Pre-service Teachers' Concept Image for Circle and Ellipse

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Concept image is proposed by Vinner and Tall (1981) to differentiate the elements and relationships that a learner constructs about a concept from formal mathematical definition for the same concept. The answers of 119 University students to an examination question are analysed to establish the concept images they have for circle and ellipse. The results show the difficulty the students have in reasoning with properties at the higher levels of Van Hiele's (2004) model of geometric thought.

Constructivist theories of learning suggest that learners build their own ideas. An implication for teaching is that data is needed about what ideas students possess and are building about the target concept. Von Glasersfeld (2010)[1] suggests that good teachers realised this necessity long before the advent of radical constructivism and abandoned the metaphor of learning through transmission. Providing effective learning experiences for students requires modifying instructional design to provoke conceptual change. Recent models of the knowledge needed for teaching mathematics include knowledge of students and learning (Ball, Thames, & Phelps, 2008; Chick, Baker, Pham, & Cheng, 2006). Teachers need to co-ordinate data about students with in-depth understanding of the mathematical content to act in the moment (Mason & Spence, 1999). Knowledge of common responses from students and cognitive obstacles they are likely to face is commonly accepted as necessary for effectively responding to students (Empson & Jacobs, 2008).

This research explores pre-service teachers' understanding of the circle and ellipse. It uses the construct of concept image to distinguish between formal definition of a concept and students' constructions of it. The differentiation between what might potentially be learned and what is actually learned reflects constructivist theories of learning. Tall and Vinner (1981, p. 2) define a concept image to be the "total cognitive structure" that the learner associates with a given concept. The ideas are evoked by a label which might be a word, like circle, or a symbol like π, and are the result of experience. So a concept image evolves in response to new stimuli. The concept image can be consistent with a formal mathematical definition or not, and may contain conflicting ideas. Some aspects of the concept image are cued more favourably over others by the learner (Pratt & Noss, 2002). Later I contrast the mathematical definitions of circle and ellipse as loci in the plane and the concept images held by students.

The data come from students' responses that may be categorised using Van Hiele's (2004) levels of geometric thought. The levels come from Pierre van Hiele's 1957 doctoral thesis and have been widely applied and critiqued. Based on Piaget's earlier work the levels framework acknowledges that students' thinking is advanced through appropriate opportunities to learn. The levels are progressive and invariant in the sense that one level builds on the previous in a predictable way. At the foundation, Level 0: Visualisation the student identifies shapes by their global characteristics. So a circle may be recognised as round like a wheel and an ellipse as a squashed circle shaped like a football. At Level One: Analysis the student identifies classes of shapes and attends to characteristics common to

[1] http://www.youtube.com/watch?v=YozoZxblQx8

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all shapes in the class. In relation to circles and ellipses the defining properties are not easy to discern from everyday experience. Understanding of a circle as a set of points with the common property of being an invariant distance from a given point (centre) is most likely to occur through practical experience such as construction or consideration of why wheels are circular.

Level Two: Informal Deduction involves students reasoning with the properties of shapes within classes and establishing relationships between classes. Understanding a circle as a type of ellipse in which the two foci coincide involves regarding circles as a sub-class of ellipses. Level Three: Deduction involves reasoning with the properties of classes to make abstract statements. Students at this level might deduce from the definition of ellipse as a set of points where the sum of the distances to two fixed points, the loci, is constant. Coincidence of the foci results in the distances from any point on the ellipse to the two foci being equal, matching the centre and radius properties of a circle. Van Hiele's highest level, Rigor, is not relevant to this study. It involves axiomatic systems.

Circles are familiar objects in the environment and the word circle is used in many different contexts like 'circle of friends' (a closed system), 'full circle' (return to the start), and 'Arctic circle' (Enclosed geographical area). Ellipse is less commonly used and students frequently refer to an ellipse as an 'oval'. These everyday experiences contribute to students' concept images in ways that are not always in harmony with the mathematical definition. The mismatch between everyday usage and mathematical definition makes the concepts of circle and ellipse fertile ground for investigating Tall and Vinner's construct of concept image.

Method

In November, 2012, 119 students at an Australian university attempted a two-hour examination. Question nine about circles and ellipses was located midway through the paper so lack of time was unlikely to be a reason for students' answers or non-attempts. The students were in their second year of study. Mathematics qualifications to Year 11 are required for entry to the course. Responses from all the students in the cohort were used so no sampling was required.

The students had opportunities during the unit to learn about the concepts of circle and ellipse. During a lecture a small group of volunteers demonstrated the construction of a circle and an ellipse using a length of rope and a piece of chalk. Both demonstrations were confined by the layout of seats in the theatre so a learning object was used to show how convergence of the foci resulted in the ellipse becoming more circular. The properties of both loci were linked to the constructions, particularly the constant sum of the distances from the foci to any point on the ellipse. Students had limited further opportunities to engage with the concepts of circle and ellipse through work on problems outside and within tutorials. Many students appeared to struggle with the relationship between classes and sub-classes of two-dimensional shapes. For example, tutors encountered sometimes perplexed and annoyed reactions from students for suggesting that, by definition, a square was a type of rectangle.

Question nine was in four parts designed to provide opportunities for students to demonstrate increasing levels of van Hiele's model of geometric thought. Parts a and c potentially established whether or not the demonstration of drawing a circle and ellipse in the lecture were features of the students' concept images. Parts b and d provided opportunities for students to connect the constructions to the formal definitions of circle and ellipse as loci. Part d assessed students' ability to reason at Levels One to Three of van

Hiele's model demonstrated by their reasoning with properties. The question parts are given below:

Question Nine

a) You have a rope, a friend and a piece of chalk.

Draw a diagram to show how you could use these resources to draw a circle.

b) Define a circle using mathematical language. How does your method in part a) match the definition?

c) You have a rope, two friends and a piece of chalk.

Draw a diagram to show how you could use these resources to draw an ellipse.

d) Is a circle a special case of an ellipse? Explain.

A coding system for answers to each question was created iteratively. The first twenty answers were coded to establish main categories. All answers were then coded using this initial taxonomy. The codes for 9c and 9d were revised to further sub-divide categories where large numbers of responses were pooled and where the descriptors were uninformative. For example, for 9c responses categorised as Foci and Equidistant were classified into three sub-categories, Single Example, Dynamic, and Definition. All answers were re-coded and the data entered into a spreadsheet. Specific responses were re-coded a third time after sorting to provide more detail about data pooled under the category of Unclear and Incorrect for questions 9c and 9d. For example with 9c, the categories Foci Points with Two Circles, Incorrect Positioning, Changing Radius and Overlapping Circles were created to separate responses in the Unclear and Incorrect category. Finally, the 11 categories for 9c and 14 categories for 9d were collapsed by looking for common features. For example, a category for 9d, Major and Minor Axis was included under Correct Inclusion Statement and Justification since the student responses in that category all contained assertions that a circle is a special case on an ellipse and used equality of the major and minor axes as justification. The results of iterative coding are provided next.

Results

The students' answers to question 9a suggest that the construction of a circle in the lecture using a rope and chalk was easily assimilated into their concept image (see Table 1). Previous experiences with drawing circles with a protractor may have assisted them to make the association between action, the construction, and result, the shape. Eighty-nine percent of the students' answers described a construction that appropriately used a person as the centre and the rope as the radius. The other eleven percent of answers involved using the rope or a person as the circumference, or part of it, using the rope as the diameter or no response at all.

Question 9b invited students to connect the constructions with the properties of a circle. Seventy-nine percent of the answers contained a correct formal definition of a circle (see Table 2). However, only 53 percent of the answers contained a link between the features of the construction and the centre and radius in the definition. Figure 1 shows an answer in this category, though in this example the student does not use the terms centre and radius.

Table 1
Answers to Question 9a about construction of a circle

Descriptor	Frequency (Percentage)
Student as centre, rope as radius	106 (89%)
Rope or person as circumference	4 (3%)
Closed curve, rope as circumference	3 (3%)
Rope as diameter	4 (3%)
No answer	2 (2%)

Table 2
Answers to Question 9b about definition of a circle

Descriptor	Frequency (Percentage)
Correct definition linked to construction	53 (44%)
Correct definition not linked to construction	42 (35%)
Incorrect or unclear use of terms	16 (13%)
Circle defined only as a closed curve	4 (3%)
No answer	3 (3%)
Rope as the diameter	1 (1%)

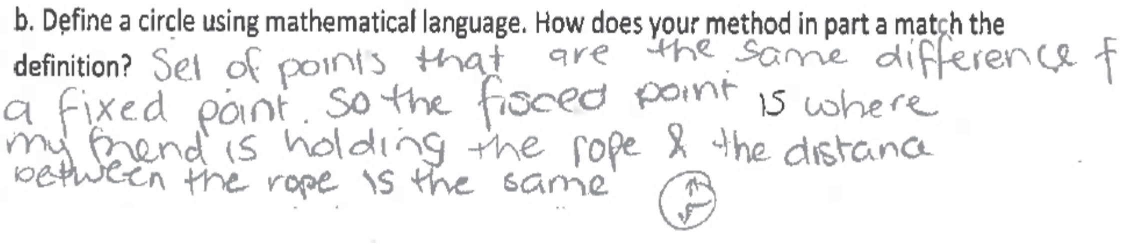

Figure 1. Linking of Construction of Circle with Formal Definition

Forty-two students gave a correct definition of a circle but did not link their definition with features of the construction. Figure 2 shows an example of this type of answer.

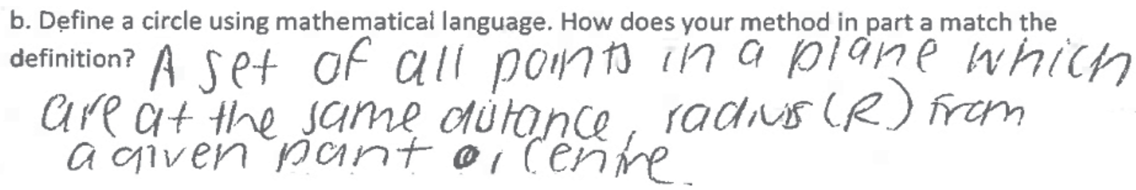

Figure 2. Correct Definition of a Circle not linked to the Construction.

Describing a circle as a closed curve and incorrect or unclear use of terms were also common. Some of the incorrect definitions were due to slips of vocabulary, such as calling the radius the circumference. Others reflected parts of the concept image that were associated, often incompatibly, with geometric ideas encountered in activities such as working with three-dimensional solids and angles.

One student wrote:

> *A circle is a round 360° object that has one face and not edges. By one friend walking around the other friend allows the rope to help form a circle object.*

The data show that relating a physical activity, such as construction, to the properties of the shape under construction is challenging for over one half of these students. While a formal definition of a circle as a locus of points is part of the concept image for many it is isolated from the experience of viewing the demonstration and from students constructing circles themselves.

The answers to questions 9c and 9d were more diverse and harder to classify than those to 9a and 9b. About one third of the answers to 9c contained an accurate description of the construction of the ellipse with students forming the two foci and the rope conserving the sum of distances (see Table 3). These correct answers were further classified into static images that showed one point on the perimeter and dynamic images that showed multiple points or movement (see Figure 3). Dynamic representations were more common than static and left the reader in little doubt that the writer had the construction method as a strong memory in their concept image. Most other responses contained some features of a correct construction that were incorrectly co-ordinated. Twenty answers contained both foci and an elliptical locus but no reference to the sum of distances. Eight students recited a correct formal definition of an ellipse with no details for construction. Some answers were attempts to answer the question using the materials provided with little understanding of the correct construction. The rope was used as the major or minor axis, as the perimeter, and as a reducing and expanding radius. Eighteen incorrect drawings of the construction had students or the rope occupying various positions inside and on the perimeter of the ellipse. Six students remembered the two foci but used the rope to draw two overlapping circles as the ‘opposite ends’ of the ellipse.

Figure 3. Static and Dynamic Representations of Foci and Equidistance.

The features of the construction of an ellipse were harder for students to assimilate into their concept image than was the case with the circle. Recalling the foci was considerably more common than the use of the rope to conserve the sum of the distances. The global appearance of an ellipse as a closed curve was a dominant feature of the data with less evidence that students understood the properties of an ellipse than was the case with a circle.

In the absence of a detailed memory of the construction of an ellipse many students hypothesised configurations of students and rope when answering Question 9c. However in responding to Question 9d lack of knowledge about the properties of an ellipse did not prohibit some students from recalling memories of the animation of the foci converging and the shape of the ellipse tending towards a circle.

Table 3
Answers to Question 9c about definition of a circle

Descriptor	Frequency (Percentage)
Foci and equidistance (dynamic)	24 (20%)
Foci and equidistance (static)	15 (13%)
Correct definition but no construction	8 (7%)
Foci and closed curve only	20 (17%)
Incorrect positioning of students and/or rope	18 (15%)
No answer	10 (8%)
Rope as axis or perimeter	12 (10%)
Foci and overlapping circles	6 (5%)
Unclear or incorrect	5 (4%)
Changing radius	1 (1%)

Fifty-six students explained that a circle is a special case of an ellipse since bringing the foci together creates a circle (see Table 4). Only five of these students linked the sum of the distances for an ellipse with the radius of a circle. All five students in this category had difficulty in expressing the connection correctly. For example one student wrote:

> Yes. In this case (part c) there are two points where the sum of the distances from these two points is constant, this is similar to the circle as the radius from the centre of the circle to any point on the circumference of the circle is always constant, that is the distance.

Though they excluded the detail that the sum of distances is twice the radius of the resulting circle as the foci converge, this student deduced the class-sub-class relation with reference to the properties of circle and ellipse. He or she reasoned with properties so was operating at Level Three of van Hiele's model. This is very unusual for students in this cohort. In one other category, Correct inclusion statement and justification, attendance to the properties of circle and ellipse are evident. One student correctly explained how a circle is an ellipse with major and minor axes equal in length.

> Yes – If you make both axis [axes] the same length in an ellipse this will form a circle.

In four responses correct inclusion statements are justified with some reference to the foci and constant distance properties of ellipses, though students find it difficult to accurately describe their ideas using mathematical language.

> Yes as the radius is the same total distance from any two points around the circle.

One student correctly associated circles and ellipse as conic sections:

> Yes. Because if the intersection of a cone is done horizontally it would not be on a tilt so it would be a perfect circle.

Correct identification of the inclusion relationship is insufficient evidence of working at Level Two or Three of van Hiele's framework. Nine students correctly stated the inclusion relationship but offered a range of incorrect justifications such as, "Yes as the ellipse is an elongated circle", and "Yes because it has never ending points". The last answer is similar to those of ten other students who justified their answers, sometimes devoid of inclusion, with reference to closed curves. An example of Closed Curve Justification follows.

> An ellipse is a closed curve that looks like an elongated circle though it is not considered to be a circle because not all points in it's [sic] plane are at the same distance.

The remaining categories of answers to Question 9d involve no justification or response, or significant misunderstanding. Incorrect inclusion statements contradict the class relationship of circle as ellipse, such as, "An ellipse is a circle with two centre points". *References to the Radius and/or Variant distance* answers provide assertions about distances that do not relate accurately to the shape properties.

> A circle is a special case of an ellipse in that instead of it's [sic] radius changing at different points, the circle's radius is the same at every point.

Answers in the References to Points and Lines and to Three Dimensions categories contain incorrect or ambiguous statements.

> Yes, it is a flattened circle so instead of having one fixed point in the middle it has two.

> It [an ellipse] is a 3D version of a circle it is a prism.

Table 4
Answers to Question 9d about a circle as a special case of an ellipse

Descriptor	Frequency(Percentage)
Closing foci results in a circle	56 (47%)
Correct inclusion statement and justification	5 (4%)
Correct inclusion statement but incorrect justification	9 (8%)
Incorrect inclusion statement	10 (8%)
Closed curve justification	10 (8%)
Reference to radius and/or variant distance	5 (4%)
Reference to points and lines	3 (3%)
Reference to three dimensions	3 (3%)
Yes with no explanation	3 (3%)
Unclear or no response	15 (13%)

In summary, the answers to Question 9d show only eight percent of these students reason with the properties of circles and ellipse, so operate at Level Three of van Heile's framework. A further forty-two percent of students rely on an experiential image of converging foci to make a correct inclusion statement. Their answers are reflective of working at Level Two. The remaining half of students call on unhelpful parts of their concept images for circle and ellipse that do not qualify inclusion, such as the shapes as closed curves, or hold confused views about the properties, or provide unclear or no answers.

Discussion

Vinner and Tall's construct of the concept image is useful in describing these data. Seeing the difference between the concepts that are intended (taught) and the images that are learned is a sobering but essential experience for a teacher. While most students are able to recite formal mathematical definitions for both circle and ellipse, fewer of them connect elements and relationships in the definition with the physical construction of these loci. This is more so for the ellipse than the circle.

The construction using fellow students, rope and chalk, supported by the computer animation, creates a powerful experience that becomes part of the concept image for half of the students. For the remaining students their lack of attendance to key features of the constructions results in an incomplete image. Attendance to the two foci of the ellipse and to the locus itself prevails over noticing the fixed distance determined by the rope. For some students the experience does not achieve status in their concept image and global characteristics such as closure, continuity and roundness are more preferred.

The data also show the challenges that reasoning with properties of shapes, operating at Levels Two and Three of van Hiele's framework, poses for these prospective teachers. Acquiring and appropriately using mathematical language is demanding as is the logic required for determining the relations between classes by reasoning with properties. An implication for teaching geometry is that dynamic construction is a powerful medium that is easily assimilated in some students' concept images for shapes. Focusing attention on the connections between elements in the construction and defining properties of the shapes seems significant to promote greater synergy between concept images and the formal definitions. An approach that asks students to consider what changes and what remains invariant during construction may prove fruitful. Wider experience in defining classes and reasoning with properties across the curriculum, for example defining species and sub-species in science, is also likely to help student to classify objects and events.

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