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Embedding Wheel-like Networks

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Abstract. One of the important features of an interconnection network is its ability to efficiently simulate programs or parallel algorithms written for other architectures. Such a simulation problem can be mathematically formulated as a graph embedding problem. In this paper we compute the lower bound for dilation and congestion of embedding onto wheel-like networks. Further, we compute the exact dilation of embedding wheel-like networks into hypertrees, proving that the lower bound obtained is sharp. Again, we compute the exact congestion of embedding windmill graphs into circulant graphs, proving that the lower bound obtained is sharp. Further, we compute the exact wirelength of embedding wheels and fans into 1,2-fault hamiltonian graphs. Using this we estimate the exact wirelength of embedding wheels and fans into circulant graphs, generalized Petersen graphs, augmented cubes, crossed cubes, Möbius cubes, twisted cubes, twisted n-cubes, locally twisted cubes, generalized

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twisted cubes, odd-dimensional cube connected cycle, hierarchical cubic networks, alternating group graphs, arrangement graphs, 3-regular planer hamiltonian graphs, star graphs, generalised matching networks, fully connected cubic networks, tori and 1-fault traceable graphs.

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2000 Mathematics subject classification: 05C60, 05C85.

1. Introduction

Graph embedding is a powerful method in parallel computing that maps a guest network G into a host network H (usually an interconnection network). A graph embedding has a lot of applications, such as processor allocation, architecture simulation, VLSI chip design, data structures and data representations, networks for parallel computer systems, biological models that deal with visual stimuli, cloning and so on [1, 2, 3, 4].

The performance of an embedding can be evaluated by certain cost criteria, namely the dilation, the edge congestion and the wirelength. The *dilation* of an embedding is defined as the maximum distance between pairs of vertices of the host graph that are images of adjacent vertices of the guest graph. It is a measure for the communication time needed when simulating one network on another. The *congestion* of an embedding is the maximum number of edges of the guest graph that are embedded on any single edge of the host graph. An embedding with a large congestion faces many problems, such as long communication delay, circuit switching and the existence of different types of uncontrolled noise. The *wirelength* of an embedding is the sum of the dilations in host graph of edges in guest graph [3, 5].

Ring or path embedding in interconnection networks is closely related to the hamiltonian problem [6–9] which is one of the well known NP-complete problems in graph theory. If an interconnection network has a hamiltonian cycle or a hamiltonian path, ring or linear array can be embedded in this network. Embedding of linear arrays and rings into a faulty interconnection network is one of the central issues in parallel processing. The problem is modeled as finding fault-free paths and cycles of maximum length in the graph [10].

The wheel-like networks plays an important role in the circuit layout and interconnection network designs. Embedding of wheels and fans in interconnection networks is closely related to 1-fault hamiltonian problem. A graph G is called f-fault hamiltonian if there is a cycle which contains all the non-faulty vertices and contains only non-faulty edges when there are f or less faulty vertices and/or edges. Similarly, a graph G is called f-fault traceable if for each pair of vertices u and v, there is a path from u to v which contains all the non-faulty vertices and contains only non-faulty edges when there are f or less faulty vertices and/or edges. We note that if a graph G is hypohamiltonian, hyperhamiltonian or almost pancyclic then it is 1-fault hamiltonian [11] and it has been well studied in [8, 11, 12].

The rest of the paper is organized as follows: Section 2 gives definitions and other preliminaries. In Section 3, we compute the dilation, congestion and wirelength of embedding onto wheel-like networks. Finally, concluding remarks and future works are given in Section 4.

2. Preliminaries

In this section we give basic definitions and preliminaries related to embedding problems.

Definition 2.1. [13] Let G and H be finite graphs. An *embedding* of G into H is a pair (f, P_f) defined as follows:

(1) f is a one-to-one map: $V(G) \to V(H)$
(2) P_f is a one-to-one map from $E(G)$ to $\{P_f(e) : P_f(e)$ is a path in H between $f(u)$ and $f(v)$ for $e = uv \in E(G)\}$.

By abuse of language we will also refer to an embedding (f, P_f) simply by f. The *expansion* of an embedding f is the ratio of the number of vertices of H to the number of vertices of G. In this paper, we consider embeddings with expansion one.

Definition 2.2. [13] Let f be an embedding of G into H. If $e = uv \in E(G)$, then the length of $P_f(e)$ in H is called the *dilation* of the edge e denoted by $dil_f(e)$. Then

$$dil(G,H) = \min_{f:G\to H} \max_{e\in E(G)} dil_f(e).$$

Definition 2.3. [13] Let f be an embedding of G into H. For $e \in E(H)$, let $EC_f(e)$ denotes the number of edges xy of G such that e is in the path $P_f(xy)$ between $f(x)$ and $f(y)$ in H.
In other words, $EC_f(e) = |\{xy \in E(G) : e \in P_f(xy)\}|$. Then

$$EC(G,H) = \min_{f:G\to H} \max_{e\in E(H)} EC_f(e).$$

Further, if S is any subset of $E(H)$, then we define $EC_f(S) = \sum_{e\in S} EC_f(e)$.

Definition 2.4. [14] Let f be an embedding of G into H. Then the wirelength of embedding G into H is given by

$$WL(G,H) = \min_{f:G\to H} \sum_{e\in E(G)} dil_f(e) = \min_{f:G\to H} \sum_{e\in E(H)} EC_f(e).$$

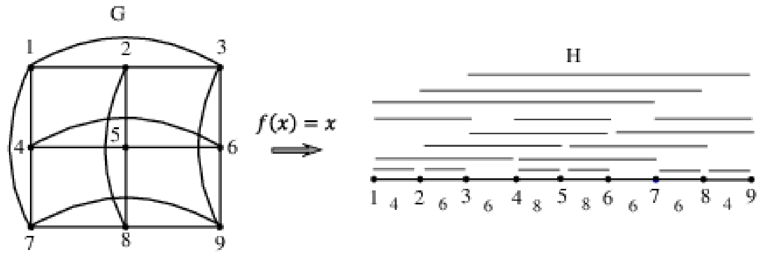

FIGURE 1. Wiring diagram of torus G into path H with $dil_f(G, H) = 6$, $EC_f(G, H) = 8$ and $WL_f(G, H) = 48$.

An illustration for dilation, congestion and wirelength of an embedding torus G into a path H is given in Fig. 1. The dilation, the congestion, and the wirelength problem are different in the sense that an embedding that gives the minimum dilation need not give the minimum congestion (wirelength) and vice-versa. But, it is interesting to note that, for any embedding g, the dilation sum, the congestion sum and the wirelength are all equal.

Graph embeddings have been well studied for a number of networks [1,2, 4–7, 11, 13–34]. Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [13, 18]. But the Congestion Lemma and the Partition Lemma [14] have enabled the computation of exact wirelength for embeddings of various architectures [14, 21, 23, 24, 32, 33]. In fact, the techniques deal with the congestion sum [14] to compute the exact wirelength of graph embeddings. In this paper, we overcome this difficulty by taking non-regular graphs as guest graphs and use dilation-sum to find the exact wirelength.

Definition 2.5. [19, 35] A wheel graph W_n of order n is a graph that contains an outer cycle or rim of order $n - 1$, and for which every vertex in the cycle is connected to one other vertex (which is known as the hub or center). The edges of a wheel which include the hub are called spokes.

Definition 2.6. [11, 36] A fan graph F_n of order n is a graph that contains a path of order $n - 1$, and for which every vertex in the path is connected to one other vertex (which is known as the core). In other words, a fan graph F_n is obtained from W_n by deleting any one of the outer cycle edges.

Definition 2.7. [36] A friendship graph T_n of order $2n + 1$ is a graph consists of n triangles with exactly one common vertex called the hub or center. Alternatively, a friendship graph T_n can be constructed from a wheel W_{2n+1} by removing every second outer cycle edge.

Definition 2.8. A windmill graph WM_n of order $2n$ is obtained by deleting a vertex v of degree 2 in T_n.

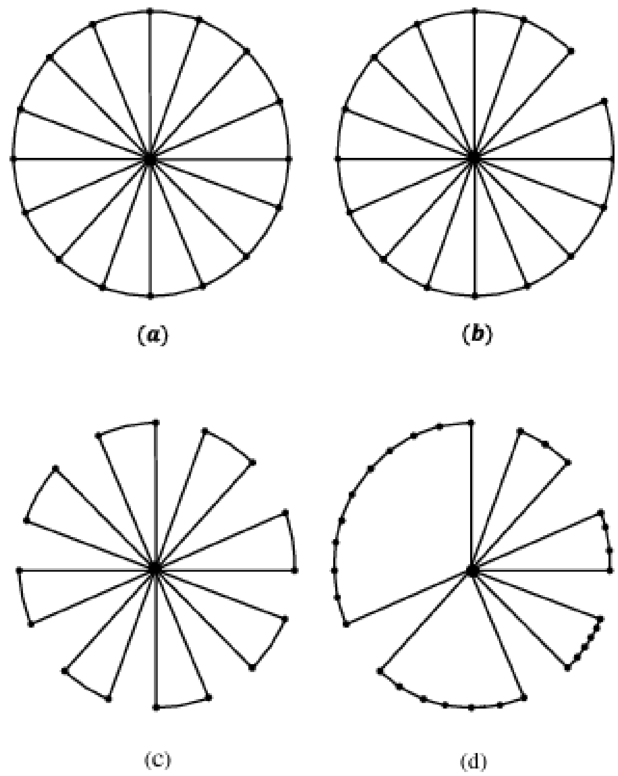

FIGURE 2. (a) Wheel graph W_{17} (b) Fan graph F_{17} (c) Friendship graph T_8 and (d) Windmill graph WM_8.

Definition 2.9. [3] A star graph S_n is the complete bipartite graph $K_{1,n-1}$.

Figures 2(a), 2(b), 2(c) and 2(d) illustrate the wheel graph W_{12}, fan F_{12}, friendship graph T_8 and windmill graph WM_8 respectively.

Definition 2.10. [37] The basic skeleton of a hypertree is a complete binary tree T_r, where r is the level of a tree. Here the nodes of the tree are numbered as follows: The root node has label 1. The root is said to be at level 1. Labels of left and right children are formed by appending a 0 and 1, respectively, to the label of the parent node, see Fig. 3(a). The decimal labels of the hypertree in Fig. 3(a) are depicted in Fig. 3(b). Here the children of the node x are labeled as $2x$ and $2x+1$. Additional links in a hypertree are horizontal and two nodes in the same level i of the tree are joined if their label difference is 2^{i-2}. We denote an r level hypertree as $HT(r)$. It has 2^r-1 vertices and $3\ (2^{r-1}-1)$ edges.

Definition 2.11. [34] For any non-negative integer r, the complete binary tree of height $r-1$, denoted by T_r, is the binary tree where each internal vertex

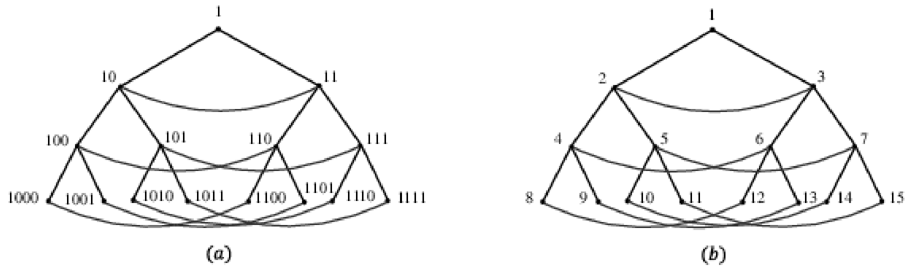

FIGURE 3. (a) $HT(4)$ with binary labels (b) $HT(4)$ with decimal labels.

has exactly two children and all the leaves are at the same level. Clearly, a complete binary tree T_r has r levels. Each level i, $1 \leq i \leq r$, contains 2^{i-1} vertices. Thus, T_r has exactly $2^r - 1$ vertices. The sibling tree ST_r is obtained from the complete binary tree T_r by adding edges (sibling edges) between left and right children of the same parent node.

Definition 2.12. The X-tree XT_r is obtained from the complete binary tree T_r by adding the consequent vertices in each level by an edge.

For illustration, the sibling tree $ST(5)$ and X-tree XT_5 are given in Figure 4.

Definition 2.13. [22, 38] The undirected circulant graph $G(n; \pm S)$, $S \subseteq \{1, 2, \ldots, j\}$, $1 \leq j \leq \lfloor n/2 \rfloor$, *is a graph with the vertex set* $V = \{0, 1, \ldots, n-1\}$ *and the edge set* $E = \{ik : |k - i| \equiv s(mod\ n),\ s \in S\}$.

It is clear that $G(n; \pm 1)$ is the undirected cycle C_n and $G(n; \pm\{1, 2, \ldots, \lfloor n/2 \rfloor\})$ is the complete graph K_n. The cycle $G(n; \pm 1) \simeq C_n$ contained in $G(n; \pm\{1, 2, \ldots, j\})$, $1 \leq j \leq \lfloor n/2 \rfloor$ is sometimes referred to as the outer cycle C of G.

Definition 2.14. [19] Let v be a vertex in G. The eccentricity of v, denoted by $\epsilon(v)$, is $\epsilon(v) = max\{d(u, v) | u \in V\}$. The maximum eccentricity is the graph diameter $d(G)$. That is, $d(G) = max\{\epsilon(v) : v \in V\}$. The minimum eccentricity is the graph radius $r(G)$. That is, $r(G) = min\{\epsilon(v) : v \in V\}$. For brevity, we denote $d(G)$ and $r(G)$ as d and r respectively.

Notation: For a graph G, the minimum degree and the maximum degree is denoted by $\delta(G)$ and $\triangle(G)$ respectively. For $u \in V(G)$, let $N_i(u)$ denotes the set of all vertices of G at distance i from u, $1 \leq i \leq d$, where d denotes the diameter of G.

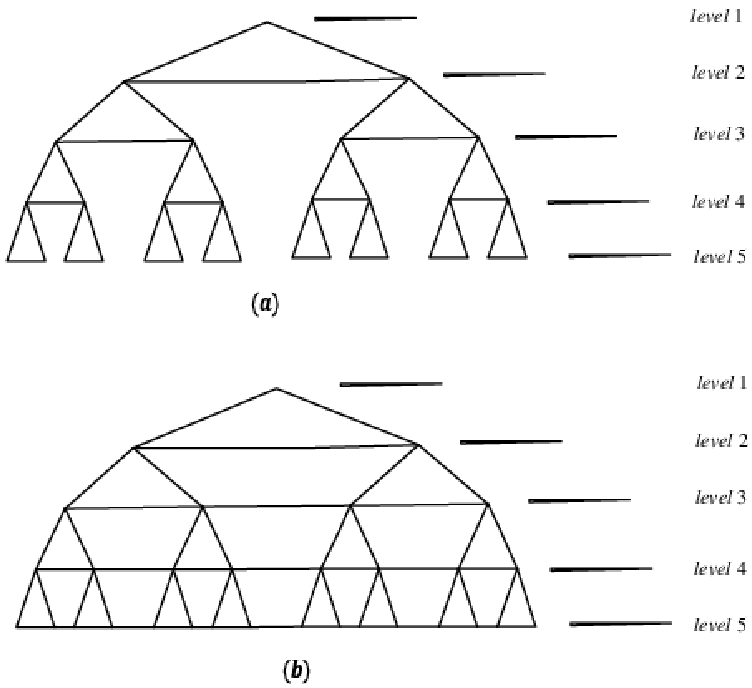

FIGURE 4. (a) Sibling tree $ST(5)$ (b) X-tree XT_5.

3. Main Results

In this section we compute the dilation, congestion and wirelength of embedding onto wheel-like networks.

3.1. **Dilation.**

Lemma 3.1. *Let G be a graph with $\triangle(G) = n-1$ and H be a graph with $|V(G)| = |V(H)| = n$. Then $dil(G,H) \geq r$, where r is the radius of H.*

Proof. Since $\triangle(G) = n-1$, there exists a vertex $u \in V(G)$ such that $d(u) = n-1$. Let f be an embedding from $V(G)$ to $V(H)$ and map $f(u) = v$. If eccentricity of v is minimum, then $dil(G,H) \geq r$. Otherwise, $dil(G,H) \geq r+1$. Hence the proof. □

Corollary 3.2. *Let G be a graph with $\triangle(G) = n-1$ and H be a vertex-transitive graph with $|V(G)| = |V(H)| = n$. Then $dil(G,H) = d$, where d is the diameter of H.*

We now compute the dilation of embedding wheel-like networks into hypertree and prove that the lower bound obtained in Lemma 3.1 is sharp.

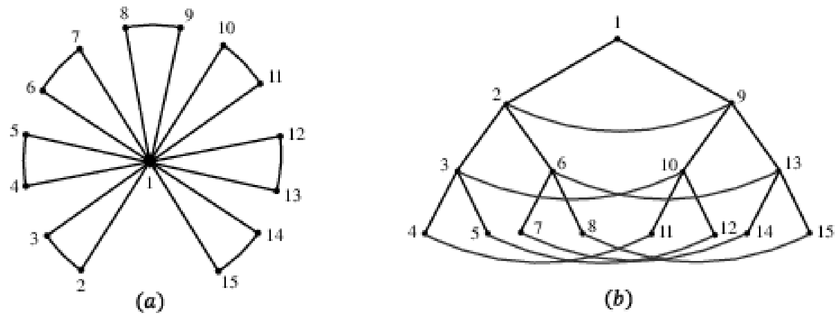

FIGURE 5. (a) Labelling of T_7 (b) Labelling of $HT(4)$.

Theorem 3.3. *Let G be W_n or F_n or $T_{\frac{n-1}{2}}$ or S_n, and H be an l-level hypertree $HT(l)$, where $2^l - 1 = n$, $l \geq 3$. Then $dil(G, H) = r = l - 1$, where r is the radius of H.*

Proof. Since $\triangle(G) = n - 1$ and by Lemma 3.1, we have $dil(G, H) \geq r$. We now prove the equality.

Label the vertices of G as follows:

- hub vertex as 1;
- outer vertices as $2, 3, \ldots, n$ consecutively start with any vertex in the clockwise or anti-clockwise direction, see Fig. 5(a).

Removal of the horizontal edges in hypertree $HT(l)$ leaves a complete binary tree T_l. Label the vertices of T_l using pre-order labeling begin with level 1 vertex, see Fig. 5(b). Let $f(x) = x$ for all $x \in V(G)$, and for $ab \in E(G)$ let $P_f(ab)$ be a shortest path between $f(a)$ and $f(b)$ in $HT(l)$.

Since the hub vertex with label 1 in $V(G)$ is mapped into a vertex $f(1) = 1$ in $V(H)$ is in level 1 gives the minimum eccentricity of H and hence any edge $e = uv \in E(G)$ with either u or v as a hub vertex is mapped into a path $P_f(uv)$ in H with dilation at most $l - 1$, which is nothing but the radius r of H.

We now claim that the outer edges of G are mapped into a path of length at most $l - 1$ in H. Since the graph H is obtained from T_l, the left and right children of any parent node in level $l - 1$ is connected by a path of length 2. By the labeling of pre-order traversal in T_l, for any parent node in level i, $1 \leq i \leq l-2$, the right most vertex of a left node and the right node of a parent node are connected by a path length at most $l - 1$ and hence the dilation of any outer edge in G is at most $l - 1$ in H. Hence the proof. □

Using the same approach, we prove the following result.

Theorem 3.4. *Let G be W_n or F_n or $T_{\frac{n-1}{2}}$ or S_n and H be a l-level sibling tree $ST(l)$ or l-level X-tree XT_l, where $2^l - 1 = n$, $l \geq 3$. Then $dil(G, H) = r = l - 1$, where r is the radius of H.*

3.2 **Congestion.**

In this section, we first obtain the lower bound for congestion of embedding onto wheel-like networks. Then prove that the lower bound obtained is sharp for embedding windmill graphs into circulant graphs. To prove the main result, we need the following result.

Lemma 3.5. *Let G be a graph with $\triangle(G) = n - 1$ and H be a graph with $|V(G)| = |V(H)| = n$. Then $EC(G, H) \geq \lceil \frac{n-1}{\triangle(H)} \rceil$.*

Proof. Since $\triangle(G) = n-1$, there exist a vertex $u \in V(G)$ such that $d(u) = n-1$, where $n = |V(G)|$. Let f be an embedding from $V(G)$ to $V(H)$ and map $f(u) = v$. Let $S = \{e : d(v, w) = 1, w \in V(H)\}$, then for any $e \in S$,

$$EC_f(e) \geq \min\left\{\frac{n-1}{\delta_0}, \frac{n-1}{\delta_1}, \ldots, \frac{n-1}{\delta_n}\right\} = \left\lceil \frac{n-1}{\delta_n} \right\rceil,$$

where δ_i is the degree of a vertex v_i in H with $\delta(H) = \delta_0 \leq \delta_1 \leq \cdots \leq \delta_i \leq \cdots \leq \delta_n = \triangle(H)$, $0 \leq i \leq n$. Thus, there is at least one edge in H with congestion $\left\lceil \frac{n-1}{\triangle(H)} \right\rceil$. Further, for any embedding g of G into H, $EC_g(e) \geq EC_f(e) \geq \left\lceil \frac{n-1}{\triangle(H)} \right\rceil$. Therefore,

$$EC(G, H) \geq \min_g EC_g(e) \geq \min_g EC_f(e) \geq \left\lceil \frac{n-1}{\triangle(H)} \right\rceil.$$

Hence the proof. □

We now compute the edge congestion of embedding windmill graphs into circulant networks and prove that the lower bound obtained in Lemma 3.5 is sharp.

Theorem 3.6. *Let G be a windmill graph $WM_{2^{n-1}}$ and H be a circulant network $H(2^n; \pm\{1, 2^{n-2}\})$, $n \geq 3$. Then $EC(G, H) = 2^{n-2}$.*

Proof. Since $\triangle(G) = n - 1$ and by Lemma 3.5, $EC(G, H) \geq 2^{n-2}$. We now prove the equality.

Label the vertices of G as follows:

- hub vertex as 1;
- pendent vertex as 2^n;
- remaining vertices as $2, 3, \ldots, 2^n - 1$ consecutively start with any vertex such that $(i, i+1)$ are adjacent, where i even and $2 \leq i \leq 2^n - 2$.

Label the consecutive vertices of $H(2^n; \pm\{1\})$ in H in the clockwise sense. Let $f(x) = x$ for all $x \in V(G)$ and for $ab \in E(G)$, let $P_f(ab)$ be a shortest path between $f(a)$ and $f(b)$ in H.

Since H is vertex transitive, map the hub vertex u, which is labeled as 1 in G into any vertex $v = f(u)$ in H. Without loss of generality, the label of v as 1 *i.e.*, $f(u) = f(1) = 1 = v$. Now, we map the edges in G into a path P_f in H using the following algorithm.

- For $(1i) \in E(G)$, let $P_f(1i)$ pass through the outer cycle of H in the clockwise direction, where $2 \leq i \leq 2^{n-2} + 1$;
- For $(1i) \in E(G)$, let $P_f(1i)$ pass through the outer cycle of H in the anti-clockwise direction, where $3 \cdot 2^{n-2} + 1 \leq i \leq 2^n$;
- For $(1i) \in E(G)$, let $P_f(1i)$ pass through an edge, which is labelled as $(1, 2^{n-2}+1)$ followed by the outer cycle of H in the clockwise direction, where $2^{n-2} + 2 \leq i \leq 2^{n-1} + 1$;
- For $(1i) \in E(G)$, let $P_f(1i)$ pass through an edge, which is labelled as $(1, 3 \cdot 2^{n-2} + 1)$ followed by the outer cycle of H in the anti-clockwise direction, where $2^{n-1} + 2 \leq i \leq 3 \cdot 2^{n-2}$.

From the above algorithm, it is easy to see that the edge congestion of each edge in H is at most 2^{n-2}. At this stage, the following edges in H have 2^{n-2} as the edge congestion and we denote the set by $A = \{(1, 2), (1, 2^{n-2}+1), (2^{n-2}+1, 2^{n-2}+2), (1, 2^n)\}$. Now, the remaining edges $(i, i+1)$, $2 \leq i \leq 2^n - 2$ and i is even in $E(G)$ is mapped into a path of length 1 in H and it will not contribute the congestion in any of the edges in A. Hence the proof. □

3.3 **Wirelength**

First, we start with the following definitions.

Definition 3.7. A graph G is hamiltonian if it has a hamiltonian cycle. A hamiltonian graph G is k-fault hamiltonian if $G - F$ remains hamiltonian for every $F \subset V(G)$ with $|F| \leq k$.

Definition 3.8. For every $v \in V$ in G, define $D(v) = \sum\limits_{u \in V} d(u, v)$, where $d(u, v)$ is the distance between u and v in G. A vertex v for which $D(v)$ is minimum is called a median of G.

Theorem 3.9. *Let G be a wheel graph W_n and H be a graph with u as a median. Then $WL(G, H) \geq n - 1 + D(u)$. Equality holds if and only if $H \setminus u$ is hamiltonian.*

Proof. Let u be the hub of W_n. Map u in G to u in H. Since u is a median of H, $D(u) = \sum\limits_{v \in V} d(u, v) = \sum\limits_{i=1}^{k} |N_i(u)|$, $k \leq d$. Suppose $H \setminus u$ is hamiltonian. Map the outer $(n-1)$-cycle in G to a hamiltonian cycle in $H \setminus u$. Thus

$$WL(G, H) = n - 1 + \sum_{i=1}^{k} |N_i(u)|, k \leq d.$$

Conversely, suppose $WL(G, H) = n - 1 + D(u)$. If $H \setminus u$ is not hamiltonian, then the cycle in G cannot be mapped onto a cycle in $H \setminus u$, a contradiction. □

Proceeding in the same way, we have the following result.

Theorem 3.10. *Let G be a fan graph F_n and H be a graph with u as a median. Then $WL(G,H) \geq n-2+D(u)$. Equality holds if and only if $H \setminus u$ contains a hamiltonian path.*

The host graphs in Theorem 3.9 and Theorem 3.10 cover a wide range of graphs. This has motivate us to identify interconnection networks which fall into this category:

Networks	**Justification for 1-fault Tolerance**
Circulant graphs $G(n;\pm S), \{1,2\} \subseteq S \subseteq \{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\}$	1-fault hamiltonian [3]
Generalized Petersen graphs $P(n,m)$	hypohamiltonian/hyperhamiltonian [3, 39]
Augmented cubes AQ_n	pancyclic [40]
Crossed cubes CQ_n	almost pancyclic [41]
Möbius cubes MQ_n	$(n-2)$-fault almost pancyclic [10, 42]
Twisted cubes TQ_n	$(n-2)$-fault almost pancyclic [10, 43, 44]
Twisted n-cubes T_nQ	1-fault hamiltonian [45]
Locally twisted cubes LTQ_n	almost pancyclic [46]
Generalized twisted cubes GQ_n	$(n-2)$-fault almost pancyclic [10]
Odd dimensional cube connected cycle CCC_n	1-fault hamiltonian [46]
Hierarchical cubic networks $HCN(n)$	almost pancyclic [47]
Alternating group graphs AG_n	$(n-2)$-fault hamiltonian [48]
Arrangement graphs $A_{n,k}$	pancyclic [49]
3-regular planar hamiltonian graphs	1-fault hamiltonian [50]
(n,k)-star graphs $S_{n,k}$	at most $(n-3)$-fault hamiltonian [51]
Generalised matching network GMN	$(f+2)$-fault hamiltonian [52]
Fully connected cubic networks $FCCN_n$	1-fault hamiltonian [53]
Tori $T(d_1, d_2, \ldots, d_n)$	fault hamiltonian [54, 55]
1-fault traceable graphs	2-fault hamiltonian [$Definition$ 3.7]

Table 1. List of 1-fault hamiltonian networks.

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