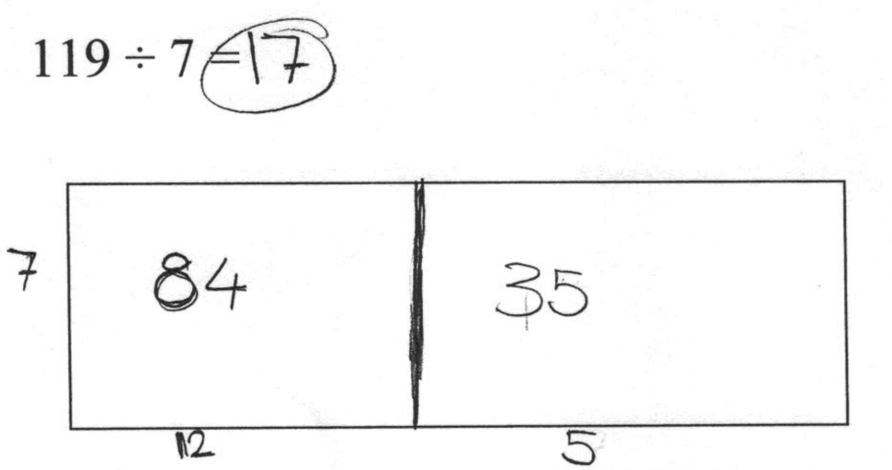
- five students seemed to struggle with one or more of the following:
    - splitting numbers into parts that each facilitated division mentally
    - reliably reversing their multiplication table facts
    - misrepresenting the division tasks as multiplication
- ten students solved both calculations correctly by first taking out ten groups of the divisor and linking the remainder directly to a table fact
- eight students solved both tasks correctly but used at least one alternative split.

For 119 ÷ 7 alternative splits to *ten lots* (70) and the remainder (49) included:

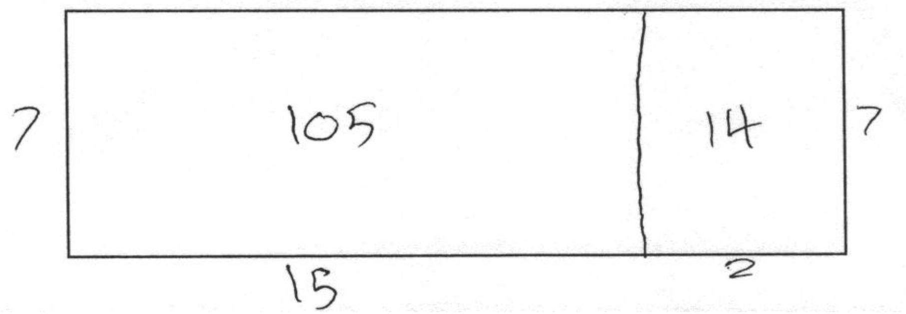
- 12 and 5
- 15 and 2
- 10 and 5 and 2.

Figure 4.3. Examples of students' use of the empty rectangle for calculating 119 ÷ 7as a reverse multiplication task.

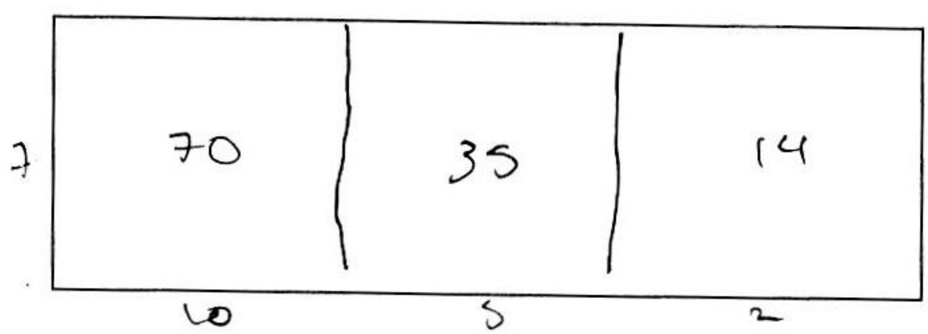




$119 \div 7 = 17$



$119 \div 7 =$



Using the empty rectangle (ER) as a visual scaffold for documenting mental strategies for calculating division tasks was an attempt to make connections to models for representing the more familiar operation of multiplication and mental strategies for doing such calculations. The intent here was to connect to the known (multiplication) as the base from which to start a path of ‘progressive mathematisation’ with division similar to that which has been used in Realistic Mathematics Education (RME) approaches in the Netherlands (Treffers and Beishuizen, 1999). The aim was for students to develop a strong concept of the operation of division and its relationship to other operations, and progressively more efficient mental and written strategies for processing an appropriate range of division calculations.

A scaffolding strategy that was used was related to lamentations by Reys and Reys (2004) about the perceived lack of emphasis on estimation in contemporary mathematics classrooms, and the positive value such an emphasis is likely to have for building aspects of number sense such as the relative magnitude of numbers and sound conceptual understanding of the four operations.

The scaffolding strategy used here was to start $m \div n$ tasks by asking the question: *would it be more or less than ten lots (or groups) of n?* This would prompt students to draw on their knowledge of multiplication facts (up to 10 x 10) and support them to make a productive start on the task:

- if *fewer* than ten groups, then the division task is either directly or indirectly related to a basic multiplication fact (for 56 ÷ 7: *ten 7s is 70, so it's less than ten times … 7 eights are 56, so the answer is 8*)
  (In this example if the 56 had been, say, 59, then the process would be similar except that the division by 7 leaves a remainder. This was beyond the targeted scope of this study but, at face value, not structurally different.)
- if *exactly* ten groups the answer should be clear; if the answer is not clear then the issue may well relate more to understanding how *ten times as many* is represented in our number system
- if *more* than ten groups, then split the original number into parts where one has ten as a factor, and the other is the remainder.

An example of using this strategy to calculate 144 ÷ 9:

- there are more than ten 9s in 144 because 10 x 9 is 90
- 144 is 90 + 54 (90 and 54 go inside the cells of an ER)
- ten 9s go into 90, and six 9s go into 54 … so the answer is 10 + 6 = 16.

It was considered important that students be asked and able to interpret the result of their calculation. An acceptable interpretation to 144 ÷ 9 would be something like: *9 goes into 144 16 times* or *if I had 144 things I could get 16 groups, with 9 things in each group.* This approach works in the same way as number sense written methods for division with larger numbers such as the chunking algorithm. The approach, however, emphasises only one of the two possible interpretations of division as an operation (the quotative model). A lack of deep conceptual understanding of how the quotative and partitive models for division are related but not the same showed itself in Session Five. Possible issues arising from failing to understand or clearly articulate both models of division are discussed in Chapter Six.

Session Five

The focus of this session was to establish meaning for calculations of the form $m \div n$ with particular emphasis on the quotative model.

Teacher C used various representations with dots, including rectangular arrays, to link language and symbols to the quotative interpretation of division tasks such as 12 ÷ 3. Most of the discussion involved the question *how many groups of …?* There was only occasional reference to division as *equal shares,* which is the partitive model. She made explicit reference to the total number of objects, the number in each group, and the number of equal groups, and linked this to multiplication tasks.

Although demonstrating reasonable competence with reverse multiplication facts, several students misinterpreted division tasks like 64 ÷ 4 as *how many times does 64 fit into 4?* Teacher C probed this language /symbol connection by asking *64 ÷ 4 and 4 ÷ 64 … are they the same or not? Explain …*

In their explanations some students started their explanation with one model for division (either quotative or partitive) and shifted mid-explanation into the other model. Students' explanations for calculating 56 ÷ 4 that drew on the partitive model included half and half again, and knowing that dividing by four is the same as finding one quarter. The ER was used again to model quotative division as the reverse of split multiplication. Several connections between numbers were exploited to facilitate the creation of component calculations that could be easily processed mentally. For example when asked to calculate 168 ÷ 7 a student 'took out' ten lots of 7 (70), doubled that (140) and added four more lots of 7 giving the correct answer of 24.

The in-class task required students to solve 23 x 6 and 136 ÷ 8 and show their thinking on separate ERs.

All of the 20 student responses had both tasks answered correctly with an appropriate explanation. Again the way each of the tasks was split varied.

Discussion

The lack of understanding of the concept of division that was exposed in Session Four came as a surprise to Teacher C and the researcher, as students appeared to have reasonable facility with calculating the answer to simple division tasks. In Session Five the emphasis on correctly explaining what was happening in simple division tasks (such as 12 ÷ 3) and then in tasks that go just beyond direct application of basic multiplication facts (such as 64 ÷ 4) confirmed to Teacher C and the researcher just how weak students' conceptual understanding of division was. Students' responses during these two sessions suggested that the interpretation of division as the reverse of multiplication was not as apparent or as meaningful to this class as might be expected.

'Cross-modelling' was the term the researcher coined to describe mixing aspects of different interpretations of a mathematical concept (in this case the quotative and partitive models for the operation of division) into the one explanation. Attempting to clarify what is happening by describing the mathematics in a different way is an understandable and perhaps in some contexts a useful teaching strategy; but cross-modelling with division clearly created or further fuelled genuine confusion. In this session Teacher C, by her own admission, was confused by her cross-modelling, and was deeply concerned about this being the case. This, and a review of the in-class tasks from this session, confirmed a belief that the operation of division was not well understood by most students, even though many students could correctly answer division tasks that were the reverse of their basic table facts (up to 10 x 10). With respect to knowing their reverse multiplication facts most of the students demonstrated a desirable level of computational fluency, which is described by Cumming (2000) as a necessary pre-requisite for success with the further learning of mathematics, and for effective applications of mathematics beyond school. But the students' shaky conceptual foundations raised doubts about their capacities to recognise contexts in which division was relevant. This was deemed to be a concern on the basis of what was known of pedagogical approaches which emphasised the importance of linking the learning mathematics to realistic or meaningful contexts, such as has been the case for decades in the Netherlands (Treffers & Beishuizen, 1999).

Teacher C's concerns about the cross-modelling that occurred during her explanatory phase of the lesson prompted an in-depth discussion with the researcher after class. The discussion pivoted around clarifying the three corners of multiplicative thinking: the number of things in a group, the number of equal groups, and the total number of things across all equal groups. It was during this discussion that the remaining parts of the framework that was eventually titled *Support notes for scaffolding number-sense written methods for multiplication and division* (Appendix E) were identified.

Session Six

The focus of this session was to introduce number sense written methods for addition and subtraction involving three-digit numbers which worked from left to right, and maintained the quantity value of component digits.

The opening comments by Teacher C to the whole class framed this session as about ways of using pen and paper to 'keep track of things you work out in your head' for calculations that are too big to do mentally.

Two calculations were written horizontally on the board: 383 + 276 and 463 – 185. Students were asked how they might calculate the answers. Different students (two for each calculation) came to the front of the class and explained their approach:

- the first student did the addition task as a formal vertical algorithm
- the second student used a number sense explanation: *300 + 200 is 500 … 80 + 70 is 150 … 6 + 3 is 9 … so 659*
- the third student did the subtraction task as a formal vertical algorithm using the decomposition method
- the fourth student used a number sense approach, first incorrectly but then she self-corrected: *463 – 100 is 363 … 363 + 15 is 378 … no, wait … 463 – 200 … then 263 + 15 is 278*.

Teacher C then described how in her first year of teaching she had tried to do a similar task in front of a class but had performed the formal vertical addition algorithm

from left to right and calculated an incorrect answer: *I was following a process with no meaning … I hadn't used it for a while, but I realized I'd done it wrong …*
She asked: *What's the 6 in 463?* (60) and *What's the 3 (on the left) in 383?* (300)

She then explained that the written methods that she was about to show would *hold the value of what those digits represent.* Also that this approach to written calculation was using the same ways of thinking about and working with numbers that they had learned and used to calculate mentally.

383 + 276 was written vertically with the component sums (500, 150 and 9) written underneath. These were then totalled mentally. Teacher C then asked the class if the same thing could be done with the subtraction task.

The calculation 463 – 185 was written vertically. Teacher C led the class through the following steps:

- *What's 400 take away 100?* (300 written underneath)
- *What's 60 take 80?* A student responded that it was – 20. Teacher C said: *That's right, I still have 20 more to take .. like 'break up/make up'* (a phrase used by Teacher C in Session Two) … - 20 was written under the 300
- *What's 3 – 5?* (-2) This was written under the -20
- Teacher C aggregated these components: *So that's 300, minus 20 … that's 280 … minus another 2 … that's 278.*

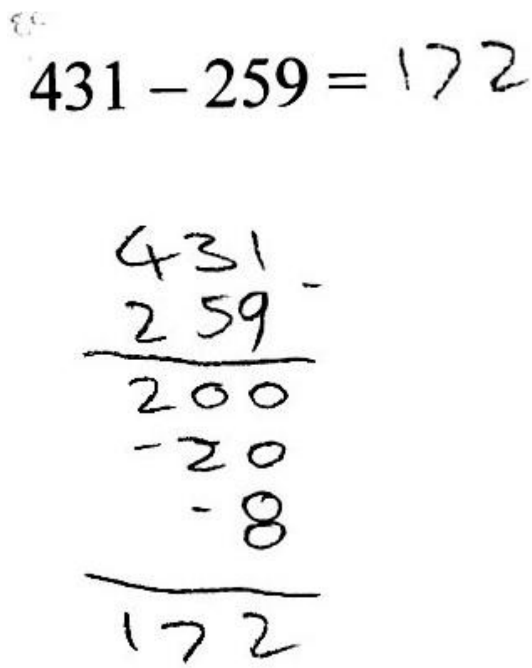
The class was given five similar calculations (368 + 293, 624 + 194, 862 – 387, 412 – 256, 926 – 468) and asked to try the approach they had just been shown. Students were asked if they had ever seen this approach to written calculation before. All students said that they had never been taught this approach, but that they had been taught other written methods for subtraction in primary school.

Five different students were then asked to share with the class their calculation process for one of the tasks. All of these students performed the number sense written approaches correctly to get the correct answer in each case. Four of the five students showed strong mental computation with the component calculations. However the remaining student repeatedly made the error of describing the component calculations in reverse. For example when explaining how he calculated 412 – 256 he said *200 take 400* and so on. Each of the three subtraction tasks above had negative answers to the component calculations of both the tens and ones.

Two further subtraction tasks were given to the class: 457 – 382 and 362 – 148. Some students inappropriately subtracted the answers to the component calculations of both the tens and ones. However when challenged to explain their thinking the students immediately self-corrected.

The in-class task had two addition and two subtraction tasks similar to the ones used earlier in the session. Students were asked to again try the number sense written methods they had been shown that day. Most students used the number sense methods for both addition and subtraction, although a few students did up to two tasks using a formal written algorithm. There was some variation in the amount of working out written for each task. However most responses tended to have only what was needed. Examples of students' responses are shown in Figure 4.4.

Figure 4.4. Examples of students' use of the written strategy of subtracting from the left while maintaining quantity value.



431 – 259 = 172

```
 431
-259
----
 200
 -70
 180
  -8
 172
```

Discussion

Students had fewer problems than expected with making sense of component subtractions that gave a negative result. Teacher C took time to connect such sections of a written subtraction task to language that made the process meaningful. For example if the component calculation was 40 – 70 the language would be *there's still 30 to take*, or *I can take 40 but I still need to take another 30*. This language mirrored the language used with V notation in Session Two. Although the students in Class C had previously engaged with more formal treatments of negative numbers, such formal work with integers was not necessary for correctly and meaningfully using negative numbers to represent the answers to component calculations in these subtraction tasks. Carroll and Porter (1998) had found that many primary students were similarly comfortable with using the negative sign informally, where appropriate, to represent "'being in the hole' or having a deficit of that quantity" (p.110).

Initially some students automatically subtracted any residuals with tens or ones. However this was addressed explicitly with a continued focus on language that supported making sense of what was happening with each stage of the overall calculation, and how the mathematical symbols (in this case a number with a negative sign) represented what was happening.

Session Seven

The focus of this session was to use the empty rectangle (ER) to represent the product of two two-digit numbers.

Teacher C opened the session by asking the class about what they had done in the previous week's intervention. She combined responses from students by saying that the methods for written calculation that were introduced last week are ways of "keep(ing) track of numbers that hold their (quantity) value. Today we are doing something similar with multiplication …".

Teacher C wrote 16 x 36 on the board. One student said this could be calculated as 10 x 30 plus 6 x 6 (336). Another student concurred. A third student said that 336 could not be the correct answer as 10 x 36 is 360. A fourth student said that there was still 6 x 36 to go. Teacher C (about the calculation so far) said "We've only done a partial answer." She then modelled the formal written algorithm for long multiplication. At least seven students said they had seen this before in primary school. Teacher C remarked: "Sometimes I forgot to put the zero".

She then asked the class how they would calculate 6 x 36. One student said he'd do 6 x 30 (180) and 6 x 6 (36) and add them (216). Teacher C modelled this on the board with an ER. She then showed 16 x 36 on an ER split as 10 and 6 vertically on the left, and 30 and 6 split horizontally on top. Component products were calculated mentally and then added to give the final answer. This was compared to the formal written algorithm which was still on the board. A second example (26 x 47) was worked through on the board in a similar way. Some students struggled to total the component products mentally. In unpacking this, students described a variety of addition sequences exploiting different connections between the numbers.

Another task (35 x 43) was put on the board. Teacher C asked "About how big should the answer be?" Students discussed various aspects of approximation and estimation such as the answer needs to be more than 1200 because 30 x 40 is 1200 and both original factors had been rounded down to get the estimate.

Teacher C probed the strategies used to calculate 30 x 40. Several students talked of calculating 3 x 4 and adding zeros. Teacher C asked what was happening when zeros were added. This led to discussing the concept of *ten times more*. Using this context 3 x 4 is 12, but 40 is ten times 4, so 3 x 40 will be ten times 12 (120). Since 30 is ten times 3 then 30 x 40 likewise will be ten times 120 (1200).

Students were asked to calculate 39 x 52 using the earlier two-by-two ERs as models. The students who shared their strategies showed a range of effective addition strategies for summing the answers to the four component products. However the strategy of adding zeros when multiplying by a multiple of ten was referred to several times. Teacher C again asked what was happening when zeros were added, and how one would know if they had added the correct number of zeros. She concluded the discussion by saying that adding a zero in this situation is about multiplying by ten.

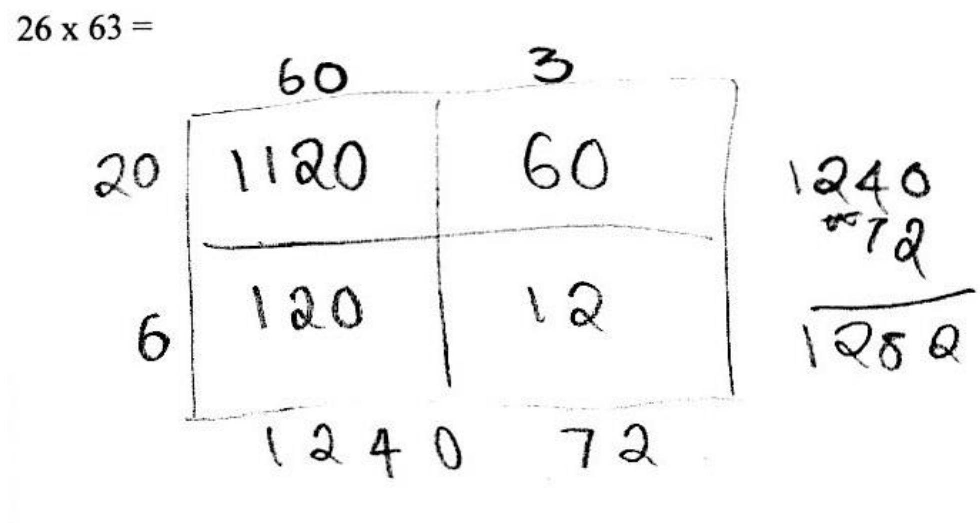
Students' in-class task involved four two-digit by two-digit products to be calculated using the ER. All items in all of the 18 student responses had been attempted using the ER. Eight responses had all four tasks calculated correctly and adequately explained. A further eight responses had one error only. These errors included:

- a component product that was out by a power of ten (such as 8 x 50 = 40)
- an error with a basic multiplication fact (such as 3 x 6 = 24 or 20 x 3 = 90)
- not totalling the component products correctly.

Two responses showed more than three errors involving basic table facts, place value or the final tallying process. However all tasks on these two responses used the ER correctly to set up the calculation process.

An example of this from the student responses is shown in Figure 4.5.

Figure 4.5. Example of students' use of the empty rectangle as a written strategy for calculating 26 x 63 which contains several errors in component calculations.



Discussion

Teacher C had already established a culture in this class of valuing number sense. This was evident in this session as Teacher C made repeated demands on students' number sense. She asked them to provide estimates of the answers to exact calculations. She also insisted that students could explain how they calculated their estimates, and interpret their estimate as either higher or lower than the exact answer. Teacher C also picked up on the *add zeros* shortcut used by some students several times in the session and pushed the class to understand and explain what was actually happening when zeros were added in this process.

It seems likely that the students who had a single error in their in-class task made a concentration error, as their complementary items showed they understood the process and could perform it correctly. However the two students whose responses each had more than three errors may have had deeper conceptual problems; if not with using the ER form of written multiplication then perhaps with mentally managing products with multiples of ten. McIntosh (1998) pointed out that when working with multiples of ten, shortcuts that involve taking off or adding on zeros do not work the same way across all of the four different operations, and this makes a "rules without reasons" (Skemp, 1976) approach to dealing with multiplying or dividing by powers of ten prone to error.

Session Eight

The focus of this session was to introduce the chunking algorithm for division.

Teacher C opened the session by showing a 4 x 3 rectangular array of counters with an overhead projector and asking questions about the group size, the number of groups and the total, and how the relationships between these could be described using mathematical symbols. She then asked for ways of working out how many groups of five are in 40. Student responses included:

- there are two fives in ten and four tens in 40, so 8
- there are four fives in 20 so double that.

The task was written as 40 ÷ 5. Teacher C explained that this could represent how many fives in 40, or that it could represent sharing 40 into five equal parts. This distinction between models for the operation of division was explicitly discussed and illustrated using a 3 x 2 array (3 rows, each row with 2 dots) to explore interpretations of the calculation 6 ÷ 2. On a *sharing* (partitive) model the two columns represented the two shares, with the three counters in each column giving the size of the equal shares. On a *groups of* (quotative) model the rows represented the groups of two, with the number of rows giving the number (3) of these same-sized groups that can be formed from the total (6).

Teacher C said that the rest of the activity in this session would involve the 'groups of' model for division. She wrote 42 ÷ 3 on the board and asked how this might be interpreted. A student responded: *how many groups of 3 are in 42?* Teacher C then asked how this might be calculated. One student said 3 x 10 is 30, and 12 left over, so 14. Teacher C modelled this on the board using an ER. Another student said that 21 is half of 42, there are seven 3s in 21, so 14. Teacher C modelled this on the board using another ER. A third student said that 3 into 15 is 5, so 5 and 5 and 4 giving 14. Teacher C modelled this on the board using a third ER, commenting on the link between representing multiplication on the ER, and on the concept of splitting numbers (like 42) so that (in this case) 3 could easily divide into the components of the split.

The meaning of 78 ÷ 3 was explored in a similar manner, as were various ways of calculating the result mentally and showing these on an ER. Teacher C then commented that in the previous two intervention sessions the focus had been on using pen and paper to keep track of what was being processed mentally when dealing with larger numbers, and that this session was going to target something similar.

She wrote 492 ÷ 6 on the board. A student offered the interpretation *how many times does 6 fit into 492?* Another student said that there could not be more than 100. A third student said there would be more than 80. This student then explained that 80 x 6 is 480, which leaves 12, so two more (sixes) makes 82.

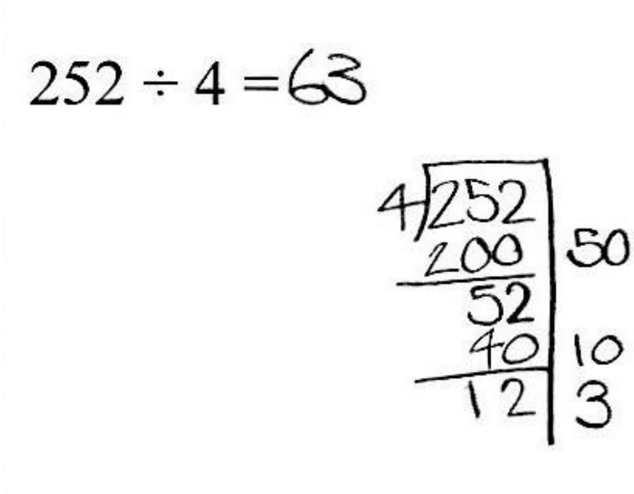
Teacher C asked if anyone had been shown ways of using pen and paper to calculate something like 492 ÷ 6. She modelled the formal written algorithm for short division and also showed how this could be represented on the ER. She then walked through the steps of the chunking algorithm for division.

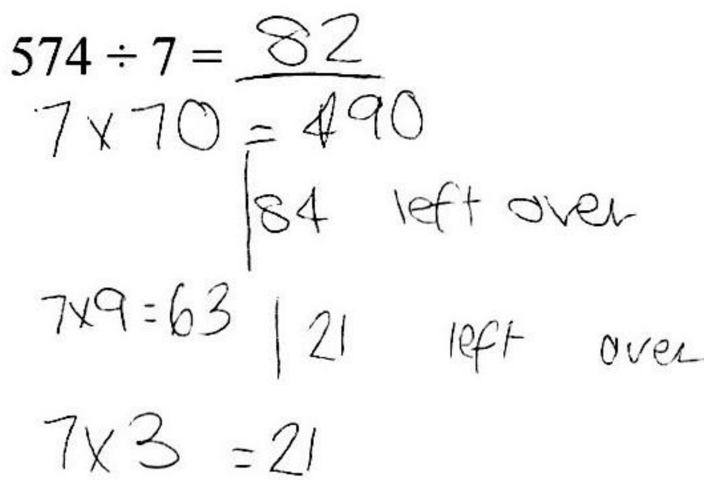
The class was asked to try the chunking algorithm to calculate 520 ÷ 8. Two students explained their respective approaches. The first said she thought of 100 lots of eight (800) but that was too much, so she took half that (50 lots) which she knew was 400. That left 120 to go. She took out another ten lots of eight (80) which left 40, which she knew was 5 x 8. She totalled the 50, 10 and 5 to give 65.

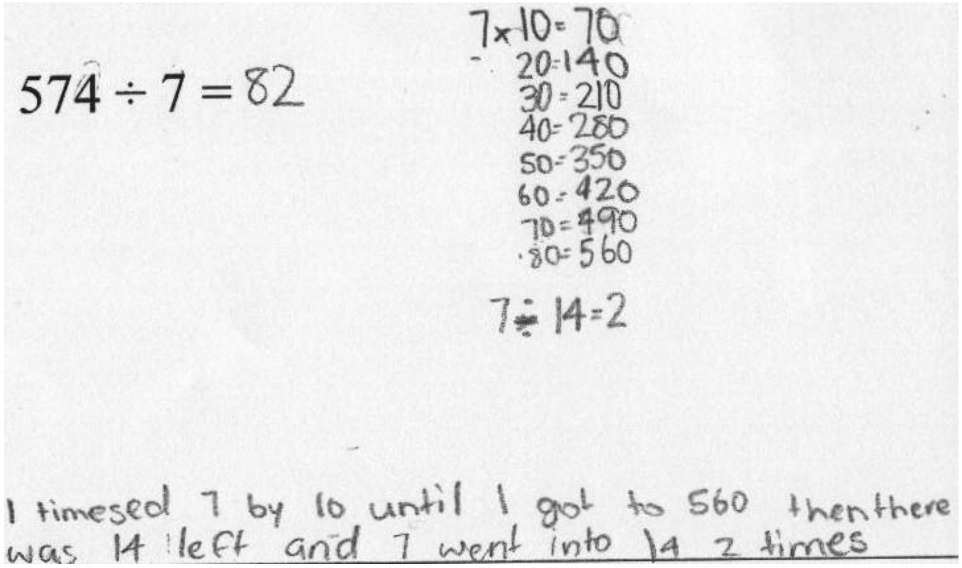
The other student recalled that 60 x 8 was 480. This left her with 40 which she also knew contained five 8s. She correctly completed and interpreted the process.

The in-class task had four items, each with a three-digit number divided by a single-digit number. Students were asked to calculate the answers using any method (mental or written) they were comfortable with. Examples of student responses to these items are shown in Figure 4.6.

Figure 4.6. Examples of students' responses to division tasks.







$574 \div 7 = 82$

7×10=70
20=140
30=210
40=280
50=350
60=420
70=490
80=560

7÷14=2

I timesed 7 by 10 until I got to 560 then there was 14 left and 7 went into 14 2 times

Discussion

Table 4.2 gives a summary of several aspects of the responses to each item on the in-class task from this session.

Table 4.2. **Summary of student performance and strategy use on in-class division tasks.**

item	**correct**	**incorrect**	**blank**	**chunking algorithm**	**empty rectangle**	**informal chunking**	**other number sense approach**	**formal written algorithm**	**no or inadequate explanation**
574 ÷ 7	13	3	1	5	-	2	2	6	1
621 ÷ 9	14	2	1	5	-	3	1	5	2
336 ÷ 3	16	-	1	3	-	5	-	6	2
252 ÷ 4	13	3	1	3	-	5	-	5	3

The first example in Figure 4.6 illustrates the setting out of the chunking algorithm. *Informal chunking* refers to responses that showed evidence that the student used the chunking process but did not set it out formally as in the second example in figure 4.6. A similar type of response to the task 621 ÷ 9 had three lines of text: 9 x 60 = 540, 81 left over, 9 x 9 = 81. *Other number sense approaches* included responses like the third example in Figure 4.6, or calculating the answer to 621 ÷ 9 by counting on by 90 seven times to 630 then taking off nine.

The 17 responses to the in-class task for this session were the most disparate over the whole intervention in terms of the range of strategies demonstrated across the class, and even within an individual student's responses to each of the four items on the task. Despite many students' earlier struggles with the concept of division as an operation during Sessions Four and Five of the intervention, on this in-class activity 56 out of 68 items (17 students x 4 items each) or 82% were answered correctly.

It was surprising that no student tried to represent any calculation on an empty rectangle. Almost half of the item responses used a number-sense approach (chunking, informal chunking or other). About one third of the item responses used the formal written algorithm for division. This suggests the division algorithm is still part of some mathematics programs in the primary or middle years of schooling, or possibly something valued by adults at home, as students had not been taught the formal division algorithm during this intervention or anytime earlier that year.

The chunking algorithm was introduced during the last session and warranted more time than it received. In spite of this, the work done with division tasks during the intervention sessions may have had an effect on students' confidence with this sort of task. It is worth comparing the initial and repeat performances on Item 10 of the student written task (1062 ÷ 9) in this regard. Of the 13 matched responses to this item only one attempt was made initially to do this task without a calculator. This was a written response that was incorrect. On the repeat task the same item drew six written responses. Only two were correct, but more students had attempted the task. The two correct responses used chunking and informal chunking. Two others attempted number-sense methods, with the remaining two responses attempting the formal division algorithm.

On the repeat written task one of the two students who had been interviewed (student C17) had used informal chunking to correctly answer Item 6 ($92 \div 4$) but had left Item 10 ($1062 \div 9$) blank. When questioned about solving $1062 \div 9$ in a similar way to $92 \div 4$, student C17 correctly applied an informal chunking approach to get the answer. The other student who was interviewed (student C21) had not attempted this item on the initial task but correctly used informal chunking on the repeat task.

It is reasonable to speculate that with some further consolidation of the chunking algorithm and similar number-sense approaches more students would have improved their understanding of and facility with written division tasks at this level.

SUMMARY

The series of eight weekly in-class interventions with Class C followed closely the intended allocation of equal emphases on explicit teaching of strategies for mental computation and number-sense written methods. Teacher C had embraced the intervention and contributed a great deal to its successful implementation with her class.

The students in Class C appeared to respond well to the explicit teaching of written calculation methods which built on the mental computation strategies that the class had practised. Almost all students, almost all of the time, attempted the somewhat novel approaches to written calculation, even if they did not feel a need to vary from more traditional methods that they had been taught earlier in their schooling.

The discussion generated, and the in-class activities given throughout this intervention repeatedly challenged, exercised or exposed aspects of students' number sense. The need to explain one's calculation strategy, or represent it in a visual as well as verbal way, or find a different strategy for a given calculation, was consistent across the work with mental and written calculation. Likewise the approaches to doing mental and written calculation were also much the same, with pen and paper used to keep track of the component calculations being done using efficient mental strategies when the numbers in the calculation meant that this was a sensible thing to do.

This emphasis on number sense drew out many examples of a disconnect between students' competence with recall or procedural tasks (such as working out that $18 \div 3 = 6$ through a connection to known multiplication facts) and their understanding of the concepts which underpinned the tasks they were doing (such as knowing what division is as an operation). This was particularly apparent with, but by no means limited to, the operation of division.

Chapter Five explores in more detail the responses to the initial and repeat written tasks which were given just before and immediately after the series of classroom intervention sessions that have been described in this chapter. Included in this exploration are selected elements from interviews with two particular students from Class C.

CHAPTER FIVE: THE STUDENT WRITTEN TASKS AND STUDENT INTERVIEWS

This chapter is focused on the performance of students from Class C on the 11-item written task (see Appendix D). The same task was given just prior to and immediately after the series of eight weekly intervention sessions. Five items showed noticeable variation between initial and repeat responses in terms of:

- the number of correct responses to the item
- the number of students who required a calculator to do the item
- the number of responses that contained an explanation of the process used to calculate the answer.

Aspects of these five items will be explored in more detail, with selected elements of interviews with two students from class C adding further to an understanding of some of the influences of building number sense through mental computation and number sense written methods that are suggested by this study.

OVERVIEW OF THE RESULTS

Table 5.1 describes the assumptions that were made about the degree of difficulty of each of the individual items in the student written task for students in Year 8 which in the ACT is the second year of high school. Students in Year 8 are typically 12 or 13 years of age at the beginning of the school year. These assumptions have been expressed in terms of the level of computational strategy that students might reasonably be expected to have capacity to use successfully at this stage of schooling (Appendix A).

Table 5.1. **Description of degree of difficulty of items in student written task.**

Assumption	**(Question)**	**Item**
Direct application of place value using only a single basic number fact	(4)	700 + 283
Mental computation using compensation	(3)	264 – 99
Mental computation using an efficient strategy with more than one step or stage in processing the calculation	(1)	68 + 44
	(2)	92 – 34
	(5)	26 x 7
	(6)	92 ÷ 4
Written calculation using number sense methods	(7)	256 + 68
	(8)	631 – 54
	(9)	143 x 6
	(10)	1062 ÷ 9
	(11)	53 x 28

On the written task students were asked to indicate several things in their response to each of the items:

- the correct answer to the calculation
- whether they calculated an answer mentally, used pen (or pencil) and paper, or if they needed to use a calculator
- an explanation of how they worked out their answer.

An explanation was some written description or indication of the strategy or method that the student used to calculate the answer. Students were given what Teacher C considered to be ample time to complete the 11 items.

Not all students completed both the initial and the repeat written tasks. Table 5.2 describes the numbers of paired responses in each Class C, which had the highest proportion of paired responses. There were equal numbers of paired responses from male and female students in this sample.

***Table 5.2*. Breakdown by gender of responses to the initial and repeat written tasks in Class C (*n* = 25).**

		Male		**Female**	
		7		**7**	**Paired responses (14)** (initial and repeat)
Initial	5	**7**	4	**4**	**Discounted (11)** (initial only, repeat only, non- or invalid attempts)
Repeat	2		**-**		
		14		**11**	

There were 14 students who attempted both the initial and repeat written tasks. Table 5.3 gives a summary by item of the responses to the initial task (in normal font) and the repeat task (in italics).

Table 5.3. **Summary of paired responses from the student written task from Class C (*n* = 14).**

	item	**circled *mentally* and correct**	**circled *mentally* and incorrect**	**circled *written* and correct**	**circled *written* and incorrect**	**no approach listed and correct**	**no approach listed and incorrect**	**circled *calculator***	**explanation given**	**total correct**
1	68 + 44	**12**				**1**			**13**	**13**
		11	***1***	***1***					***13***	***12***
2	92 - 34	**9**	**3**			**1**			**13**	**10**
		9	***3***	***1***		***1***			***13***	***11***
3	264 - 99	**7**	**3**	**3**					**12**	**10**
		9	***4***	***1***					***13***	***10***
4	700 + 283	**11**							**12**	**11**
		13							***13***	***13***
5	26 x 7	**10**	**1**	**2**				**1**	**13**	**12**
		8	***2***	***3***				***1***	***13***	***11***
6	92 ÷ 4	**5**		**2**				**6**	**7**	**7**
		9		***3***				***2***	***12***	***12***

Table 5.3 continued

	item	**circled *mentally* and correct**	**circled *mentally* and incorrect**	**circled *written* and correct**	**circled *written* and incorrect**	**no approach listed and correct**	**no approach listed and incorrect**	**circled *calculator***	**explanation given**	**total correct**
7	256 + 68	**6**	**1**	**3**		**1**	**1**		**12**	**10**
		9		***4***		***1***			***12***	***14***
8	631 - 54	**3**	**6**	**1**	**1**	**1**	**1**		**13**	**5**
		6	***2***	***3***		***2***		***1***	***13***	***11***
9	143 x 6	**4**		**2**	**1**	**1**		**3**	**8**	**7**
		4	***3***	***2***	***3***			***2***	***10***	***6***
10	1062 ÷ 9				**1**			**12**		**1**
				2	***4***			***8***	***4***	***2***
11	53 x 28		**2**	**1**	**4**		**3**	**2**	**9**	**1**
			4	***6***	***2***				***13***	***6***

Table 5.4 compares the number of correct responses to each item between the initial and repeat written tasks.

***Table 5.4.* Comparison between initial and repeat tasks of numbers of correct responses by item (*n* = 14).**

item number	***item***	**mentally correct**		**written correct**		**other correct**	
		initial	*repeat*	initial	*repeat*	initial	*repeat*
1	68 + 44	12	*11*	0	*1*	1	*0*
2	92 – 34	9	*9*	0	*1*	1	*1*
3	264 – 99	7	*9*	3	*1*	0	*0*
4	700 + 283	11	*13*	0	*0*	0	*0*
5	26 x 7	10	*8*	2	*3*	0	*0*
6	92 ÷ 4	5	*9*	2	*3*	0	*0*
7	256 + 68	6	*9*	3	*4*	1	*1*
8	631 – 54	3	*6*	1	*3*	1	*2*
9	143 x 6	4	*4*	2	*2*	1	*0*
10	1062 ÷ 9	0	*0*	0	*2*	0	*0*
11	53 x 28	0	*0*	1	*6*	0	*0*
	totals	**67**	***78***	**14**	***26***	**5**	***4***

Total of correct responses to items on the student written task:

Initial	86
Repeat	108

There was a 26% increase in the overall number of correct responses between the initial and repeat written tasks. It would be unreasonable to attribute this solely to the number sense focus of the intervention sessions as it could be argued that any regular weekly focus on number (say using formal written algorithms) might also increase overall performance on these tasks in a similar way. Nevertheless at face value the fact that about half the increase in correct responses is from greater use of mental calculation (as attributed by the students) suggests that there was an impact on performance related to building number sense in this way.

At the individual item level the limited number of paired responses makes expressing differences and proportions as percentages inappropriate. It also makes extended quantitative analysis of limited if any real value. Nonetheless some observations about the results as a whole are worth noting:

- Only one item (Item 7 on the repeat task) was answered correctly by the 14 students in this group. The item that was assumed to be almost trivial to calculate (Item 4: 700 + 283) was incorrectly read as subtraction by at least one student on both the initial and the repeat tasks. This error would be better characterized as a concentration error than as an error in a particular approach to calculating the answer, or as an indicator of a lack of capacity to calculate items of this difficulty. Errors of this type do not seem strongly related to the choice and use of specific strategies for computation, although perhaps different computational strategies may help minimize the occurrence of these or other concentration errors. In this study the criterion chosen for selecting items for deeper analysis attempted to reduce the influence of this class of error on discussions of variations in student performance that can fairly be attributed to the number sense intervention.
- The number of responses that indicated a need for a calculator suggests that, particularly on the initial task, students were much less comfortable with the two items that involved the operation of division (Items 6 and 10).

The criterion for selecting items for further exploration related to the purposes of this study was *a variation of more than two between initial and repeat responses* in:

- the number of correct responses for an item (Items 6, 7, 8 and 11)
- the number of students who required a calculator to do the item (Items 6 and 10)
- the number of responses that contained an explanation of the process used to calculate the answer (Items 6, 10 and 11).

This criterion (a variation of more than two) was deemed by the researcher as suggesting the possibility of an effect resulting from the number sense intervention and not just a somewhat random variation related to confounding factors like concentration errors when recalling basic number facts. If there were any evidence for an intervention effect it might be found by interrogating the characteristics of the student responses to those items more closely. These characteristics could include changes in the proportions of calculations using written methods, or changes in the nature of the written methods used by students towards more frequent use of approaches that make use of efficient mental computation strategies and number sense.

ANALYSIS OF SELECTED ITEMS

Tables 5.6 to 5.10 contain descriptions of selected characteristics of the 14 paired responses to each of the five items that were deemed to have enough variation between the initial and repeat responses to be considered priorities for further exploration. Each table is followed by a discussion of the characteristics which relate to the research question in this study. Later in this chapter several insights from interviews with two students from Class C (students C17 and C21) will be described which add to the understanding of the analysis of the five selected items.

Table 5.5 defines the headings and abbreviations used in Tables 5.6 to 5.10.

***Table 5.5.* Headings and abbreviations used in Tables 5.6 to 5.10.**

M√	circled *mentally* and gave the correct answer
Mx	circled *mentally* and gave an incorrect answer
W√	circled *written* and gave a correct answer
Wx	circled *written* and gave an incorrect answer
?√	no approach listed but the correct answer given
?x	no approach listed and an incorrect answer given
C	circled *calculator*
exp	explanation given
FWA	formal written algorithm
ENL	empty number line
ER	empty rectangle

Analysis of Item 6: 92 ÷ 4

Table 5.6. **Detailed comparison of responses to Item 6 (92 ÷ 4).**

Student		**M√**	**M**x	**W√**	**W**x	**?√**	**?**x	**C**	**exp**	***Observations***
C11	initial							1		
	repeat	**1**							**1**	**used 'half of a half' strategy**
C12	initial	1							1	listed as written but explains '4 fits into 80 twenty times … which leaves 12 …'
	repeat	**1**							**1**	**evidence of FWA**
C13	initial	1							1	knew 25 into 100 and 20 into 100, then combined 3 x 4 = 12 and 20 x 4 = 80
	repeat	**1**							**1**	**used 'half of a half' strategy**
C14	initial	1							1	reverse multiplication in parts: 10 x 4 = 40 and 13 x 4 = 52
	repeat	**1**							**1**	**knew 4 x 20 = 80 and 4 x 3 = 12**
C16	initial			1					1	FWA
	repeat			**1**					**1**	**FWA**
C17	initial							1		
	repeat	**1**							**1**	**knew 4 x 20 = 80 and 4 x 3 = 12**
C19	initial									crossed out/no response
	repeat			**1**					**1**	**5 lots of 4 per 20, built up by 20s to 80**
C21	initial							1		
	repeat	**1**							**1**	**wrote 4 ÷ 80 = 20, 4 ÷ 12 = 3, 20 + 3 = 23**

Table 5.6 continued

Student		M√	Mx	W√	Wx	?√	?x	C	exp	*Observations*
C22	initial							1		
	repeat			**1**					**1**	**4 x 10 (40), 4 x 10 (40), 4 x 3 (12)**
C23	initial	1							1	used 'half of a half' strategy
	repeat							**1**		
C24	initial							1		
	repeat	**1**							**1**	**used 'half of a half' strategy**
C25	initial							1		
	repeat							**1**		
C27	initial			1					1	used 'half of a half' strategy
	repeat	**1**							**1**	**guess and check back from 4 into 100**
C28	initial	1							1	5 fours in 20, 4 lots of that to get 80, then 3 more fours (23)
	repeat	**1**							**1**	**knew 4 x 20 = 80 and 4 x 3 = 12**

This item showed a shift from seven to 12 correct responses.

Some students showed no evidence of any impact of the number sense intervention in the way they responded to the item. Student C16 correctly performed the traditional short division algorithm both times, while student C25 circled *calculator* on both tasks.

Several students used what have been described in this study as number sense methods on both tasks (students C13, C14, C27 and C28). Students C14 and C28 used a quotative strategy (finding the number of fours in 92) both times, while students C13 and C27 used a partitioning strategy (split or share 92 into four equal parts) on one occasion and a quotative strategy on the other.

Two students had paired responses that were somewhat anomalous and are perhaps best viewed as challenges to tendencies to read too much into any single response. Student C12 initially used a number sense strategy successfully on this item but on the repeat task used the formal written algorithm, possibly suggesting that s/he was comfortable with either strategy and used whichever approach felt right at the time. Student C23 initially used the 'half of a half' strategy but then circled *calculator* on the repeat task. The initial response was reasonable evidence that s/he was capable of calculating this item mentally but for some reason on the repeat task either forgot how to do this or chose not to. It might be reasonable to infer that this student needs more consolidation of the 'half of a half' strategy but it would be inaccurate, on the basis of the response to the repeat task alone, to infer that s/he did not have any effective strategies to do this calculation mentally.

Of particular interest was the number of students who either did not attempt or circled *calculator* on the initial task, but then successfully calculated the answer on the repeat task (students C11, C17, C19, C21, C22 and C24). Two students used the 'half of a half' strategy (C11 and C24). Between the remaining four students there were three slightly different ways of using part/whole equivalences for 92 to construct convenient multiples of four:

- knew 4 x 20 = 80 and 4 x 3 = 12
- 5 lots of 4 per 20, built up by 20s to 80
- 4 x 10 (40), 4 x 10 (40), 4 x 3 (12).

The greater number of responses that used a number sense approach for this item and the coincident improved performance suggest a positive impact from the intervention.

Analysis of Item 7: 256 + 68

Table 5.7. **Detailed comparison of responses to Item 7 (256 + 68).**

Student		**M√**	**Mx**	**W√**	**Wx**	**?√**	**?x**	**C**	**exp**	***Observations***
C11	initial									blank
	repeat	**1**							**1**	**250 + 60, 6 + 8**
C12	initial						1		1	subtracted instead of added
	repeat			**1**					**1**	**FWA for vertical addition**
C13	initial	1							1	added 250 and 68 (318) then added 6
	repeat	**1**							**1**	**250 + 60, 6 + 8**
C14	initial	1							1	250 + 50 = 300 + 10 = 310 then 6 + 8 = 14 (324)
	repeat	**1**							**1**	**250 + 60, 6 + 8**
C16	initial		1						1	250 + 60 = 310, 6 + 68 = 74, 310 + 74 = 384 (double counted the 60)
	repeat	**1**							**1**	**250 + 60, 310 +(6+8)**
C17	initial			1					1	FWA for vertical addition
	repeat	**1**								**6+8, 50+60, +200**
C19	initial			1					1	FWA for vertical addition
	repeat			**1**					**1**	**FWA**
C21	initial	1							1	200 + 68 = 268 (then) 268 + 50 = 318 … 318 + 6 = 324
	repeat	**1**							**1**	**250 + 60, 6 + 8**

Table 5.7 continued

Student		**M√**	**M**x	**W√**	**W**x	**?√**	**?**x	**C**	**exp**	***Observations***
C22	initial	1							1	compensation: 256 + 70 (326) minus 2 (324)
	repeat			**1**					**1**	**FWA**
C23	initial					1			1	stated they used FWA but did not show this on the response sheet
	repeat			**1**					**1**	**on ENL add 60 to 256 (316) then +8 (324)**
C24	initial	1							1	250 + 60 = 310, 310 + 8 + 6 = 324
	repeat	**1**							**1**	**200 + 68 (268), + 50 (318) + 6 (324)**
C25	initial	1							1	‘I did 200 + 56 + 68 and got 324'
	repeat	**1**							**1**	**200 + 60 (260), 56 + 8 (64) then 260 + 64 (324)**
C27	initial			1					1	250 + 60 (310) then 6 + 8 (14) then added
	repeat	**1**							**1**	**compensation: 256 + 70 - 2**
C28	initial									subtracted correctly (188)
	repeat					**1**				**confused explanation: added 60 and 60 (120) then added 8 and 5 (13) 'and added both numbers together and it equeled (sic) up to 324'**

This item showed a shift from 10 to 14 correct responses. This becomes less impressive when one considers the three incorrect responses on the initial task that were probably due to concentration errors. Students C12 and C28 subtracted instead of adding the two numbers and student C11 left the initial response blank.

Addition is probably the most intuitively obvious, and perhaps most practiced, of the four basic operations with number, and the number of incorrect responses to this item (effectively nil) supports this point of view. Of particular interest with this item is the number of initial responses that attempted number sense methods to calculate the answer mentally (students C13, C14, C16 incorrectly, C21, C22, C24 and C25) or with some pen and paper support (C27). Note that this was prior to the series of eight regular weekly intervention sessions. This is likely to have been the result of earlier work with this class stemming from Teacher C's interest in mental computation. Strangely there was almost just as much use of the traditional written algorithm (where the addition proceeds from the right) on the repeat task as there was on the initial task, with no apparent evidence of the number sense written method for addition modelled in the intervention sessions (where the addition proceeds from the left). One possible reason for this is that many of the students in Class C tended to be capable of genuinely doing this task mentally, and others (such as student C19) were already comfortable with the formal written addition algorithm and felt no reason to change.

Analysis of Item 8: 631 – 54

Table 5.8. **Detailed comparison of responses to Item 8 (631 – 54).**

Student		**M√**	**Mx**	**W√**	**Wx**	**?√**	**?x**	**C**	**exp**	***Observations***
C11	initial									blank
	repeat					**1**			**1**	**630 - 50, took 3**
C12	initial						1		1	4 – 1 = 3, 50 – 3 0= 20 then 600 - 23 but wrote 588
	repeat	**1**							**1**	**631 - 50 (581) then take 4**
C13	initial	1							1	took 50 from 600 (550) then added 31 (581) and 'minused' 4
	repeat	**1**							**1**	**600 - 50, 31 - 4, 550 + 27**
C14	initial	1							1	630 - 30 = 300 - 20 = 580 - 3
	repeat	**1**							**1**	**631 - 30 (601), take 20 (581) - 4 (jump back strategy)**
C16	initial		1						1	631 - 50 = 579, 579 - 54 = 525 (double counted the 50)
	repeat			**1**					**1**	**FWA**
C17	initial	1							1	‘I minused the 4 away from 1 and got 627 then I minused the 50 away from 627 to get 577'
	repeat							**1**		**misread the calculation as division**
C19	initial		1						1	‘630 - 50 =580 plus 5 which = 585'
	repeat		**1**						**1**	**630 - 50 = 580 - 5 (575)**

Table 5.8 continued

Student		**M√**	**M**x	**W√**	**W**x	**?√**	**?**x	**C**	**exp**	***Observations***
C21	initial		1						1	600 - 50 + 550, 550 - 55 = 495
	repeat	**1**							**1**	**631 - 50 (581), 581 - 4**
C22	initial		1						1	attempt at compensation: take 54 to 50 (and) 631 to 630 … 630 - 50 = 580 + 4 + 1 = 585
	repeat			**1**					**1**	**vertical but used negative numbers to record 'how much more to take': components 6 - 20 -3 interpreted as 600 - 23 (577)**
C23	initial		1						1	possible minor concentration error: "I took 31 from 54 than (sic) I took 24 from 600 (576)'
	repeat			**1**					**1**	**jump back on ENL: take 1 (630), take 50 (580) then - 4 (576) then add 1**
C24	initial		1						1	630 - 30 = 600, 600 - 20 = 580, 580 - 4 - 1 = 175
	repeat	**1**							**1**	**630 - 50 = 580 + 1 - 4**
C25	initial			1					1	FWA with tally marks suggesting decomposition
	repeat		**1**						**1**	**600 - 54 (546), then take 31 to get 515**
C27	initial				1				1	630 - 50 (580) then took 5 (575)
	repeat	**1**							**1**	**jump back: 631 - 50 (581) - 4 (577)**

Table 5.8 continued

Student		**M√**	**M**x	**W√**	**W**x	**?√**	**?**x	**C**	**exp**	***Observations***
C28	initial					1			1	confusing explanation: 'take 30 from 631 = 601 then take 5 = 96 then 20'
	repeat					**1**			**1**	**confusing explanation: took 31 from 631 (600) 'then just took the rest off …'**

This item showed a shift from 5 to 11 correct responses.

Evidence from the written responses to this item strongly suggests that some errors were highly likely to be the result of lapses in concentration. Student C17 misread the item as division, and student C12 wrote the answer to 600 – 23 as 588. In some responses it is not possible to discern whether an incorrect response was due to a concentration error or a conceptual error (such as both responses from student C19 which may indicate a lack of understanding of compensation within subtraction tasks).

Two students (C13 and C14) showed evidence of having used extended mental computation to correctly calculate the answer to this item on both tasks. Of particular interest was the number of students who initially answered this item incorrectly but on the repeat task answered it correctly with evidence of using number sense approaches:

- Students C21 and C24 initially had stumbled through a string of sometimes erroneous mental calculations leading to incorrect answers. On the repeat task they used efficient mental strategies to calculate the correct answer and represented these compactly and clearly on paper.
- Student C22 used the written approach modelled in the intervention sessions where a negative number is used to represent how much more to subtract in a component calculation, and C23 used the empty number line to represent efficient thinking.
- Student C27 initially used an informal written strategy incorrectly, but on the repeat task used the jump strategy to correctly calculate the answer mentally.

This appears to be evidence of a greater use of efficient number sense methods supporting a positive shift in performance on this item.

Analysis of Item 10: 1062 ÷ 9

Table 5.9. **Detailed comparison of responses to Item 10 (1062 ÷ 9).**

Student		**M√**	**M**x	**W√**	**W**x	**?√**	**?**x	**C**	**exp**	*Observations*
C11	initial							1		
	repeat							**1**		
C12	initial							1		
	repeat				**1**				**1**	**FWA with minor error**
C13	initial							1		
	repeat							**1**		**crossed out/no response**
C14	initial									crossed out/no response
	repeat				**1**				**1**	**attempted FWA**
C16	initial				1			1		used estimate: 9 x 12 = 108 then divided 1062 by 108 to get an estimate of 'around 10'
	repeat			**1**					**1**	**chunking algorithm**
C17	initial							1		
	repeat							**1**		
C19	initial									crossed out/no response
	repeat							**1**		
C21	initial							1		
	repeat			**1**					**1**	**informal progressive chunking: 900 ÷ 9 = 100, 162 ÷ 9 = 18, 100 + 18**

Table 5.9 continued

Student		M√	Mx	W√	Wx	?√	?x	C	exp	*Observations*
C22	initial							1		
	repeat				**1**			**1**		**attempted informal chunking: 9 x 100 (900), 9 x 9 (81), 9 x 1 (9), 9 x 1 (9) … 111**
C23	initial							1		
	repeat							**1**		
C24	initial							1		
	repeat							**1**		
C25	initial							1		
	repeat									**blank**
C27	initial							1		
	repeat				**1**			**1**		**attempted informal chunking with minor arithmetic error**
C28	initial							1		crossed out/no response
	repeat									**crossed out/no response**

This item showed an encouraging shift from 12 to 8 responses that indicated a need for a calculator to perform the calculation.

On both the initial and repeat tasks, eight students (C11, C13, C17, C19, C23, C24, C25 and C28) indicated that they required a calculator for this task, or left a blank response. However five of the remaining six students attempted some form of written calculation for this item on the repeat task after having made no attempt to do so on the initial task. Strategies evident in the repeat responses included the formal long division algorithm (students C12 and C14) and informal chunking (students C21, C22 and C27). Student C16 had attempted to estimate the answer to this item on the initial task, but correctly used the chunking algorithm on the repeat tasks. Student C21 was the only other student to correctly calculate the answer to this item on the repeat task.

The interesting difference between the initial and repeat responses to this item was the number of attempts at a written approach to the calculation on the repeat task. As described previously in Chapter Four it became apparent during the latter intervention sessions that the students in Class C had a very limited understanding of division as a concept. This is consistent with the results for this item from the initial task where 13 out of the 14 paired responses either circled *calculator* or were a non-attempt. Although less time than was intended was spent during the second half of the series of intervention sessions on written methods for long division tasks, the repeat responses to this item suggest that students had developed greater confidence and capacity to engage with division tasks of this type. The nature of the strategies they used on the repeat task also suggests that the chunking method for these tasks resonated with quite a few students. This was surprising given that there had been very limited explicit teaching of the chunking algorithm. There had been, however, substantial explicit teaching around using the empty rectangle as a visual organiser for reframing division as reverse multiplication, and then completing division tasks in steps by the strategic partitioning of the divisor. Interestingly there was no written evidence in the repeat task of students using the empty rectangle this way with the items that involved division.

Analysis of Item 11: 53 x 28

Table 5.10. **Detailed comparison of responses to Item 11 (53 x 28).**

Student		**M√**	**M**x	**W√**	**W**x	**?√**	**?**x	**C**	**exp**	***Observations***
C11	initial				1				1	wrote 50 x 20 = 2000 but multiplied in parts maintaining quantity value in component calculations
	repeat				**1**				**1**	**used ER correctly but wrote 20 x 3 as 600 to get 2024**
C12	initial		1						1	wrote 3 x 8 = 24 then 50 x 20 = 100 (giving) 124
	repeat				**1**				**1**	**appears to have done 3 x 8 (24) and 5 x 20 or 2 x 50 to get 100, then added to 124; used a cross notation that was shown for 'break up/make up' with addition/subtraction tasks**
C13	initial				1				1	multiplied in parts maintaining quantity value but missed 3 x 8
	repeat		**1**						**1**	**53 x 20 = 1300, 8 x 53 = 64, 1364 (wrong answers to components but correct split)**
C14	initial									crossed out/no response
	repeat			**1**					**1**	**ER**
C16	initial			1					1	FWA for long multiplication
	repeat			**1**					**1**	**ER**

Table 5.10 continued

Student		**M√**	**Mx**	**W√**	**Wx**	**?√**	**?x**	**C**	**exp**	***Observations***
C17	initial						1			in parts with quantity value but only did 3 x 8 and 50 x 20 (1024)
	repeat			**1**					**1**	**listed then correctly totalled the four component products maintaining quantity value**
C19	initial									crossed out/no response
	repeat		**1**						**1**	**50 x 20, 3 x 8 (1024)**
C21	initial						1		1	in parts with quantity value but only did 50 x 20 then 3 x 8 (1024)
	repeat			**1**					**1**	**20 x 53 (1060), 10 x 53 (530), 530 - 106 (424) 1060 + 424**
C22	initial				1				1	50 x 20 = 100 (sic), 3 x 8 = 24, 20 x 3 = 60, 50 x 8 = 400, 100 + 60 = 160, 400 + 24 = 424, 424 + 160 = 584
	repeat		**1**						**1**	**50 x 20 = 100 (sic) and 3 x 8 (124)**
C23	initial		1						1	only calculated 50 x 20 and 3 x 8 then added (1024)
	repeat			**1**					**1**	**correctly calculated 58 x 23 using ER**
C24	initial							1		
	repeat		**1**						**1**	**5 x 20 and add a zero, 3 x 8 (1024)**

Table 5.10 continued

Student		**M√**	**M**x	**W√**	**W**x	**?√**	**?**x	**C**	**exp**	***Observations***
C25	initial				1				1	partial FWA giving 144
	repeat		**1**						**1**	**3 x 8 (24) the 50 x 20 to get 100 (sic)**
C27	initial						1		1	50 x 20 = 1000 ad 3 x 8 = 24 (1024)
	repeat			**1**					**1**	**ER**
C28	initial							1		crossed out/no response
	repeat									**crossed out/no response**

This item showed a shift from one to six correct responses.

Eight of the 14 repeat responses to this item showed evidence of some sort of number sense strategy:

- four students (C14, C16, C23 and C27) used the empty rectangle (ER) correctly
- one student (C17) listed all four component products and added them correctly but without the visual organiser of the ER
- one student (C11) used the ER correctly but made an arithmetic error on one of the component products (20 x 3 = 600) which was carried through correctly
- two students (C13 and C21) used ways of reframing the calculation which strongly suggested that they each understood the operation of multiplication and how to interpret the numbers in the product.

Five of the remaining responses (C12, C19, C22, C24 and C25) indicated a fundamental conceptual error that had surfaced during the intervention session targeting long multiplication with two, two-digit numbers. This error involved processing only two of the four component products: the tens components were multiplied together (in this item 50 x 20), the ones components also multiplied together (here 3 x 8) with the sum of only these two products being offered as the overall result (here 1000 + 24 = 1024).

The empty rectangle (ER) had been modelled explicitly in the intervention sessions for mental computation and written methods. It was chosen for its likely value as a visual organiser that would help students identify and keep track of all the component products generated by a split and recombine strategy for multiplication tasks. When compared to the responses to the in-class activity from Session Seven where the explicit teaching about this application of the ER occurred (as described in Chapter Four) the number of attempts at using the ER and the number of correct responses were both much less than anticipated.

However, it is worth remembering that on the initial written task there was only one correct response to this item (student C16 used the formal written algorithm for long multiplication) and only three other responses (students C11, C13 and C22) which showed any evidence of some underlying correct strategy. This was despite the fact that it was clear from students' verbal interactions in Session Seven that many if not all of them had seen and practised written approaches to long multiplication in primary school. The relatively poor performance on this item on the initial written task may be the result of a lack of consolidation and ongoing use of the formal algorithm since that time. But if students are unable to perform these tasks correctly a couple of years after having been shown and (at the time) practiced in the formal algorithm, what value should that earlier effort be credited from a cost/benefit point of view?

The positive shift in performance on the repeat item was encouraging. It may be that this group needed more consolidation with using the ER in this way, and for exploring and understanding multiplicative relationships. The errors on the repeat task where only two of the four component products were captured in the calculation seem less likely to be the result of limited consolidation with the ER, and more to do with conceptual flaws related to understanding multiplicative thinking. It is still worth considering that the use of the empty rectangle to represent the long multiplication process might help both conceptual development and performance with these calculations (Siemon et al, 2005). Research by Seah and Booker (2005) highlighted just how poor students' understanding of and facility with using multiplicative thinking can be, and suggested that poor performance on calculation tasks was related to inadequate conceptual understanding.

Further research that offered more consolidation at the point of learning the empty rectangle as a visual organiser for multiplication tasks, and a longer gap than was available in this study between learning this method and having to do related tasks, would be valuable.

STUDENT INTERVIEWS

Two students from Class C were interviewed by the researcher with the aim of adding some further detail around their choice of response to selected items on the initial and repeat written tasks, and any observations or reflections they might have had about participation in the number sense intervention.

The interviews were conducted within a few days of the students completing the repeat written task. The researcher had compared the responses to the initial and repeat written tasks and selected students C17 and C21 as having differences in their paired responses that appeared potentially useful to explore. These differences included:

- a greater use of number sense methods in the repeat task
- variation in the amount and nature of the explanations the accompanied the responses to the items on the written tasks.

Each interview was conducted in a relatively private space in School C where there were few if any visual or auditory distractions for the students. The interviews were recorded on a digital audio recording device.

Tables 5.11 and 5.12 describe in detail selected aspects of the initial and repeat responses from both interviewed students to each item on the written task. The comments were derived from the comparison of written responses to the items, augmented at times with details furnished by the students during the one to one interviews.

Student C17

***Table 5.11.* Description of responses by student C17 to the initial and repeat written tasks**

	item	***initial***			***repeat***			***comments***
		√/x	*stated strategy*	*evidence of strategy*	√/x	*stated strategy*	*evidence of strategy*	
1	***68 + 44***	√	mental	split method: started with 8 + 4 (12) then 60 + 40 (100) then added 12 to 100 (112)	√	mental	split method: added 60 and 40 (100) then added 8 and 4 (12) then added 12 to 100 (112)	The initial response was processed in the order associated with formal written algorithms for addition (units then tens). The repeat response works from the left with the tens first then the ones. Both responses were correct but the repeat response is consistent with the emphasis in the classroom intervention on first working from the left to foreground the order of magnitude of the answer.

Table 5.11 continued

	item		***initial***			***repeat***		***comments***
		√/x	*stated strategy*	*evidence of strategy*	√/x	*stated strategy*	*evidence of strategy*	
2	***92 – 34***	√	mental	took 4 from 92 (88) then took 30 (58)	√	mental	took 4 from 2 (-2) then took 30 from 90 (60) then took 2 (58)	The repeat response appears to use the split method for subtraction with regrouping which was explicitly taught during the teaching intervention.
3	***264 – 99***	√	mental	rounded 99 to 100, took 100 from 264 (164) then added 1 (165)	√	mental	rounded 99 to 100, took 100 from 264 (164) then added 1 (165)	The correct use of a compensation strategy is identical on both tasks.
4	***700 + 283***	√	mental	split method: 200 + 700 (900) then 80 + 0 (80) then 3 + 0 (3)	√	mental	knew 700 + 200 (900) then added 83	The explanation for the repeat response appears to exhibit better number sense in that with only the number of hundreds changing, any processing of the tens and ones, as in the initial explanation, is redundant.

Table 5.11 continued

	item		***initial***			***repeat***		***comments***
		√/x	*stated strategy*	*evidence of strategy*	√/x	*stated strategy*	*evidence of strategy*	
5	***26 x 7***	√	blank (text suggests mental)	6 x 7 (42) then 7 x 20 (140) then added 42 and 140 (182)	x	mental	20 x 7 (140) then 6 x 7 incorrectly (41) then added (181)	The repeat response used the correct strategy with what appears a concentration error with the table fact 6 x 7. No apparent connection to the principle that any multiple of an even number must be even. The repeat response again differs from the initial response in that the tens are treated first then the ones.

Table 5.11 continued

	item	initial			repeat			comments
		√/x	*stated strategy*	*evidence of strategy*	√/x	*stated strategy*	*evidence of strategy*	
6	***92 ÷ 4***	NA	calculator	blank response	√	mental	estimated the answer to be more than 20 as 4 into 80 goes 20 times leaving 12; three 4s are 12; then added 20 and 3 (23)	When interviewed it was clear that this student understood division as 'how many lots or groups of' and the informal chunking strategy he used here. The initial estimation became an exact answer to a component division (80 ÷ 4) which made the remaining component division (12 ÷ 4) obvious. The correct aggregation of 20 and 3 as component responses to 'how many lots or groups of' is further evidence that the response to the repeat task was based on a full understanding of the relationships between the numbers involved and this interpretation of the operation of division.

Table 5.11 continued

	item	initial			repeat			comments
		√/x	*stated strategy*	*evidence of strategy*	√/x	*stated strategy*	*evidence of strategy*	
7	***256 + 68***	√	written	formal written algorithm: right justified vertical setting with carry marks	√	blank (text suggests mental or number sense written method)	added 6 and 8 (14), then added 50 ad 60 (110), then added 200 (310) then added 14 (324)	The initial response suggests this student was competent with the traditional vertical algorithm for addition. The repeat response differs from the initial response in that each of the component products maintains its full quantity value. The brevity of the initial response suggests that the explanation of the repeat response is for a mental calculation.

Table 5.11 continued

	item		*initial*			*repeat*		*comments*
		√/x	*stated strategy*	*evidence of strategy*	√/x	*stated strategy*	*evidence of strategy*	
8	***631 – 54***	√	mental	took 4 away from 1 (sic) (627) then took 50 away from 627 (577)	NA	calculator	blank on the repeat response interview response: take 50 from 630 (580) then took 4 from 1 (-3) then took 3 from 580	When interviewed, this student said he had misread the operation as division. When asked how he would have done the correct task he calculated the correct answer using the strategy taught as a number sense written method for subtraction. All component calculations were done mentally with the recording of his explanation appearing to also act as a way of keeping track of the component calculations. The pulling together of the component calculations to give the final answer was done mentally.

Table 5.11 continued

	item	initial			repeat			*comments*
		√/x	*stated strategy*	*evidence of strategy*	√/x	*stated strategy*	*evidence of strategy*	
9	***143 x 6***	√	written	formal written algorithm: right justified vertical setting with carry marks	√	mentally	multiplied 100 by 6 (600) then 40 x 6 (240) then 3 x 6 (18) then added 18 onto 840 (858)	The repeat response reflects the mental strategy for multiplication that was modelled during the intervention. It differs from the initial response in that each of the component products maintains its full quantity value.
10	***1062 ÷ 9***	NA	calculator	blank response	NA	calculator	blank	In the interview this student was questioned about his repeat responses to this task and to Item 6 (92 ÷ 4). When prompted to consider using the same informal chunking strategy he used successfully with the repeat Item 6 he was able to correctly complete this task.

Table 5.11 continued

	item	initial			repeat			comments
		√/x	*stated strategy*	*evidence of strategy*	√/x	*stated strategy*	*evidence of strategy*	
11	***53 x 28***	x	blank (text suggests mental)	multiplied 3 by 8 (24) then multiplied 50 by 20 (1000) then added to get 1024	√	written	50 x 20 (1000), 3 x 20 (60), 50 x 8 (400), 3 x 8 (24) then added 1060 and 424 (1484)	The repeat response identifies all of the component products which are then aggregated in two steps (apparently firstly by proximity) to give the correct total. The initial response perhaps suggests he inappropriately applied an additive strategy (tens with tens, ones with ones) to perform a multiplication task.

Discussion

The interview with student C17 was particularly enlightening around the impact of the number sense intervention on his understanding of and performance with division tasks.

During the interview with student C17 the researcher asked about the responses to Item 6 (92 ÷ 4). The specific question was: *what changed so that you could answer this question the second time?* Student C17 said that he had previously used “breaking up” but had never really understood it. When asked what he meant by “breaking up” he referred to working with the tens in the number. When asked if this change in his understanding had been related to the number sense intervention student C17 was positive and definite that it had.

The researcher then asked about the responses to Item 10 (1062 ÷ 9) which had been marked by student C17 as a calculator item on both tasks. His responses to Item 6 and his explanations strongly suggested that he clearly understood the strategy of separating a number into manageable components that had the divisor as a factor, and how this strategy could be used to complete the initial calculation. He also had been part of the lesson where the chunking algorithm for division with larger numbers was explicitly taught. Student C17 said that he did try to do the task in his head, but he did not really understand what was going on (with the chunking algorithm) when the teacher was showing what to do in class. He mentioned that at the time he did not fully appreciate what questions like *would there be more or less than 10 (or 100) lots of (the divisor) in (part of the dividend)?* were aiming to achieve.

The researcher suggested that the same strategy which student C17 used successfully on the repeat response to Item 6 could be used here with Item 10. The researcher then invited student C17 to work through Item 10 using that strategy and provided a series of prompts to this effect:

- Would there be more or less than 100 nines in 1062? (*more*) How do you know? (*100 nines is 900*)
- If we took them (the 100 nines) out, what would be left? (*162*)
- How do you know? (*I take the 900 away from 1062*)

- Would there be more or less than 10 nines in 162? (*more …10 nines is 90 … that leaves 72*)

Student C17 completed the calculation this way and said that it made sense to him now.

Although this example is limited in its focus it encapsulates the rationale behind this study and offers modest support for the belief that number sense written methods warrant consideration in a balanced approach to computation in a calculator age. Student C17 initially showed no facility for mental or written calculation with the two division items on the written task. The number sense intervention appeared to contribute to genuine conceptual growth and improved performance on a task that could reasonably be expected to be calculated mentally (Item 6: 92 ÷ 4). The interview suggested that it would be possible to build a similar level of performance using conceptually aligned number sense methods such as the chunking algorithm to deal with division tasks deemed to be just outside the reasonable scope of mental computation (such as Item 10: 1062 ÷ 9), if more time were available for initial explicit teaching and adequate consolidation.

This student was also asked about his repeat response to Item 8 (631 – 54). Initially he had done the calculation correctly, but in his second response he had indicated the need for a calculator. Student C17 stated that he misread the repeat item as division. When invited by the researcher to attempt the item correctly, he used the integer approach to subtraction that had been modelled in one of the intervention sessions. He said that representing (in this case) *1 take 4* as -3, using the interpretation "3 more to take", made sense. These comments, together with the greater number of repeat responses that exhibit aspects of number sense (such as Item 7 where the explanation on the repeat task uses full quantity value in the component sums) suggest that aligning written calculation with number sense is at least workable and may lead to improved overall student performance with calculations of this level.

Student C21

Table 5.12. **Description of responses by student C21 to the initial and repeat written tasks**

	item	***initial***			***repeat***			***comments***
		√/x	*stated strategy*	*evidence of strategy*	√/x	*stated strategy*	*evidence of strategy*	
1	***68 + 44***	x	mental	incorrectly read the operation as subtraction but calculated the correct difference: 8 – 4 (4) then 60 – 40 (20) then 20 + 4 (24)	√	mental ‘written’ circled but scribbled over	60 + 40 (100) then 8 + 4 (12) then 10 + 12 (112)	Although different calculations, the initial response worked with the ones first (8 – 4) then the tens (60 – 40) while the repeat response worked first with the tens (60 + 40) then the ones (8 + 4)
2	***92 – 34***	√	mental	wrote 92 = (sic) 4 = 88 and 88 = (sic) 30 = 58	√	mental	90 – 30 (60) then 2 – 2 (-2) then 60 – 2 (58)	The repeat response uses the split strategy for subtraction involving regrouping which was shown during the intervention sessions.

Table 5.12 continued

	item		***initial***			***repeat***		***comments***
		√/x	*stated strategy*	*evidence of strategy*	√/x	*stated strategy*	*evidence of strategy*	
3	***264 – 99***	√	mental	wrote 264 = 100 = 164 + 1 = 165 (incorrectly circled 265 as his final answer distinctly to the left of his other written record which shows the correct answer of 165)	√	mental	264 – 100 (164) then 164 + 1 (165)	Strategy use in both responses seems identical.
4	***700 + 283***	x	mentally	incorrectly read the operation as subtraction but calculated the correct difference: wrote 700 – 200 = 500 then 500 – 83 = 417	√	mental	700 + 200 (900) then 900 + 83 (983)	
5	***26 x 7***	√	mentally 'calculator' had been circled but then scribbled over	7 x 20 (140) then 7 x 6 (42) then 140 + 42 (182)	√	mental	5 x 26 (130) then 2 x 26 (52) then 130 + 52 (182)	In the initial response 26 was split as 20 and 6. In the repeat response the 7 was split as 5 and 2. When interviewed this student said he had calculated 5 x 26 as 'half (of 26) then add zero'.

Table 5.12 continued

	item		***initial***			***repeat***		***comments***
		√/x	*stated strategy*	*evidence of strategy*	√/x	*stated strategy*	*evidence of strategy*	
6	***92 ÷ 4***	NA	calculator	blank	√	mentally	4 ÷ 80 (20) then 4 ÷ 12 (3) then 20 + 3 (23)	Although the numbers in the division calculations on the repeat task were reversed (4 ÷ 80 instead of 80 ÷ 4) when interview this student showed that that he understood what he was working towards
7	***256 + 68***	√	mentally	200 + 68 (268) then 268 + 50 (318) and 318 + 6 (324)	√	mentally	250 + 60 (310) then 6 + 8 (14) then 310 + 14 (324)	The explanation of the repeat task shows a more efficient mental strategy.
8	***631 – 54***	x	mentally	600 – 50 (550) then 550 – 55 (495)	√	mentally	631 – 50 (581) then 581 – 4 (577)	
9	***143 x 6***	√	mentally ‘calculator’ had been circled but then scribbled over	100 x 6 (600) then 40 x 6 (240) then 3 x 6 (18) the correct total of 858 is circled and appears to the right of the three component calculations	√	written	143 x 3 (429) then 429 x 2 (858) no written evidence of how this student calculated 143 x 3	When interviewed this student was asked how to work out 143 times 7 and 143 times 9: x7 was x3, double then plus one x9 was x3 three times then add these, or x10 minus 143

Table 5.12 continued

item		***initial***			***repeat***			***comments***
		√/x	*stated strategy*	*evidence of strategy*	√/x	*stated strategy*	*evidence of strategy*	
10	***1062 ÷ 9***	NA	calculator	blank	√	written	900 ÷ 9 (100) then 162 ÷ 9 (18) then 100 + 18 (118)	When interviewed this student was asked how he calculated 162 ÷ 9. He said he split 162 into 90 and 72 and divided each by 9 mentally
11	***53 x 28***	x	'calculator' had been circled but then scribbled over	50 x 20 (1000) then 3 x 8 (24) 1024 circled and to the right	√	written	20 x 53 (1060) then 10 x 53 (530) then 530 – 106 (424) then 1060 + 424 (1484)	The initial response showed an error shared by several students. However the repeat response exploits an equivalence of 28 as 20 + 10 – 2.

Discussion

Student C21 was less descriptive in his comments than student C17. Similarly, student C21 did not use any words in his descriptions or explanations of his calculation strategies on his written tasks. His notation in the repeat response was more accurate and mathematically correct. He showed evidence of competence with basic addition and multiplication facts but also a variety of well-developed number sense strategies.

When asked to explain his thinking around his repeat response to Item 6 (92 ÷ 4) it was evident that he now understood exactly what he was doing in this calculation ("how many fours are in 92") even though he had reversed the order of the numbers in his explanation. When asked what had helped him do this he answered "dunno … (Teacher C) showing us". Similarly the initial response to Item 10 (1062 ÷ 9) was blank, with the repeat response showed thinking perfectly consistent with the chunking algorithm. However, the written response was not set out the way it had been shown during the number sense intervention. Getting detail from this student was difficult, and the researcher offered too many leading questions. Despite this, student C21 did make it clear that he understood what he was doing in answering this item, what each component calculation meant, and how the component calculations related to the answer to the overall calculation. For example when asked how he calculated 162 ÷ 9 he talked through the correct partitioning of 162 into 90 and 72 and competently explained how this led to a quotient of 18.

The repeat responses from student C21 to the items that involved multiplication warrant further discussion. The initial responses to the first two of the multiplication items involved what could reasonably be deemed as conventional splits:

- Item 5: 26 x 7 7 x 20 (140) then 7 x 6 (42) then 140 + 42 (182)
- Item 9: 143 x 6 100 x 6 (600) then 40 x 6 (240) then 3 x 6 (18).

The larger (and in both of these cases, the first) of the two numbers was split into manageable parts, each of which was multiplied by the second number. Such partitioning was explicitly taught during the intervention sessions that focused on multiplication and division. The initial responses from student C21 showed that he was confident and competent with this approach prior to the intervention sessions, at least when multiplying by a single digit number. His initial response to the third multiplication item (53 x 28) incorrectly split the product into only 50 x 20 (1000) and 3 x 8 (24).

The three repeat responses used strategies that were different from the initial responses, and also different from each other. On the repeat items student C21 used unconventional manipulations of the *second number* in each product, in each case exploiting connections within and between the given numbers which demonstrated an enviable level of number sense:

- 26 x 7 was calculated mentally as "half of (26) then add a zero" plus double 26
- 143 x 6 was calculated mentally as double 143 x 3
- 53 x 28 was calculated with pen and paper support as 20 x 53, then another 10 x 53 minus 106 (double 53).

When questioned by the researcher about the strategies used with the repeat responses to these items student C21 clearly and correctly explained what he did and why. But when asked why his approaches to the repeat tasks differed from his initial responses he shrugged his shoulders and said "dunno". What was strongly apparent from his written and verbal explanations was a sound conceptual platform around the operations of multiplication and division, and considerable fluency with using part/whole relationships and place value.

SUMMARY

The comparison of the paired responses from selected students in Class C to the items on the written task showed:

- a reduction in the number of items that were deemed by students to require a calculator (from 24 to 14)
- an increase in the number of items that had an adequate explanatory component (from 112 to 129)
- an increase in the number of correct responses calculated mentally (from 67 to 78)
- an increase in the number of correct responses calculated with written support (from 14 to 26)

- an increase in the overall number of correct responses (from 86 to 108).

It would be unreasonable to suggest that the variations between the initial and repeat responses described above were solely due to having a number sense orientation, as it could be argued that a sustained and regular focus on number using more traditional emphases such as formal written algorithms might achieve the same result. However the emphasis on building number sense through mental computation during the regular weekly intervention, and the exploration of number sense written methods, certainly *did not hinder students' performance* on the repeat written task. At the very least this would suggest that there is no reason not to reconsider the traditional place and expression of written calculation in the mathematics curriculum in the middle years of schooling.

Chapter Six describes selected aspects of the implementation of this study, and its findings and conclusions.

CHAPTER SIX: CONCLUSIONS, LIMITATIONS AND IMPLICATIONS FOR FURTHER RESEARCH

SUMMARY OF THE STUDY

Three teachers and their respective Year 8 classes were included in the initial stages of this project during term two of the 2008 school year. Students in Year 8 are typically 12 or 13 years of age at the beginning of the school year. However, the major focus of this study was Teacher C and her students, as this case provided substantially more robust data for further analysis.

The study used a mixed methods approach, emphasising case study methodology. The rationale for doing this related to the importance for addressing the research questions of engaging with the participants' understandings of and attitudes towards what they knew about, and could do within, this part of the mathematics curriculum. This emphasis was however tethered to a quantitative measure of the more demonstrable issue of performance on tasks.

Students initially were given a written task made up of 11 calculations of varying complexity. The students were asked to state how they could do the calculation (mentally if possible, using pen and paper if unable to calculate mentally, or a calculator if necessary), provide the correct answer to the calculation where possible, and describe the strategy they used to calculate their answers.

The written task preceded a series of eight regular weekly classroom intervention sessions, each between 30 and 50 minutes long. These sessions focused on the explicit teaching of strategies for mental computation and number sense methods for written calculation. Written responses from students to weekly in-class activities which related to the focus of each session were collected by the researcher, who also observed every intervention session and recorded aspects of the behaviour of, and interactions between, the participants in the study. Aspects of the planning and review sessions involving Teacher C and the researcher were also documented.

The same written task was given to the students at the end of the series of weekly intervention sessions. Students' performance on each item and the evidence of their strategy use were compared. Two students were interviewed by the researcher, with an audio record made of both of the interviews.

The sources of data that have been the primary focus of further analysis are:

- paired responses to the initial and repeat written tasks from students in Class C
- interview responses from two students from Class C
- summaries of the researcher's observations of the eight classroom intervention sessions with Class C with reference to the in-class tasks that accompanied these sessions and related discussions between the researcher and Teacher C when planning and debriefing these sessions.

ISSUES RELATED TO COLLECTING THE DATA

Mental computation assessment

During the preliminary planning discussions all teachers stated that they already had a reasonable sense of where to start the planned interventions with their classes. Therefore it was decided, for different reasons in the case of each class, not to administer the initial or repeat assessments of mental computation performance.

Teacher A was familiar with the instrument for assessing mental computation performance and had some of this data on his class from School A's use of the instrument for its own purposes. This was also the case with Teacher C as the mental computation data is pivotal to the functioning of the Learning Assistance (LA) program in her school. Teacher B felt that the assessment process would be disturbing to her students and that she already had enough of an idea of their capacities to plan appropriate activities for the first intervention session.

Within the first two weeks of the research period it became clear to the researcher that the student written task in particular, and the weekly student work samples and lesson observations, would generate the data required to investigate the research question in this context.

Student written task

There were several issues regarding the structure of, and students' engagement with, this instrument.

- Classes varied considerably in their familiarity with mental calculation, and in their capacity to communicate strategies used for calculation. This appeared to have an impact on both the amount and quality of explanation or description they provided as a written record of their thinking.
- The degree of difficulty of the calculations seemed to alienate some students from attempting a response to some or almost all of the 11 items.
- Several students in Class B received help from Special Teaching Assistants (STAs) who were assigned to them to explain the items or to transcribe the student's verbal responses.
- The length of the task (11 items) seemed too long for some students. Some students appeared to get tired or bored and stopped trying. There was no capacity within the instrument to determine if and when students were capable of responding to an item but had simply left it out as a non-attempt.
- Some students covertly used a calculator to work out some answers. Some students plagiarised other students' work.
- Responses to some items were only partial, others were ambiguous. For example some responses had an answer and explanation, but no word circled. Others had an answer and no explanation. Some had two words circled.
- At times there was an apparent clash between a student's stated strategy (indicated by circling one of the words *mentally*, *written* or *calculator* which were above each item) and the written response by the student to an item. For example some students circled *mentally* as the stated strategy for calculating 143 x 6. However it is highly unlikely that they could have done this if they had not written out the answers to the component products 100 x 6, 40 x 6 and 3 x 6 as part of the explanation of their calculation approach. Such a response would have been more appropriately designated as *written*, since although the component calculations may well have been done mentally, the use of pen

and paper was necessary to hold the components together to calculate the final answer.

Some of these things, such as the level of difficulty of the calculations in the written task, and partial or ambiguous responses, were visible in the responses from all classes and, to varying degree, eroded the data from the student written task. Class C was the group that was least affected by these factors. Things such as the influence of adult help, plagiarism or the use of calculators to find answers were not observed at all in Class C. This class also had the most developed and established culture of valuing and explaining the strategies that individuals used to solve mathematical problems in general, and calculations with numbers in particular. The work samples generated each week by the students in Class C became a significant complementary source of data; this data has been explored in depth in Chapter Four.

Student interviews

It was anticipated that students from each of the three classes would be interviewed. The intended purpose of these interviews was to draw out from selected students further detail about their approaches to calculation and how they felt about the work they had done on number sense methods for written calculation.

Due to time and availability constraints, only three students were interviewed. Two were from Class C and one was from Class A. They were selected because their responses to the initial and repeat written tasks showed some evidence of a shift in performance (number of tasks answered correctly) as well as a shift towards the use of number sense written methods. All three students were male.

Other sources of data

The researcher took notes during all classroom sessions that were observed, as well as during and immediately after all discussions with teachers. All teachers in this study collected samples of student responses to tasks that focused on the material targeted in the number sense intervention. These tasks came out of discussions between the teachers and researcher as the practicalities of the classroom interventions were being decided. The work samples contributed to effective planning of ensuing sessions and gave many insights into aspects of students' strategy development.

CHANGES IN FOCUS OF ANALYSIS

Several factors had driven a change in the number and type of data collecting instruments and approaches from what originally had been proposed for this study. These factors included:

- the mental computation assessment was not used with all classes
- concerns about the quality of the data from the student written tasks from Classes A and B
- the large collection of in-class activities and notes on lesson observations from Class C
- practical limitations around exploring connections within and across the various sources of data.

ADDRESSING THE RESEARCH QUESTIONS

In Chapter Two, it was argued that using number sense written methods within a balanced approach to computation would contribute more effectively to building and strengthening students' number sense, and that this was likely to produce positive learning and performance outcomes for more students. This led to the investigation in this study of the following research question:

> What are some of the effects on teachers and students within a junior high school setting, of aligning written calculation with a strategies approach to teaching and using mental computation?

Of specific interest, this study investigated any effects related to:

- conceptual understanding of number and operations with numbers
- facility with performing calculations.

Effects on the teacher

There was evidence in this study that aligning written calculation with a strategies approach to teaching and using mental computation had substantial positive effects on Teacher C. These effects included the following:

- By the end of the data collection phase of this study, Teacher C had increased her capacity for planning explicit teaching across several aspects of the core curriculum related to building students' number sense. She had increased her repertoire of strategies for mental and written calculation, and these were connected and mutually reinforcing. Using these strategies had given her a natural context for addressing other aspects of number sense, including the use of approximations to mentally calculate estimates to exact calculations that were carried out using pen and paper or electronic calculators. All of this work was in keeping with her expectation in teaching mathematics, that she and her students should understand and be able to communicate what was happening in what they were doing.
- The depth of treatment afforded by this coherent approach, coupled with the emphasis on making sense of numbers and the processes for calculating with them, built her confidence and competence with teaching this aspect of the curriculum. During the weekly intervention sessions, as well as during planning and review meetings with the researcher, Teacher C made references repeatedly to the fact that she now understood so much more about numbers, and was much more confident working with them. Previously she had never really felt comfortable with them. Some of this was the result of having to confront her lack of deep understanding of the operation of division. This confrontation was directly related to the commitment Teacher C had made to building students' number sense, and the imperative that she believed this placed on her to do likewise.

Teacher C developed better conceptual understanding of, and facility with using, the four operations with whole numbers. Her existing interest in mental computation was augmented by a coherent and related set of pedagogical strategies, practices and resources for working with students in the middle years of schooling around developing a balanced approach to computation.

Effects on students

There was evidence in this study that aligning written calculation with a strategies approach to teaching and using mental computation also had substantial positive effects on students in Class C. These effects included the following:

- Students improved their performance with mental and written calculation with tasks within the range of complexity targeted in this study
- Students were more confident and able to attempt these calculations mentally or with written strategies. This was suggested by the reduced need for using a calculator to complete the calculations.
- The focus on developing students' number sense meant that a variety of misconceptions with foundational aspects of number were exposed and therefore able to be addressed. Additionally this focus prompted considerable discussion of alternative but valid perspectives on the targeted content that added richness and depth to this part of the curriculum.

For the students in Class C, aligning mental and written computation added to their development of conceptual understanding of some of the big ideas in number. This was especially evident with the operations of multiplication and division. The group also showed improved performance with mental and written calculation, and a reduced need for using a calculator to do tasks in this range.

FORMING CONCLUSIONS: THE LITERATURE REVISITED

The process of distilling the various sources of data into a set of general conclusions suggested three main themes for which the data from this study provided demonstrable support:

- The importance of building conceptual understanding, illustrated most clearly in this study around understanding the operation of division.

- The value of representing mathematical concepts in multiple and meaningful ways; typically (but not limited to) words, objects or pictures, and mathematical symbols.
- The value of explicit teaching of mental computation for building number sense and improving performance with mental and written calculation.

The identification of these themes prompted the researcher to return to the literature, but this time looking particularly through these lenses. Several additions to the literature described in Chapter Two are worth mentioning in this context, in that they provide a backdrop to considering the trustworthiness of the conclusions drawn by the researcher.

The three conclusions stated above are interrelated: a principle (building conceptual understanding); a useful strategy for achieving this (multiple representation); and a workable action (explicit teaching around mental computation). Despite the emphasis implied in their respective titles, the following discussions contribute in different ways to engaging holistically with the set of conclusions from this study.

Seah and Booker (2005) reported on a study of 143 students in their first year of high school (Year 8) in a socially disadvantaged area of Brisbane. The study focused on the students' knowledge of number concepts (such as place value and regrouping) and multiplication, and applications of these to solving problems. Overall the study found that most of the students' limited discernible knowledge related to performing procedural tasks, and there was considerable evidence that students had very little conceptual understanding of the targeted number concepts or the operation of multiplication:

> A lack of conceptual understanding of numeration and other previously learned mathematical ideas, together with little understanding of the interconnections among them, are the source of students' difficulties with multiplication. This can be seen by the way many students mixed addition and subtraction strategies when completing the multiplication algorithm and their inability to use place value knowledge to rename the numbers that resulted from multiplication of the partial products involved in the algorithm.
> Word problems posed the greatest difficulties, with a majority of students not able to use any appropriate strategy to come to terms with the problems let alone provide an appropriate answer. (p.95)

The authors had initially stated their beliefs around the importance of a sense-making approach to teaching and learning mathematics. This flavour was clearly evident in their description of the implications of their study for teaching approaches which were limited to recalling facts and applying routine procedures:

> The data collected and analysed in this study shows the consequences of such an approach. A lack of understanding of the conceptual knowledge underpinning mathematics and the interconnectedness among this knowledge causes major difficulties at the initial level of applying elementary ideas to straightforward problems and also fails to provide a basis for further learning. (p.96)

Siemon, Breed and Virgona (2005) drew attention to the importance of multiplicative thinking, and cited the transition from additive thinking to a capacity for multiplicative thinking as “one of the major barriers to learning mathematics in the middle years” (p.1). They gave several examples of responses to tasks where additive thinking had been applied to solving a problem which was fundamentally multiplicative in its structure. Not only could this result in relatively inefficient solution strategies, but more importantly it may imply that a student has not developed the capacity for multiplicative thinking.

They suggested a range of strategies and pedagogical approaches, from early primary school through to the middle years of schooling, to support this transition from additive to multiplicative thinking. These included developing the *groups of* concept by counting collections by a fixed number (such as in twos, threes, fives or tens) and exploring systematic *sharing* (such as, how many ways can 18 lollies by shared equally?). Developing appropriate language that helps students engage with the concepts is important, as are concrete and visual representations of the concepts. In this regard they noted that in Victoria the emphasis in practice in primary schools is with the quotative model of division (that is, how many groups of …?) rather than with the suggested approach in system curriculum documents of emphasising the sharing (partitive) model.

Array models using discrete materials such as counters show totals structured as repeated equal groups, and readily link to area models such as the empty rectangle (ER). The ER provides an abstracted symbolic representation of how multiplication distributes over addition, and can be used conveniently with some larger whole numbers and decimals to model the relationship between factors and products. Other aspects of multiplicative thinking described were the concepts of *for each* (the Cartesian product) and the notion of division as finding a missing factor. Interestingly it was suggested that limiting the teaching and interpretation of multiplication and division to just a *groups of* model would leave students under-resourced to deal with the many incarnations of multiplicative thinking elsewhere in the curriculum.

By contrast, when describing some of the issues in teaching multiplication and division, Anghileri (1999) acknowledged the interpretation of division as sharing, but warned of the limitations of this model for understanding number patterns involving multiples, and their role in developing facility with the more powerful (in her view) model of division as using grouping strategies:

> Procedures for grouping and sharing will help identify the concepts of multiplication and division but these will remain impotent unless the related number patterns are an early focus. In later schooling pupils who can use number facts and relationships will have a considerable advantage over those who continue to 'share' and 'group' with objects or with visual imagery. (p.185)
>
> When pupils limit their understanding to a sharing model it can hamper progress … The notion of division as 'sharing' is difficult to reconcile with a grouping procedure, but 'grouping' and the use of multiplication facts will provide the key to success in most division calculations as this way of 'breaking down' numbers can be related to the 'building up' of multiplication. (p.186)

Her later work with the chunking algorithm (Anghileri, 2001; 2006) suggests that the comments quoted here about the advantages, in that context, of grouping strategies over a sharing model are likely to be correct.

Discussion about which conceptualisation of the operation of division (partitive or quotative) is more relevant than the other is valuable in that it perhaps suggests that a *sound understanding of both models* is important for developing the full concept of division and its connections to the other three main arithmetic operations. Expedient tricks such as treating division as reverse multiplication are obviously useful in practice; but limiting students' conceptual development by only engaging with division at this arguably shallow level means they may be hindered from grasping the fuller concept adequately. This was certainly evident in this study, particularly in the lack of understanding of the similarities and differences between the partitive and quotative models for division that was initially shown by Teacher C, and frequently visible in students' written work and verbal explanations.

CONCLUSIONS

Building conceptual understanding: division as an operation

Weaving mental computation and written calculation together in this study helped Teacher C and her students to develop a much better understanding of multiplication and division as concepts. This conceptual development supported improved performance by students on calculations that involved multiplication and division.

Neither the majority of the students in Class C, nor Teacher C herself, appeared initially to understand the concept of division as an operation. This fact was exposed in Sessions Four and Five of the intervention where students struggled with performing division tasks that went beyond reverse multiplication facts, and where Teacher C, on her own admission, muddled her explanation, cross-modelling the quotative and partitive models for division.

Prior to the intervention sessions that were part of this study, students' work with division had almost certainly been limited to reverse table facts and, for some at least, use of the formal written algorithm. By describing division as the reverse of multiplication, students may (quite sensibly) use known multiplication facts (such as 6 x 4) to calculate the answer to related division tasks (such as 24 ÷ 6). They could also perform the division algorithm with pen and paper, without needing to understand division as an operation. The expectation that one should be able to explain one's strategy for doing a calculation accompanied the strategies approach to teaching and using mental computation, and it was this expectation that brought this particular conceptual weakness into the open. Whatever success students had previously achieved with interpreting division as reverse multiplication may well have masked their lack of understanding of division as an operation.

If developing sound conceptual understanding is as important to learning and using mathematics as has been suggested in Chapter Two, then leaving the concept of division as just the reverse of multiplication may be a cause for concern inasmuch as it can camouflage a failure to understand division as a concept in its own right. One could speculate on the impact of this conceptual deficit on learning other topics in the school mathematics curriculum, such as developing sound fraction concepts, or procedural facility with algebraic expressions that involve division.

From the point of view of authenticity of assessment it is important to separate skill with the recall of reverse-multiplication facts from genuine understandings and applications of the operation of division. This is particularly significant now, as most involved and complex calculations outside school are processed electronically (Brinkworth, 1985). Here the real work is in understanding the relationships between the components of the calculation and setting the correct sequence of operations to represent those relationships, so that the electronic tool will process the intended calculation.

The quotative model of division seems strongly linked to understanding and using the chunking algorithm, and applications to interpreting calculations with benchmark fractions (such as 3 ÷ ½) as in the support materials for teachers developed by McIntosh and Dole (2004). However the partitive model of division appears common in initial explanations of division as equal shares, and in establishing foundational fraction concepts, such as marking a given length into four equal lengths, with three of those end to end representing ¾ of the original length. It is hardly surprising that what has been labelled in this study as cross-modelling occurs, and that working simultaneously across both models promotes confusion.

The concept of division as equal shares (the partitive model) warrants an adequate level of exposure for several reasons:

- Sharing tasks can reasonably occur in day to day living, such as sharing $70 in tips between five wait staff at the end of a shift.
- It has been described in national and state and territory curriculum documents as a suggested teaching strategy in the primary years (ACT DET, 2007; AEC, 1990) and has therefore almost certainly been seen (if not understood) by most teachers and high school students.
- It underpins aspects of a number of contemporary approaches to establishing foundational fraction concepts, and teaching mental computation with fractions (as in module five in McIntosh & Dole, 2004).

One further reason, related more to the teaching than the learning of mathematics, stands out as somewhat different to those listed above. When debriefing Teacher C after her cross-modelling episode in Session Five, it seemed necessary to describe clearly both division as equal shares, and division as groups of, to unpack her misconceptions and to help her develop a robust concept of division as an operation. Teacher C had earlier accepted the importance of understanding subtraction as take away and as difference, and wanted a similar level of understanding with the equivalent situation with division. To not go into the similarities and differences between the partitive and quotative models for the operation of division would have been to avoid supporting Teacher C's development of deep understanding of that operation, as suggested as important in the framework for number sense used by McIntosh et al. (1997). It would also have avoided an opportunity to build the sort of connections within Teacher C's content-pedagogical knowledge that Askew (1999) described as the mark of effective teachers of numeracy.

Multiple representations: words, pictures, symbols

In this study there was a commitment to linking words, objects or pictures, and mathematical symbols, on the belief that this would help students make meaning around the mathematics they were using. There were several examples from this study of approaches that appeared to be effective in doing this in the context of improving students' facility with mental computation. They included:

- using the empty number line (ENL) to represent subtraction tasks as both *take away* and as *difference*
- using V notation to visually illustrate the split strategy with addition or subtraction, intuitive notions of negative numbers, and catch phrases such as 'break up/make up' to represent a split strategy for subtraction tasks with two two-digit numbers which required regrouping
- using the empty rectangle (ER) to model a split strategy for multiplication of a two-digit number by a one-digit number
- using the ER to model division tasks as finding the missing factor, and as a way of organising sensible splits of the dividend to facilitate efficient applications of the chunking strategy.

In terms of this study, there were at least two main purposes for improving students' mental computation. Firstly, that mental computation is useful in practical terms and for building number sense. Secondly, sound mental computation is integral to using the number sense methods for written calculation that were explored. The ER worked very well in representing calculations that were done mentally, and as a visual organiser for keeping track of the component calculations when using pen and paper to process long multiplication tasks. The difficulties that students in this stage of schooling have in moving from additive to multiplicative thinking and the significance of sound conceptual development in achieving improved performance with calculation, both suggest that the ER in particular should have a higher profile in the mathematics classroom.

Building robust conceptual understanding of division is but one instance of a commitment to valuing conceptual understanding within a sense-making approach to teaching, learning and using mathematics. This is a commitment shared by many (McIntosh et al., 1997; NCTM, 2000; Seah & Booker, 2005; Trafton, 2004). However, within the typical period allocated to K – 12 schooling the nature of the concepts may change. This is evident when one compares simple arithmetic tasks with division (such as 24 ÷ 3), which can be modelled physically or visually, to division with algebraic fractions, which very quickly becomes something that defies any similar instructive physical or visual representation. This shift in the nature of the concepts to be understood in the school mathematics curriculum is pertinent to the discussions by Star (2005) around reconceptualising our view of procedural knowledge, and Pierce and Stacey (2001) in their explorations of a framework for algebraic insight.

That we cannot create physical or visual representations which complement verbal and symbolic representations for *all* topics in the school mathematics curriculum (such as division with algebraic fractions) is no reason not to do so when we can. The lack of a robust understanding of division that was identified in the participants in this study was indicative of a concept that *can* be modelled both physically and visually. It was precisely this modelling, augmented by descriptive language and the appropriate mathematical symbols that helped Teacher C improve her understanding of this aspect of the curriculum. This modelling contributed to students' improved performance with calculation, as identified in the improved performance on division tasks overall, and confirmed in the student interviews. Other work by the researcher (separate to this study) into the use of physical and visual models to support the development of initial understandings of symbolic algebra with Year 7 students, suggests that more research into the use of these models would be valuable.

Mental and written calculation: student performance

Any reform to the way calculation is taught and used in school must consider the efficacy of the proposed changes on students' ability to perform calculations correctly and get the right answers. Unless discussion of the importance of conceptual understanding also relates to improvements in learning and performance it is more philosophical than useful. However, it is important in this regard to articulate explicitly the performance standards against which any judgments are made, and to frame these standards sensibly within the social and developmental contexts in which they are to be applied.

In this study the repeated written task gave some indication of students' performance with a range of calculations prior to and immediately after the eight week series of intervention sessions. The comparison of the paired responses to the 11 items on the written task showed:

- a reduction in the number of items that were deemed by students to required a calculator (from 24 to 14)
- an increase in the number of items that had an adequate explanatory component (from 112 to 129)
- an increase in the number of correct responses calculated mentally (from 67 to 78)
- an increase in the number of correct responses calculated with written support (from 14 to 26)
- an increase in the total number of correct responses (from 86 to 108).

The 26% increase in the number of correct answers suggests that the emphasis on mental computation, in its own right and as a basis for using alternate number sense written methods of calculation, enhanced students' facility with performing calculations correctly. The concurrent increases in the number of calculations done mentally, and with written support of any kind, imply students' number sense and capacity to do calculations in the range targeted by this study improved.

This conclusion is also supported by the reduction in the number of repeat responses which nominated that a calculator was needed to perform a calculation. Of the total of 38 responses that required the use of a calculator over both administrations of the written task, only one response was for an additive task. The remainder all related to the items that involved multiplication and division. This is hardly surprising, and is in keeping with the difficulties that students have with growing into multiplicative thinking (Siemon et al., 2005).

The reduction in nominated calculator use here is commensurate with an increased engagement with mental or written methods for these calculations. This suggests that valuing and using mental computation and number sense written methods does lead to less reliance on electronic calculation, particularly for those calculations for which it is reasonable to expect students not to need a calculator to perform. The complementary data from the lesson observations, described in detail in Chapter Four, suggest that a focus on improving number sense can improve performance on this type of task, and improve students' understandings of foundational number concepts. This concurrent benefit, of performance on a limited but reasonable range of calculations, and richer conceptual development, is the main reason to consider reframing the treatment of written calculation with whole numbers around the use of the number sense written methods described in this study.

LIMITATIONS OF THIS STUDY

Number of participants and sources of data

Initially a study comparing three classes in different schools was planned. Several factors, including a large amount of rich data from School C, prompted a variation from the number and the type of data collecting instruments and approaches originally proposed for this study.

As a result the remainder of this study focused mostly on the teacher and the 25 students in Class C. The sources of data that became the primary focus of analysis were:

- 14 paired responses to the initial and repeat written tasks from students in Class C
- interview responses from two students from Class C

- summaries of the researcher's observations of the eight classroom intervention sessions with Class C
- the written responses from students to the in-class tasks from each intervention session
- discussions between the researcher and Teacher C when planning and debriefing the intervention sessions.

The unforeseen development of weekly in-class tasks related to the intervention topics provided a rich source of data which, although initially unplanned, provided greater depth to this study than originally anticipated. The inherent flexibility of case study methodology allowed the data collection process to exploit the dynamic opportunities in the interactions between the various participants, many of which could not have been foreseen. Video recording of the classroom sessions, the discussions between the researcher and Teacher C, and the student interviews would have been extremely valuable, but were beyond the practical scope of this study.

Coding the items on the written task

For each of the 11 items on the written task used in this study, students were asked:

- if possible, to calculate the answer mentally and explain the strategy they used, then circle 'mentally'
- to use pen and paper to work out the answer only if they could not calculate the answer mentally, and then circle 'written'
- to circle 'calculator' only when they had no other option for calculating the answer.

Interpreting and applying this coding protocol consistently became difficult. For example, on the initial task student C12 nominated a written strategy for question 3, but gave this explanation: "I took 99 off 200, which leaves me with 101, then plus 64 = 165". In the absence of any record of a written process (such as answers to component calculations) this suggests she quite likely calculated this answer mentally. In the initial response to question 9 (143 x 6) she circled 'mentally' but lists the three component products which are then totalled. This is unlikely to have been possible without using pen and paper to keep a record of the answers to the component calculations, suggesting that this response might more appropriately be seen as involving a written strategy, albeit a number sense one. However this is speculative. A one-to-one interview would have been useful for clarifying this issue but due to time constraints student C12 was not one of the students interviewed.

Similarly student C21 added 256 and 68 this way on the initial instrument: 200 + 68 = 268, 268 + 50 = 318, 318 + 6 = 324. He circled ‘mentally’ which might accurately reflect that he did all those calculations in his head. However it seems at least plausible that he was only able to do this because the process of writing down his explanation served the concurrent purpose of carrying his running total through several stages. Hence for this student, and perhaps in the thinking of others, circling the term ‘mentally’ might not exclude using pen and paper as an *aide memoir*. The original intent of the researcher was to classify such use of pen and paper as a written strategy, but one that was of a different nature to formal algorithmic processes. On her initial response to question 5 (26 x 7) and question 9 (143 x 6) student C22 circled ‘written’ but gave the same explanation as many students who had circled ‘mentally’. While each situation might have in fact been categorised correctly by the students, this does seem unlikely. In future this problem may be reduced by investing more time into explaining what the terms ‘mentally’ and ‘written’ were to signify, but that possibility would need to be balanced against providing so much guidance that the students get caught up in doing what they were shown, rather than showing how they think and can perform with the given tasks.

Another difficulty with the scoring protocol was how to code the correct calculation of an alternate question. For example student C21 correctly mentally calculated 68 – 44, using the split method, and explained this adequately. However he should have added the two numbers. Although he provided evidence of several key indicators of number sense (such as correctly separating tens and ones, processing subtraction correctly then recombining the parts to give his answer) he did not answer the same question as most of the other students.

IMPLICATIONS FOR FURTHER RESEARCH

This study generated a raft of further questions related to this area. These included the following:

- Would leaving out formal written algorithms for whole numbers adversely impact on contemporary needs for operating with fractions, and procedural aspects of operating with algebra? Or are the contexts for using fractions now different, with operating tasks now dominated by using equivalent decimals, and framing answers to calculations using them as approximations? Is providing whatever students need for algebra simply a matter of re-working how we teach algebra? How then could the teaching of introductory concepts of algebra be reframed from a sense-making perspective?
- Does a focus on multiple representations of mathematics in the early years of schooling (using words, objects or pictures, and mathematical symbols) help when learning more abstracted but related concepts that are not readily modelled in this way?
- Are students up to and including the middle years of schooling better served by number sense written calculation, or would a serious and sustained program of mental computation provide a platform for more effective application of formal written algorithms?

Work done by the researcher outside of this study, on a multi-representational approach to introducing algebra to students in their first year of high school, probed a sense-making approach in this area of the curriculum. This involved using words, and objects or pictures, to augment symbols as a way of making sense of foundational algebraic conventions. An approach of this type to learning introductory concepts in algebra seemed more than feasible. This, plus the capacity for scientific and other calculators to both perform fraction calculations and illustrate some of the conceptual underpinnings of the processes used, suggest that reframing the teaching of algebra and fractions to complement number sense written calculation would be workable and of benefit to students.

AND SO …?

The range of calculations for which mental computation and number sense written methods are appropriate is outlined in Appendix A, and does not differ significantly from that described by Plunkett (1979).

The rationale behind this study was that number sense methods for written calculation, with whole numbers at least, appear to have much more to offer in terms of developing number sense in students in the middle years of schooling than the formal written algorithms typically seen in the delivered curriculum. The advent and widespread adoption over the last forty years or so of electronic calculation has meant that the development of number sense has greater social utility than facility with written calculation. This shift towards valuing number sense has been visible in mathematics curriculum documents over that time, but there is uneven uptake of this in practice. The amount and type of written calculation in the delivered curriculum (usually involving a disproportionally large emphasis on formal algorithms) is there because teachers still put it in there, perhaps for the lack of a coherent, balanced view of how calculation is used in contemporary society. The amount of written calculation required outside of mathematics lessons appears to be marginal at best, making the time spent mastering formal written algorithms in mathematics lessons a very questionable use of the time allocated to learning mathematics in school. If written calculation is to continue to be part of the mathematics curriculum, its purpose should be more than just getting answers to calculations that will, for the most part, only ever be done outside of school by pushing buttons. The phrase "less but better" captures the findings of this study.

This study has suggested that the teaching and use of number sense written methods, for a limited range of calculations with whole numbers, can contribute to strengthening conceptual understandings of place value and the four operations. And it is these understandings that help students work meaningfully and successfully with numbers, whatever medium they might use for processing a calculation.

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APPENDICES

APPENDIX A

Complexity of calculations for the classroom number sense initiative

Tables A1.1 and A1.2 describe and give examples of what was considered, for the purposes of this study, to be fair and reasonable expectations for the complexity of calculations with whole numbers to be done mentally and by number sense written methods. This represents a level of facility with calculations involving whole numbers that aligns closely with the ideas outlined in Plunkett (1979) and reflects the prevalence and efficacy of electronic means for calculation in contemporary society.

Some people can calculate mentally with larger or more complex numbers. Similarly, some people may choose to use number-sense written methods to calculate the answer to more involved calculations. However, these standards for mental computation and written calculation seem achievable and desirable. When seen as a complement to the effective and efficient use of electronic calculation, this level of performance with mental and written calculation seems more than adequate for facilitating further learning of mathematics in school, and applied calculation in other contexts.

Table A1.1.Complexity of calculations to be done mentally.

Operation	Performance level	Examples
Addition	2-digit number plus 2-digit number	48 + 37
Subtraction	2-digit number minus 2-digit number	63 – 29
Multiplication	2-digit number multiplied by 1-digit number	34 x 6
Division	2-digit number divided by 1-digit number	72 ÷ 4

Table A1.2. Complexity of calculations to be done using number sense written calculation.

Operation	Performance level	Examples
Addition	3-digit number plus 2- or 3-digit number	254 + 68
		434 + 267
Subtraction	3-digit number minus 2- or 3-digit number	214 – 67
		431 – 259
Multiplication	3-digit number multiplied by 1-digit number	234 x 7
	2-digit number multiplied by 2-digit number	82 x 29
Division	3- or 4-digit number divided by 1-digit number	544 ÷ 8
		1014 ÷ 3

APPENDIX B

Mental computation assessment tasks with whole numbers[2]

Table B.1. Whole numbers/addition and subtraction

A	**B**	**C**	**D**
1) 4 + 7	1) 6 + 13	1) 50 + 70	1) 43 – 12
2) 9 + 5	2) 36 – 5	2) 140 – 60	2) 33 + 15
3) 10 – 8	3) 21 + 4	3) 60 – 13	3) 92 – 34
4) 8 – 3	4) 58 – 3	4) 30 + 22	4) 27 + 25
5) 17 – 7	5) 3 + 48	5) 76 + 40	5) 105 – 26
6) 11 – 5	6) 57 + 9	6) 54 – 20	6) 264 – 99
7) 6 + 6	7) 42 – 6	7) 65 – 35	
	8) 31 – 4	8) 15 + 25	
basic facts	*2 and 1 digit*	*bundles of 10*	*mixed*

Table B.2. Whole numbers/multiplication and division

A	**B**	**C**	**D**
1) 6 x 9	1) 2 x 40	1) 30 x 5	1) 27 x 3
2) 21 ÷ 3	2) halve 15	2) 80 ÷ 4	2) 5 x 12
3) 20 ÷ 4	3) 60 x 2	3) 200 ÷ 5	3) 150 ÷ 6
4) 8 x 3	4) halve 46	4) 7 x 200	4) 26 x 7
5) 72 ÷ 9	5) 17 x 2	5) 13 x 20	5) 92 ÷ 4
6) 7 x 8	6) 12 x 10	6) 40 x 70	6) 72 ÷ 3
7) 24 ÷ 6	7) 10 x 19		
	8) 28 x 10		
	9) 10 x 32		
basic facts	*doubles and x 10*	*extended basic facts*	*mixed*

[2] As in use at school C in 2005

APPENDIX C

Student written task: initial and repeat

Name	*Class*
Teacher	*School*

Here are a few calculations to do. Thank you for having a go at them. ☺

Please do as many calculations as you can.

Write your answers ***and*** how you worked them out in the boxes that look like this:

Try and work out each answer in your head. If you can

- circle the word *mentally*
- then explain how you worked out the answer.

If you can't work it out in your head, you may use pen (or pencil) and paper to help you ***any way you like***.

If you use pen (or pencil) and paper

- circle the word *written*
- then show how you worked out the answer.

You can write a few words to help explain what you did.

If you feel you would need a calculator to do the calculation

- circle the word *calculator*
- then move on to the next question.

1

68 + 44 *mentally* *written* *calculator*

2

92 – 34 *mentally* *written* *calculator*

3

264 – 99 *mentally* *written* *calculator*

4

700 + 283	*mentally*	*written*	*calculator*

5

26 x 7	*mentally*	*written*	*calculator*

6

92 ÷ 4	*mentally*	*written*	*calculator*

7

256 + 68	*mentally*	*written*	*calculator*

8

631 – 54	*mentally*	*written*	*calculator*

9

143 x 6	*mentally*	*written*	*calculator*

10

1062 ÷ 9	*mentally*	*written*	*calculator*

11

53 x 28	*mentally*	*written*	*calculator*

thank you

APPENDIX D

Efficient mental calculation strategies for addition and subtraction tasks

Table D.1 was adapted from Thompson (1999) and gives a selection of efficient mental calculation strategies for addition and subtraction tasks. It gives names for, and examples of, the efficient strategies that were evident in the classroom observations and student work samples that were the focus of this study.

The examples involve two two-digit numbers. However, these strategies could be used with numbers that had more or fewer digits. The ease with which such calculations could be done mentally would depend on the connections between the particular numbers involved in each case.

Table D.1. Selection of efficient mental calculation strategies for addition and subtraction tasks.

Strategy	Description	Examples
Split [also called the *partitioning* method]	The numbers in the calculation are split into multiples of ten and ones.	***46 + 17*** 40 plus 10 is 50 … 6 plus 7 is 13 … together that makes 63
	Typically the multiples of ten are processed together first, then the ones.	***85 – 31*** *(direct subtraction)* 80 take 30 is 50 … 5 minus 1 is 4 … so 54
	Intuitive interpretations of negative numbers as 'more to take' are workable.	***72 – 25*** *(with regrouping)* 70 take 20 is 50 … 2 minus 5 is -3 which means take 3 more away … 50 minus 3 is 47

Table D.1 continued

Strategy	Description	Examples
Jump [also called the *sequencing* or *cumulative* method]	Start with one of the numbers and move in manageable parts of the other number [or the answer, depending on the task]. Efficient use of this strategy typically involves moving [or *bridging*] to a tens number, and moving in steps of one or more tens [*move to a ten* and *move by a ten*]. This strategy can be represented easily on the empty number line [ENL].	***37 + 25*** I add 3 onto 37 to get 40, then 22 to get 62 ***54 – 17*** *(as take away)* I take 4 off 54 to make 50 … then another 10 to get to 40 … then back 3 more to 37 ***54 – 17*** *(as difference)* I start at 17 … add 3 to get 20, add 30 to get to 50, and there's 4 more … 3 + 30 + 4 = 37
Split/jump [also called the *mixed* method]	This is a combination of the split and jump methods.	***75 + 26*** 70 plus 20 is 90 … 95 plus another 5 makes 100, then one more …
Compensation [also called the *over-jump* method]	Add or subtract a near multiple of ten, then adjust.	***54 – 19*** 54 take 20 is 34 … I took one too many so I add one back to get 35

APPENDIX E

Support notes for scaffolding number sense written methods for multiplication and division

This material was developed by the researcher during term two 2008. The purpose of this material was to describe an effective teaching sequence that supported students in developing sound concepts of multiplication and division with whole numbers which would facilitate becoming competent with number sense written methods (NSWM) for these operations.

The main strategy used was splitting numbers into parts that were easily manipulated mentally. Pre-requisite understanding and skills include:

- mental computation with basic multiplication facts (up to 10 x 10 and reverse)
- a knowledge of place value.

This material was used by teacher C to develop her own content-pedagogical knowledge of this aspect of the mathematics curriculum as she worked with the researcher in planning, implementing and evaluating some of the eight intervention sessions that are the focus of this study. Table E.1 lists the sequence of focus areas and related strategies used in the resource.

Table E.1. Focus areas and related strategies for teaching multiplication and division.

	Focus	***Main strategy***
1	Establish multiplication as repeated equal groups	*rectangular arrays*
2	Explore visual representations of multiplicative part/whole relationships	*split arrays*
3	Consolidate symbolic representations of multiplicative part/whole relationships	*empty rectangle*
4	Develop language and notation for extended multiplication tasks	*empty rectangle*
5	Use an abridged version of the same sequence to establish meaningful connections between multiplication and division	*empty rectangle and reverse multiplication facts*
6	Introduce the chunking algorithm for division	*building with multiples of powers of ten*

In this appendix the material has been reproduced in close to its original format.

1 Establish multiplication as repeated equal groups

	representation of quantity	example	key points
1	*random display*	12 counters tipped onto the table … some might even overlap	lack of structure almost certainly makes it difficult to accurately and reliably determine the total likely approach is ‘count by ones’ but without moving the items keeping track of which items have been counted is not easy
2	*linear display*	line the 12 counters up in a single row … straight or curved	linear structure still suggests ‘count by ones’ but assists in keeping track of which items have been counted only one group is represented; the total grouped structure represents the total as a collection of equal-sized groups
3	*bunch display*	make (say) four distinct groups, each group with three counters each group of three counters is like a little triangle	bunch display suggests skip counting to find the total: 3, 6, 9, 12 unless the orientation of each group of three is the same the equality of the groups may not be obvious

	representation of quantity	example	key points
4	*rectangular array*	each group of three is placed as a horizontal line	group size and the equality of the repeated groups is arguably represented more directly
		the four horizontal lines are stacked directly on top of each other to form three vertical groups of four counters	the rectangular grouping suggests a *two-dimensional multiplicative count* as the product of the number of repeated groups (vertical count) and group size (horizontal count): 4 x 3 = 12
		* * * * * * * * * * * *	typical language here is ‘four groups of three’ this display structure provides a clear representation of basic multiplication facts
			rotating the array can help show what is the same and different in partnered multiplication facts (such as 4 x 3 and 3 x 4) • what is the same? (the total) • what is different? (the group size and the number of repeated groups)
			sense-making approaches to teaching mathematics in the early years such as *Count Me In Too* make extensive use of rectangular arrays as a basis for constructing meaning for the operation of multiplication

Notes

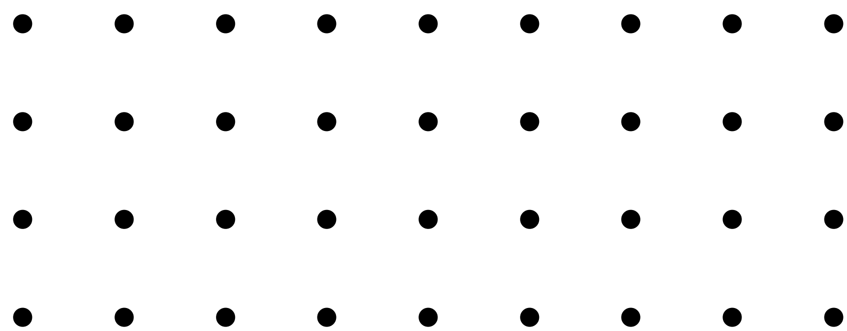
- A variation of the bunch display (category 3 above) is the *iconic display*. This involves culturally significant display structures that represent quantity pictorially. An example is the way five dots are typically represented on a die. Such displays have been connected to *subatising* (the recognition of quantity without needing to count). The amount of cultural construction of meaning to the ‘picture’ of the number suggests these display structures are significantly different in nature to ‘ordinary’ bunch displays.
- A variation on rectangular arrays (category 4 above) is *irregular arrays*. Counters are placed in a sequence of rows but not all rows are equal. For example

```
* * * *
* *
* * *
* * *
```

  This representation suggests an *additive* strategy for finding the total. The value of this representation is that it contrasts with the *multiplicative* thinking modelled by rectangular arrays.
- Although multiplication and division are strongly related there may be value in initially emphasizing multiplication as a way of establishing *multiplicative thinking*. Once there is evidence of students being grounded with multiplicative thinking there is perhaps a more effective and workable basis for representing division.

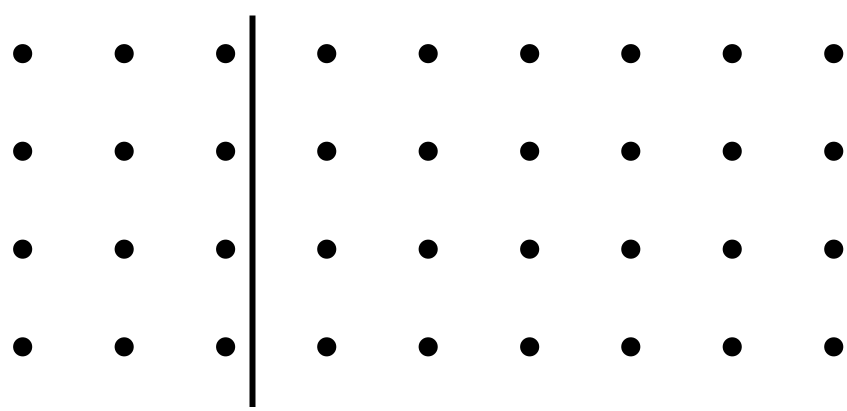
2 Explore visual representations of multiplicative part/whole relationships

Rectangular arrays can be used to give a structured visual representation of part/whole relationships involving products. A term to describe this is ‘split arrays’. Consider the rectangular array representation of 4 x 9 as four rows each with nine dots:

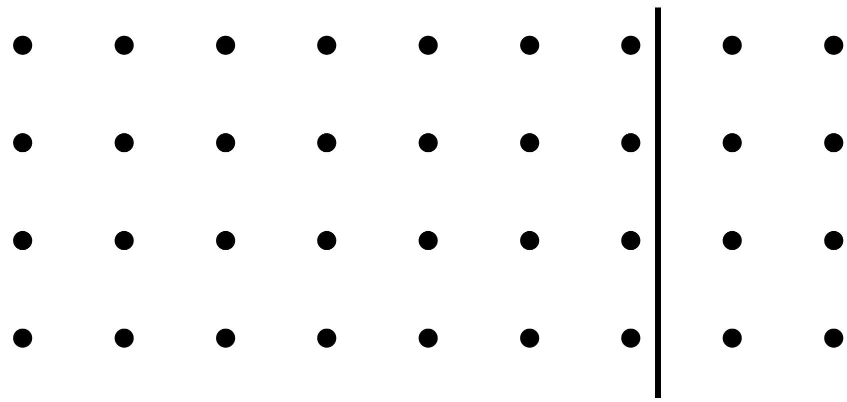


This basic array can be ‘split’ with horizontal or vertical lines to represent a range of equivalent mathematical representations of the total of 36. Each alternate representation will be the sum of two or more component products. For example:

36 is *four groups of* ***three*** (12) plus *four groups of* ***six*** (24)

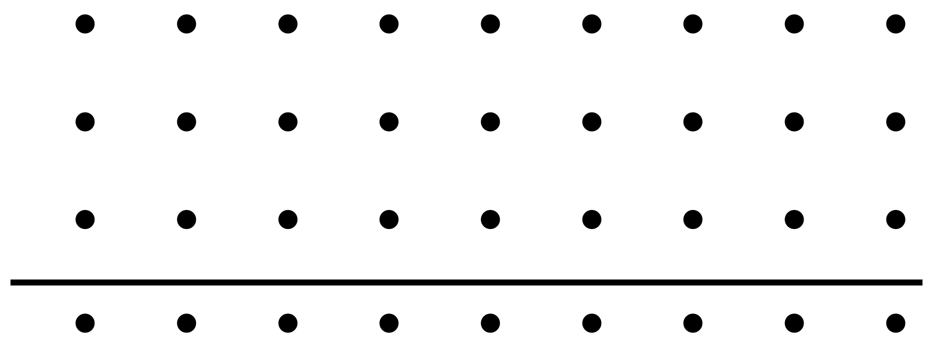


36 is also *four groups of* ***seven*** (28) plus *four groups of* ***two*** (8)



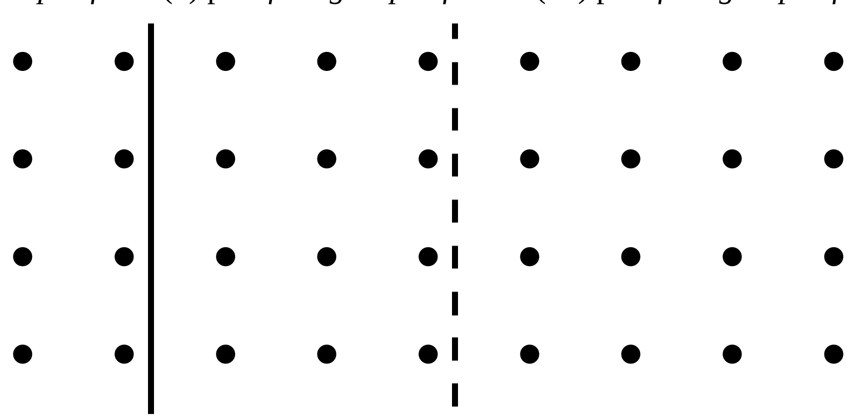
Both of these examples split the array *vertically*. The array could also be split *horizontally*.

36 could be represented as *three groups of* ***nine*** (27) plus *one more group of* ***nine*** (9)



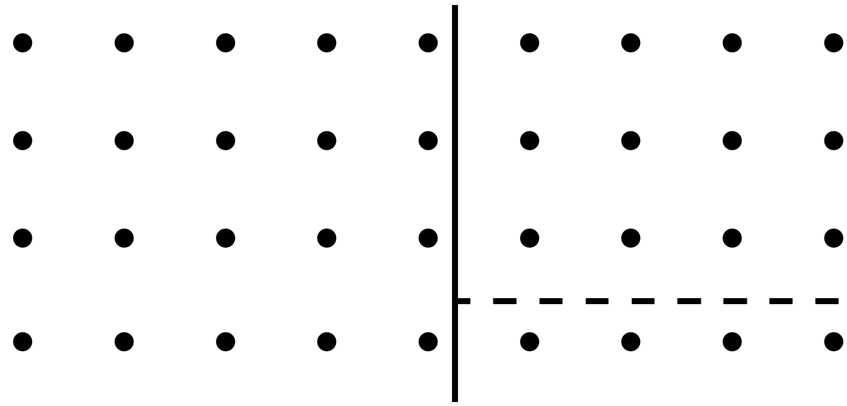
The examples of spilt arrays so far have each had two component products. Further representations of 36 could be:

four groups of ***two*** (8) plus *four groups of* ***three*** (12) plus *four groups of* ***four*** (16)



or

*four groups of **five*** (20) plus *three groups of **four*** (12) plus *one more group of **four*** (4)



Possible strengths of activities of this type include:

- consolidation of component number facts for multiplication
- exercise in mentally adding several one-digit or two-digit numbers
- meaningful visual representation of expressions such as 4 x 7 + 4 x 2 as an alternative to rule-based approaches for introducing conventions related to the order of processing arithmetic operations
- providing a visual representation that illustrates the distributive law and gives meaning to conventional notation used to represent collecting common factors (consider 36 as 3 x 9 plus 1 x 9 which is (3 + 1) x 9 which is 4 x 9).

3 Consolidate symbolic representations of multiplicative part/whole relationships

Rectangular arrays give a direct visual representation of quantity. The empty rectangle builds on the structural aspects of rectangular arrays but leaves behind the direct visual representation of quantity. Instead the empty rectangle conceptualises quantity as lengths and areas (but not to scale) and works with a *symbolic* representation of quantity.

It appears essential for all students to initially establish meaning through representing quantity *directly*. This is particularly important for students struggling with foundational concepts in number. However it is also important to support students' development so that they can *eventually leave direct representations of quantity behind* and develop reliable facility with *symbolic* representations of quantity.

A likely indicator that it is the appropriate time to shift away from quantity representations is that a student can mentally perform simple calculations such as 7 x 9 quickly and correctly but when asked to, can also give an explanation that contains a quantity representation for that task.

An indicator of inadequate facility with basic multiplication facts is if a student cannot reliably or relatively quickly recall or calculate the answer to task such as 7 x 9 without the support of a physical or visual representation of that task.

	strategy	example	key points
1	*represent rectangular arrays for basic multiplication facts as empty rectangles*	show the array for 7 x 9 draw next to the array a rectangle with 7 on the left-hand side and 9 on top draw a vertical line to split the array into 7x 5 and 7 x 4 draw a vertical line just to the right of half way across the rectangle write 5 below the rectangle to the left of the vertical line, and 4 below the rectangle to the right of the vertical line write the component products within the two parts of the rectangle write the total to the right of the rectangle	discuss the meaning of the numbers with students here the 7 x 5 means 7 groups of 5 dots (or whatever symbol is used in the original array) although this is conceptually congruent with indirectly calculating the area of a rectangle for its dimensions there is no need to link to this interpretation; doing so at this stage could confuse some students however this approach to understanding part/whole relationships in numbers to ten and the operation of multiplication is likely to support students in later making meaning for the formula A = l x b

	strategy	example	key points
2	*use the empty rectangle to represent two-digit by one-digit products*	show the calculation 6 x 14 model this on an empty rectangle (6 on the left, 14 on top) ask for ways of splitting 14 explore ways of splitting 14 that might make calculation easy calculate component products and the total of those products	several strategies might make the mental calculation of the total 'easy': • splits of 10 + n exploit the relative ease of multiplying by ten • splitting the 14 in this example into two sevens could combine a possible known fact (6 x 7 = 42) and doubling review the equivalence in terms of the overall product of reverse representations (6 x 14 and 14 x 6)

4 Develop language and notation for extended multiplication tasks

	focus	example	key points
1	*one-digit by two-digit products mentally*	calculate 6 x 14 mentally by • splitting into 6 x 10 (60) and 6 x 4 (24) then adding to get 84 • calculating 6 x 7 (42) and doubling • some other efficient strategy	previous work with the empty rectangle provides an image that can support the creation and managing of component calculations mentally
2	*two-digit by two-digit products with a number-sense written method (empty rectangle)*	represent 27 x 14 on a 2 by 2 empty rectangle as (20 + 7) x (10 x 4) giving four component products: • 20 x 10 • 7 x 10 • 4 x 20 • 4 x 7	component products are calculated mentally; if students cannot do this readily then this task is unsuitable the answer to each component product (here 200, 70, 80 and 28) is recorded in its respective cell the total (378) is then calculated mentally and recorded to the right of the rectangle

5 Establish meaningful connections between multiplication and division

Multiplication and division both involve *repeated equal groups:*

	I know	*I know*	***I work out***	*sample tasks*
multiplication	the number of groups	the number in each group	**the total**	I have 5 groups of 3 things How many do I have altogether? There are 6 packets in each box I have 4 full boxes How many packets are there altogether?
division as 'equal sharing' (partitive)	the total	the number of groups	**the number in each group**	I have 12 minties I give the same number of minties to each of 4 people How many minties does each person get? I share 20 things equally between 5 people How many things does each person get?
division as 'repeated subtraction' or 'how many groups of …' (quotative)	the total	the number in each group	**the number of groups**	There are 18 counters How many groups of 3 counters can I make? I have $200 I want to give $5 to each person How many people can I give $5 to?

Students need to relate division to multiplication and have experience with both interpretations of division (*equal sharing* and *repeated subtraction*).

This can be initially modelled then supported with counters. Students then develop written *connected solutions* to a task using words, pictures and symbols.

	strategy	example	key points
1	*construct rectangular arrays to represent equal sharing*	here are 24 counters …	
		if I share them equally among six people (place six counters in a *vertical column*) …	start by representing the number of groups as a *column*
			the number of groups is represented by the number of rows (which is the number of dots in a single column of a rectangular array)
		how many counters will each person get? (complete each row evenly)	connect this to 24 ÷ 6 = 4 and 6 x 4 = 24
		how can I use the counters to show sharing these among eight people? (place eight counters in a vertical column … complete each row)	

	strategy	example	key points
2	*construct rectangular arrays to represent how many groups of …*	here are 24 counters … how many groups of four counters can I make? (place four counters in a *horizontal row* … repeat making rows of four) what if I only wanted three counters in each row? how many groups of three can I make?	start by representing the number of things in one group as a *row* the number of groups is represented by the number of rows connect this to $24 \div 4 = 6$ and $6 \times 4 = 24$

	strategy	example	key points
3a	*describe an array for a given division fact*	describe an array that represents 24 ÷ 4	using rectangular arrays exploits connections to a known structure
		partitive: to share 24 counters between four people I put four counters in a *column* to represent four rows … . I fill out the rows evenly … each row has six counters … so each of the four people would get six counters	students need to develop language related to visual representations of both the partitive and quotative models/interpretations of division
		quotative: to show how many groups of four I can get I put four counters in a *row* … I keep making rows of four underneath the first row … I get six rows which means there are six groups of four in 24 …	representing this could be done with an 'L' sketch which shows the number in each group (a full row) and how many groups (a full column): * * * * * * * * *
3b	*construct an empty rectangle to represent 'missing multiplication'*	(draw an empty rectangle … put four on the left and 24 inside) … what must the top number be? explain your thinking how could we use an empty rectangle to represent 30 ÷ 6?	using empty rectangles provides a connection to another known structure also connect 24 ÷ 4 to 4 x __ = 24

	strategy	example	key points
4	*split rectangles to represent extended two-digit by one-digit division*	put 42 ÷ 3 on the board how can we split 42 so that each bit is easily divided by three? (probably 30 + 12 … maybe 21 and 21 …) for a 30 + 12 split draw an empty rectangle with two horizontal cells … put three outside on the left, 30 in the left cell, 12 in the right cell, 42 outside to the right use equivalent language to identify the required values: • how many threes in 30? • if I share 30 three ways how much is each share? • what do I multiply by three to get 30? put 10 on top of the cell above the 30; repeat the process for the remaining cell combine the component calculations (10 + 4 = 14) to give the answer to the original calculation	promote splits that can be calculated mentally use equivalent language to build conceptual understanding of both models of division, and the link between division and multiplication: • how many ___ in _ _ _ ? • if I share _ _ _ ___ ways how much is each share? • what do I multiply by ___ to get _ _ _?

6 Introduce the chunking algorithm for division

	strategy	key points
1	*establish meaning for 'how many groups of …'*	**visual representation of quantity** Put a 4 x 3 array of counters on the OHP • how many counters are there? (12) • how many groups of three are there? (each row has three counters … there are four rows) • how many groups of four are there? (each column has four counters … there are three columns) • why does the array makes it easy to answer these questions? (the groups (either rows or columns) are all equal and arranged so this is obvious) **symbolic representation of quantity** • how many groups of five are in forty? (8) • how do you know? (there are two fives in every ten … four tens are forty … two times four is eight …perhaps show this on a hundred chart on the OHP) • how would we write this calculation in mathematical symbols? (40 ÷ 5 = 8) Explain: when we write a calculation like **a** ÷ **b** there are two interpretations • share **a** things **b** ways • how many groups of **b** (draw a circle around the **b**) are in **a** (draw an arrow from the **b** over the top to the **a**)? It's this interpretation of division we'll keep working with because it's the most common … Show: 40 ÷ 5 = 8 … circle the 5 … draw an arrow over the top to the 40 • what are some other ways of working out how many fives are in forty? (skip count in fives, or reverse multiplication facts like 5 x 8 or 4 x 5 is 20 and double the four …) Explain: multiplication and division are closely related …

	strategy	key points
2	*MC division using splits*	what mental strategy have we used to calculate 42 ÷ 3 in our head? (split 42 into parts that each are easily divisible by three … perhaps 30 + 12 … perhaps 21 twice …) model the language of ‘how many groups of three in … ’ by working through each (or another) split on an empty rectangle (ER) what about this calculation? (78 ÷ 3) … how could we interpret this calculation? (how many threes are in 78) what split could we use here? (perhaps 30, 30 and 18 … model the language as ‘how many threes in 30? (10) … same in the next 30 (another 10) … there are six threes in 18 … that’s 10, another 10 and six … 26 threes in 78 … so 78 ÷ 3 is 26 …)
3	*P&P division as chunking*	Explain: let’s look at using the same sort of thinking with bigger numbers … we’ll use pen and paper (P&P) to keep track of the steps we work out in our head … Put the calculation 492 ÷ 6 on the board • how could we interpret this calculation? (how many sixes are in 492) • are there more or less that 10? (a lot more … ten sixes are only 60) • are there more or less that 100? (less) … how do you know? (100 sixes would be 600 .. 492 is less than 600) • how about fifty sixes? (link this to half of 100 sixes) … that’s 300 … . if we take that out of the 492 what’s left? (192) Write this on the board using chunking notation … • what’s another easy chunk of sixes? (ten lots, double ten, half fifty ….) Follow through with the chunking algorithm … Ask: so, how many sixes altogether are there in 492 ... exactly? (add up the numbers on the right to get 82)

Notes

The chunking algorithm is flexible. Different ‘chunks’ of the divisor can be progressively withdrawn depending on the student’s facility with splitting.

Some writers suggest that the phrase ‘lots of … ‘ is not good … but it is what many teachers and students are used to.

Scaffolding tasks

Useful chunks are multiples of powers of tens (10, 100, 1000 …) and halves and doubles of these (five lots is half of ten lots, 50 lots is half of a hundred lots …)

As warm-ups perhaps give students tasks like:

Using the fact that 10 lots of seven is 70, how could you easily work out

- 5 lots of seven?
- 20 lots of seven?

Using the fact that 100 lots of seven is 700, how could you easily work out

- 50 lots of seven?
- 25 lots of seven?
- 75 lots of seven?

Using what you know about 10 lots and 100 lots of seven, how could you easily work out

- 15 lots of seven?
- 60 lots of seven?
- 80 lots of seven?