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Shifting the ways prospective teachers frame and notice student mathematical thinking: from deficits to strengths

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Abstract

Noticing the strengths in students' mathematical thinking is a critical skill that teachers need to develop, but it can be challenging due to the prevalence of deficit-based thinking in mathematics education. To address this challenge, a teacher education course was designed to encourage prospective teachers to engage in critical reflection on their own and others' framings of students' thinking and shift their focus towards noticing students' strengths. The study analyzed written responses from the prospective teachers, collected at the beginning and end of the course, to investigate their framing and noticing of students' mathematical thinking. The analysis focused on the aspects of students' thinking that the prospective teachers paid attention to, the stances they took when interpreting students' thinking, and the instructional moves they proposed in response to their thinking. Furthermore, the study established a spectrum of deficit-based and strength-based framings on students' mathematical thinking. This spectrum allowed for the identification of each participant's written noticing responses within a range of possibilities, contributing to a more nuanced understanding of the changes in teachers' framing and noticing of students' thinking over time.

Keywords Deficit thinking · Framing theory · Secondary mathematics teacher education · Student mathematical thinking · Teacher noticing · Mathematical strengths

1 Introduction

Over the past 20 years, the education research literature has placed significant emphasis on the concept of *teacher noticing* (see König et al., 2022). Teacher noticing refers to the capacity to attend, interpret, and respond to classroom events. This ability is particularly critical in mathematics education reform, as it underscores a student-focused, responsive teaching approach (Jacobs & Spangler, 2017). Teachers must be highly attentive to their students' ideas and accurately interpret them to make informed in-the-moment decisions (Mason, 2002; Schoenfeld, 2011). Studies have shown that increased attention to students'

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thinking results in more opportunities for student learning (Jacobs et al., 2007; Kersting et al., 2012; Santagata & Yeh, 2014).

Moreover, tailored professional development programs enable teachers to become more attentive to their students' thinking (van Es & Sherin, 2008). However, the literature raises concerns about the factors that foster changes in noticing (Fernández et al., 2020; Haj-Yahya, 2022; Rotem & Ayalon, 2023). While some studies suggest that changes in noticing are linked to the specificity with which teachers see a phenomenon (van Es, 2011), others suggest that changes are linked to how teachers frame the object of attention (Russ & Luna, 2013).

The concept of framing has become increasingly crucial in understanding teacher noticing and developing effective opportunities for teachers to learn how to notice students' thinking (Louie et al., 2021; Scheiner, 2021; Sherin & Russ, 2014). Deficit-based framing, which views students' thinking as shortcomings or failures, is widespread in mathematics education (Adiredja & Louie, 2020) and reinforces social and educational inequalities (Martin, 2009; Phillips & Chin, 2004; Valencia, 2010). This framing approach obstructs the development of a positive mathematical identity among students (Aguirre et al., 2013). Thus, there is a growing recognition of the necessity for alternatives to deficit-based framing of students' thinking (Crespo, 2000; Frade et al., 2013), particularly in the pursuit of equity in mathematics education (Byun, 2023; Hand, 2012; Nasir et al., 2014; Shah & Coles, 2020; van Es et al., 2022).

One of the alternatives to deficit-based framing is strength-based framing, which considers students' thinking as an asset or resource instead of a weakness or deficit (Bannister et al., 2018; Crespo et al., 2021; Johnson et al., 2022; Jilk, 2016; Kalinec-Craig et al., 2021). This approach highlights the positive contributions students make to the classroom and is critical to reducing the stigmatization of ability, particularly among traditionally disadvantaged and marginalized groups (Adiredja, 2019). However, strength-based framing does not overlook the various qualities in students' thinking or the existence of cognitive or epistemological obstacles that students may encounter when learning mathematics. Instead, it offers an alternative perspective that focuses on identifying and leveraging students' strengths.

The skill of noticing students' mathematical strengths is complex and challenging to learn, especially given that deficit thinking is deeply ingrained in mathematics education (Adiredja & Louie, 2020; Peck, 2021). To address this issue, the author developed a course for prospective secondary mathematics teachers to promote critical reflection on their and others' framing of students' mathematical thinking. The course aimed to shift from a deficit-based to a strength-based approach when noticing students' thinking. This shift is particularly significant for secondary mathematics teachers as research has shown a decline in students' mathematical achievement and motivation upon entering secondary school (Jacobs et al., 2002; Köller et al., 2001).

This paper aims to contribute to the literature on teachers' noticing by examining how prospective teachers frame students' mathematical thinking and develop their noticing of students' mathematical strengths. The study has two main objectives: First, it aims to characterize changes in prospective teachers' modes of attending, interpreting, and responding to students' thinking. Second, the study seeks to establish a spectrum of specific framings of students' thinking that allow for the identification of where participants' noticing responses fall within a range of possibilities.[1]

[1] Preliminary results of this study were presented at the 2022 Annual Meeting of the American Educational Research Association in San Diego, USA, and the 45th Conference of the International Group for the Psychology of Mathematics Education in Alicante, Spain. This paper offers a comprehensive and detailed analysis of the changes in prospective teachers' framing and noticing of students' mathematical thinking, which were not extensively reported on in previous papers. This paper specifically presents and discusses the aspects that prospective teachers attended to, the stances they took in interpreting, and the instructional moves they suggested in responding to students' thinking. Additionally, the paper offers insight into the specific framings of students' thinking, contributing new knowledge to the field.

2 Theoretical framework

The present study is informed by the extensive literature on teacher noticing and framing. A considerable body of literature has focused on understanding teacher noticing (see Amador, 2019; Dindyal et al., 2021; Scheiner, 2016; Stahnke et al., 2016). Noticing is typically defined as the capacity to attend to critical aspects in the classroom, interpret them, and respond appropriately (Jacobs et al., 2010; Kaiser et al., 2017; Sánchez-Matamoros et al., 2019; Stahnke & Blömeke, 2021; van Es & Sherin, 2002). Researchers have studied teachers' noticing through these three components (attending, interpreting, and responding), with a particular emphasis on noticing students' mathematical thinking (Sherin et al., 2011). In this study, attending is defined as identifying noteworthy aspects of students' reasoning or work, interpreting as adopting a specific stance about their understanding, and responding as proposing instructional moves in response to the observed thinking. Thus, it is recognized that noticing is value-laden: what teachers notice depends largely on what they value (Schoenfeld, 2011).

However, the current approaches to teacher noticing tend to overlook the broader, historically and culturally constituted ways in which noticing is organized and shaped (Louie, 2018; Scheiner, 2021). This study thus adopts framing theory (Goffman, 1974) to provide a more comprehensive perspective on noticing as a component of a more extensive activity that sets the context and importance of what is being noticed while also shaping it (Scheiner, 2021). According to Goffman (1974), framings offer "principles of organization" that direct events, particularly social ones, and our participation in them (pp. 10–11). These principles of organization serve as "schemata of interpretation," enabling us to understand how events and activities are perceived, identified, and named, thus attributing significance, structuring our experiences, and directing our actions (Goffman, 1974, p. 21). Framing affects perception by highlighting certain aspects, values, and other considerations, making them more pertinent to the issue than they would appear if framed differently (Hammer et al., 2005). As a result, framing both enables and constrains what we see and how we interpret it.[2]

Framings are not static constructs imposed on a given situation and adopted by participants. Instead, they are co-constructed through social and discursive practices (Greeno, 2009) and are shaped by broader orientations, such as cultural attitudes towards the nature of mathematics and its teaching and learning (Schoenfeld, 2010). In this sense, framings can be seen as socially and culturally pre-structured attunements that orient participants towards certain affordances and constraints, with culture evident in the language, frameworks, and tools employed to identify what to pay attention to (Holzkamp, 1983).

In this study, an integrated view of framing and noticing, as shown in Fig. 1, was adopted following Levin et al. (2009) and Russ and Luna (2013). This view recognizes that the three processes of noticing—attending, interpreting, and responding—shape, and are shaped by, the way teachers frame the object of attention, which is in turn often shaped by broader orientations, such as addressing deficiencies in students' thinking (see Louie et al., 2021; Scheiner, 2021). Thus, framing and noticing often reinforce each other, underscoring the need for critical reflection on framing and its impact on noticing.

[2] In recent years, there has been growing interest in the application of framing theory across various domains of mathematics education research (see e.g., Alvidrez et al., 2023; Louie et al., 2021; Scheiner, 2021; Schou & Bikner-Ahsbahs, 2022).

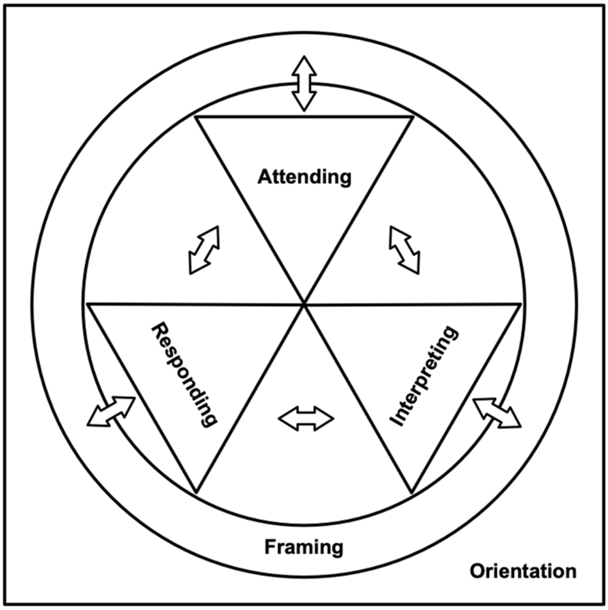


Fig. 1 An integrated view of teacher noticing and framing (adapted from Scheiner, 2022, p. 397). Note. Bidirectional arrows between attending, interpreting, and responding indicate their interrelated nature, which shapes, and is shaped by, how teachers frame the object of attention, shown by bidirectional arrows between framing and noticing. Framing and noticing are embedded within broader socio-cultural orientations

Framings supported or reinforced by socially, culturally, or historically dominant orientations, such as deficit thinking in mathematics education, are often taken for granted and seldom subjected to critical reflection. Disrupting these framings may require extensive and ongoing critical reflection, both at an individual and societal level (Hand et al., 2012). To this end, a mathematics teacher education course was developed to encourage prospective secondary mathematics teachers to critically reflect on their individual and collectively shared framings of students' mathematical thinking, ultimately changing their orientation towards noticing students' strengths.

The study sought to answer the following research questions:

1. What aspects, stances, and instructional moves can be identified among prospective teachers when noticing students' mathematical thinking, and how do their modes of attending, interpreting, and responding change in the mathematics teacher education course?
2. What framings do prospective teachers use when noticing students' mathematical thinking, and how do their framings change in the mathematics teacher education course?

3 Research design and method

In this study, framing theory was adopted as a framework in a teacher education course to promote systematic reflection on individual and shared framings of students' thinking and to support the development of prospective teachers' ability to notice students' strengths. Learning to notice students' strengths was regarded as an enculturation process that involved engaging prospective teachers in social and discursive practices focused on questioning their own and others' framings of students' thinking, as well as exploring alternative framings. This enculturation process was considered crucial in developing reflective practitioners (Brookfield, 1995) who can effectively notice and appreciate the strengths in students' mathematical thinking.

3.1 Design of the mathematics teacher education course

The course was part of a 2-year master's program in mathematics teacher education at a major German university, spanning a full semester and comprising of 14 3-h face-to-face sessions. The course took a *transformative* approach to teacher learning (Mezirow, 2000), which involved critical reflection on one's own and others' practices in noticing students' mathematical thinking through collaborative inquiry.

The prospective teachers, who were the participants, worked together with the course instructor, who was also the author and researcher of the study, towards a shared goal of moving away from deficit-based orientations and towards strength-based orientations in noticing students' mathematical thinking. The instructor's role was to support the prospective teachers' transformation process using methods of critical reflection (Liu, 2015).

The course design and exemplary learning activities are summarized in Table 1, which presents the different phases of critical reflection on framing students' mathematical thinking informed by Liu's (2015) model of critical reflection for transformative learning.

The course initiated a critical examination of established framings of students' mathematical thinking, recognizing that these framings are socially constructed and influenced by cultural contexts. The prospective teachers scrutinized both individual and shared framings, using their written responses to specific noticing tasks regarding students' mathematical

Table 1 Phases, objectives, and exemplary learning activities of critical reflection on framing students' thinking in the teacher education course

Phase	Objective	Learning activity
(1) Unfolding, analyzing, and questioning individual and collectively shared framings	Prospective teachers identify their own and others' framings of student thinking and question the validity of these framings against their own learning experiences	Writing noticing responses on student thinking; analyzing and discussing examples of individual and shared framings of student thinking; sharing personal learning experiences through storytelling
(2) Raising awareness of the social and cultural situatedness of framings	Prospective teachers recognize the social, cultural, and historical conditioning of their individual and shared framings	Consulting and discussing critical readings that question traditional narratives about student thinking
(3) Exploring alternatives in framing student thinking	Prospective teachers explore alternative ways of framing student thinking that highlight student strengths	Reflective and collaborative writing on critical readings with alternative narratives of student thinking as strengths; applying alternative narratives to established ways of framing student thinking
(4) Reflective skepticism	Prospective teachers question the grounds of alternative framings and the claims made for any universal validity of framing student thinking	Developing counter-narratives by collecting counter-examples of student thinking to refute traditional deficit-based narratives
(5) Reflection-based noticing of student thinking	Prospective teachers implement their (new) framings in noticing student thinking	Writing noticing responses on student thinking; sharing and discussing noticing responses
(6) Reflecting on the potential effect of reflection-based noticing	Prospective teachers reflect on the social, affective, and cognitive impact of their (new) framings on student learning	Reflective writing on (new) framings' impact on promoting student mathematical learning and positive identity

thinking (for the specific noticing tasks, see Sect. 3.3), to raise their awareness of deficit orientations that devalue students' knowing and understanding.

To counteract deficit thinking, the prospective teachers engaged in storytelling about their personal experiences of learning mathematics, emphasizing achievements and positive language of mathematical accomplishment. The course also consulted critical writings that questioned the traditional narrative positioning students' thinking as having weaknesses and shortcomings, such as Smith et al.'s (1994) work on reconceiving misconceptions, to recognize the persistence of deficit thinking and explore the social, cultural, and historical dimensions that influence prospective teachers' framings.

Next, prospective teachers sought alternatives to established ways of framing students' mathematical thinking. They engaged in reflective writing on assigned readings, such as Crespo (2000) and Jilk (2016), which provided alternative narratives positioning students' thinking as strengths and assets instead of deficits and obstacles. Collaborative writing activities encouraged "ideological becoming" (Bakhtin, 1981) and empowered prospective teachers to develop and define their own voices for framing students' mathematical thinking.[3] They also developed counter-narratives to refute traditional narratives about students' limitations in mathematics by collecting examples of students' mathematical learning in the literature that omitted or distorted students' strengths and gathering counter-evidence to celebrate and support their success.

Finally, the prospective teachers put their new framings into practice by creating written noticing responses to students' mathematical thinking once more (see Sect. 3.3). They also reflected in writing on how their new framings could potentially enhance students' mathematical learning and cultivate positive mathematical identities among students.

3.2 Participants

Participation in this study was voluntary and extended to all fifteen prospective teachers enrolled in the mathematics teacher education course. Data from nine prospective teachers (hereafter referred to as participants) were included in this study, who provided written consent for their data to be used for research purposes. Participants in this study had obtained a bachelor's degree in mathematics and were currently in the first year of a 2-year master's program in mathematics teacher education. To ensure confidentiality, pseudonyms were used for all participants mentioned in this paper.

3.3 Data collection

While previous studies have frequently used video materials to support teachers learning to notice students' thinking (Amador et al., 2021; Santagata et al., 2021; Walkoe et al., 2020), this study employed a different approach by utilizing students' written reasoning and work. This method allowed participants to thoroughly examine and reflect on students' thinking without the pressure of providing an immediate response, which can be particularly challenging for prospective teachers. However, it is important to acknowledge that this approach did not make the task of noticing students' thinking any

[3] Bakhtin's (1981) concept of "ideological becoming" suggests that the intersection of diverse perspectives, ideas, and voices can have a transformative effect on an individual's thinking, resulting in an internally persuasive discourse.

easier or less complex. For any prospective teacher, knowing what to pay attention to and how to interpret students' mathematical thinking, whether in written or other forms, is a challenging task (Baldinger, 2020).

The noticing tasks were carefully designed to provide insights into participants' noticing of students' thinking about limits, a topic relevant to their future work as secondary school teachers. Figure 2 displays an example of a noticing task used in the study.

Emma's thinking about limits

In a lesson on the limit of sequences, students are presented with the following sequence:

$$a_n = (-1)^n\left(1 + \frac{1}{n}\right)$$

A student named Emma gives the following reasoning when asked if this sequence has a limit.

Teacher:	Does this sequence have a limit?
Emma:	Okay. (Emma places the epsilon strip parallel to the x-axis with the centre at $y = 1$; see Figure A). There are infinitely many points that can be covered by an epsilon strip around the value 1. So 1 is the limit of the sequence.
Teacher:	Can you determine whether -1 is the limit of the sequence?
Emma:	Um. (Emma moves the epsilon strip down so that its centre is aligned with $y = -1$; see Figure B). I mean, it's exactly the same argument: -1 is a limit of the sequence because infinitely many points can be covered by an epsilon strip. So, -1 is a limit of the sequence.

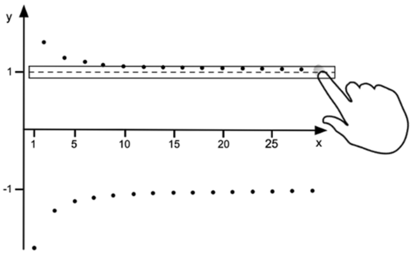


Figure A: Emma places the epsilon strip with its centre at $y = 1$

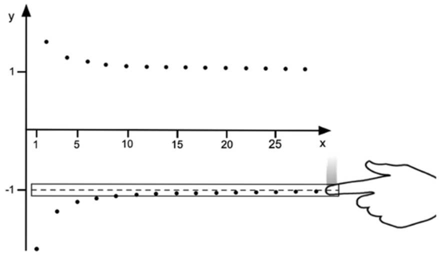


Figure B: Emma moves the epsilon strip so that its centre is aligned with $y = -1$

Noticing tasks

(a) Attend to Emma's thinking: What do you find noteworthy about Emma's mathematical reasoning and work?

(b) Interpret Emma's thinking: What have you learned about Emma's mathematical understanding and how can you interpret her understanding?

(c) Respond to Emma's thinking: Suppose you are Emma's teacher, what and how would you respond to Emma's mathematical thinking?

Fig. 2 Noticing task derived from Emma's case study (based on Roh, 2008, p. 223)

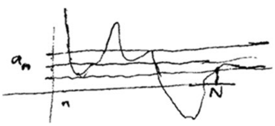
To maximize the effectiveness of the noticing tasks, they were grounded in a diverse range of case studies from the research literature on students' thinking about limits. The chosen case studies aimed to cover a wide array of students' thinking and provide participants with realistic and authentic situations to engage with. Table 2 presents an overview of the case studies used in the noticing tasks, along with a brief description of the student's thinking.

The noticing tasks were intentionally designed to have a comparable level of difficulty and structure, but with variations in the sequences considered and the students' thinking about limits. Each task comprised a brief teacher-student exchange and an illustration of the student's work. Subsequently, participants were presented with a series of questions to guide their noticing of the student's thinking. Following Jacobs et al. (2010), each noticing task included three activities: attending to the student's thinking ("What do you find noteworthy about the student's mathematical reasoning and work?"); interpreting the student's thinking ("What have you learned about the student's mathematical understanding and how can you interpret the student's understanding?"); and responding to the student's thinking ("Suppose you are the student's teacher, what and how would you respond to the student's thinking?"). These noticing activities aimed to determine what aspects of the student's thinking participants identified as noteworthy, what stances they adopted in their interpretation of the student's understanding, and what instructional moves they proposed in response to the student's thinking.

The data for the study consisted of participants' written responses to the noticing tasks, collected during the first and last sessions of the course. In the first session, participants were presented with four noticing tasks (based on the case studies referred to in Table 2) and asked to choose one, providing written responses on their noticing of the student's thinking. This approach aimed to enhance the possibility of each participant finding at least one noticing task of interest to explore in depth. In the final session of the course, participants were given the same noticing task they had previously chosen, and they were asked to provide written responses again.

Note the study's focus was on understanding the specific ways in which participants' framing and noticing changed, rather than merely ascertaining if changes occurred. By employing the same task in both sessions, consistency in content and context was

Table 2 Overview of the case studies and descriptions of student thinking

Student name	Sequence under consideration	Description of student thinking	References
Brian	$a_n = 1$ and $b_n = \begin{cases} 1, n\ even \\ 1 - \frac{1}{n}, n\ odd \end{cases}$	Brian thinks that constant sequences do not have a limit since a limit has to be approached	Roh (2008)
Chris		Chris considers limits in terms of a graphical, generic representation of a converging sequence	Scheiner and Pinto (2019)
Emma	$a_n = (-1)^n\left(1 + \frac{1}{n}\right)$	Emma thinks that the given sequence has two limits	Roh (2008)
Isabell	0.9, 0.99, 0.999, …	Isabell thinks that 1 is the limit of the sequence 0.9, 0.99, 0.999, … but that the numbers 0.999, …, and 1 are different	Przenioslo (2004)

maintained, allowing for a precise comparison of participants' written responses at two different time points. The 13-week gap between the first and final sessions reduced the chances that participants would recall their initial responses, thereby mitigating the impact of task repetition on any observed changes in framing and noticing. The teacher education course was designed to transform participants' framing of students' mathematical thinking through critical reflection, making it reasonable to expect changes in their framing and noticing. The extended time gap and the course's design, combined with the complexity of the noticing tasks, suggest that any observed changes in framing and noticing were primarily due to the teaching intervention.

3.4 Data analysis

The analysis of written noticing responses that participants provided during the first and last sessions of the course was conducted in three phases.

3.4.1 Phase 1: identifying aspects, stances, and instructional moves

The initial step of the study was to standardize and blind the written noticing responses for analysis. The unit of analysis was the noticing statements related to how participants attended to, interpreted, and responded to students' mathematical thinking. The noticing tasks were designed intentionally to capture participants' attending, interpreting, and responding, following the approach by Jacobs et al. (2010). However, when participants gave a single response to the noticing task, their responses were segmented into three units: attending, interpreting, and responding.

Participants' written noticing statements that focused on noteworthy aspects of students' thinking as identified by the participants were classified as *attending*. Statements that went beyond identifying noteworthy aspects and instead referred to the participants' thought processes and sense-making of students' understanding were classified as *interpreting*. These statements revealed a particular stance, such as an evaluative stance, in making sense of students' understanding. Lastly, written noticing statements that proposed instructional moves resulting from observation and interpretation were classified as *responding*. Statements that referred to more than one category were included in all relevant categories.

Next, fine-grained analysis was conducted at the level of each written noticing response unit (attending, interpreting, and responding), involving an in-depth line-by-line examination of the data, similar to knowledge analysis (diSessa et al., 2016). This enabled the development of an initial set of codes to capture what participants paid attention to and how they interpreted and responded to students' thinking.

Regarding the attending units, the focus was on the *aspects* participants identified as noteworthy, such as mistakes made by students in their mathematical work. For the interpreting units, the focus was on the *stances* participants adopted in interpreting students' understanding, such as an evaluative stance evident in their interpretation of students' understanding. Lastly, the focus of the responding units was on the *instructional moves* participants proposed in their responses to students' thinking, such as flagging or correcting errors identified in students' thinking.

Through an iterative process involving constant comparison method (Glaser & Strauss, 1999), a total of seven aspects (see Sect. 4.1), eight stances (see Sect. 4.2), and nine instructional moves (see Sect. 4.3) were identified and categorized as either deficit-based, strength-based, or uncommitted (i.e., neither deficit-based nor strength-based).

Aspects, stances, or instructional moves were categorized as *deficit-based* when participants attended to shortcomings in students' mathematical reasoning; interpreted them as indicative of deficiencies in students' mathematical understanding; or responded in ways that aimed to address, resolve, or avoid such shortcomings. On the other hand, aspects, stances, or instructional moves were categorized as *strength-based* when participants attended to strengths in students' mathematical reasoning; interpreted them as evidence of abilities or resources in students' mathematical understanding; or responded in ways that aimed to enrich, expand, or build upon such strengths. If participants' noticing statements did not fall into either the deficit-based or the strength-based category, they were categorized as *uncommitted*.

3.4.2 Phase 2: coding data for aspects, stances, and instructional moves

The second phase involved using the identified aspects, stances, and instructional moves to code the attending, interpreting, and responding units separately. To ensure the reliability of the coding process, two coders double-coded all attending, interpreting, or responding units for the presence or absence of each aspect, stance, or instructional move. Inter-rater reliability was high, exceeding 80% for all categories (aspects, stances, and instructional moves) and response units (attending, interpreting, and responding). Any disagreements were resolved through discussion.

During the coding process, participants could be found to identify several aspects, adopt different stances, or propose multiple instructional moves within a single response unit. This allowed for the identification of both deficit-based, strength-based, and/or uncommitted aspects, stances, or instructional moves within a single response unit.

The number of participants who identified each aspect, adopted each stance, or proposed each instructional move was recorded, enabling a comparison of their occurrence in the first and last session of the course. This provided an overview of the similarities among participants in terms of their attention, interpretation, and response to students' thinking.

3.4.3 Phase 3: inferring framings on students' thinking

The third and final phase involved analyzing the coded noticing response units (attending, interpreting, and responding) for patterns among aspects, stances, and instructional moves to infer participants' framings. To this end, the entire noticing responses of each participant were examined to obtain more comprehensive patterns for making inferences about their framings. This approach, as detailed by Russ and Luna (2013), required identifying "local patterns" in prospective teachers' noticing.

However, inferring participants' framings from these local patterns was not always a straightforward process and often involved making inferences at different levels of granularity. Local inferences were made by accentuating patterns between the most salient aspects, stances, and instructional moves within participants' attending, interpreting, and responding units. These local inferences provided clues to support the larger inferences about the framings underlying participants' noticing of students' mathematical thinking.

Through the comparison and contrast of individual participants' noticing responses, a spectrum of specific framings was developed (see Sect. 4.4). This spectrum enabled the identification of where each participant's written noticing responses fell within a range of possibilities, contributing to a deeper understanding of the changes in teachers' framing of students' mathematical thinking.

The data analysis approach aimed to balance reductionism and holism. In the first two phases, the unit of analysis consisted of attending, interpreting, and responding statements made by prospective teachers, which corresponded to the way the construct of noticing has been conceptualized. However, it is important to recognize that these processes are interconnected and not isolated within the activity of noticing. To address this, the third phase expanded the unit of analysis to encompass the entire written noticing response provided by prospective teachers. This change was necessary to more accurately capture the framing of prospective teachers, as it could not be inferred from individual processes of noticing alone. By adopting this approach, the analysis facilitated broader inferences and acknowledged that the whole is greater than the sum of its parts.

4 Results

This section provides a summary of what participants attended to in students' reasoning and work (i.e., their aspects), how they interpreted students' understanding (i.e., their stances), and how they responded to students' thinking (i.e., their instructional moves). Furthermore, it presents a spectrum of different framings that could be inferred from participants' written noticing responses. The aspects, stances, and instructional moves, as well as the framings, are categorized according to their orientations: deficit-based, strength-based, and uncommitted. Selected examples from the data are used to illustrate each aspect, stance, and instructional move. In addition, the number of participants who identified each aspect, adopted each stance, or proposed each instructional move in the first and last session of the course is reported. This information demonstrates the extent to which participants shifted from a deficit-based to a strength-based orientation in their noticing of students' thinking.

4.1 Aspects participants highlighted in attending to students' mathematical reasoning or work

Table 3 presents the aspects that participants focused on when attending to students' mathematical reasoning and work, organized according to their orientation. The table displays the number of participants who highlighted each aspect in the first and last session of the course.

There were seven aspects identified, four of which were deficit-based, two were strength-based, and one was uncommitted. In the first session, eight out of nine participants highlighted deficit-based aspects, while only two mentioned strength-based aspects. Three participants attended to the uncommitted aspect. However, in the last session of the course, only three participants pointed out deficit-based aspects, while eight participants identified strength-based aspects in students' reasoning or work. Five participants attended to the uncommitted aspect in the last session.

Table 3 also provides brief descriptions and examples of the aspects participants highlighted, such as pointing out errors or mistakes in students' work, attending to aspects students do not demonstrate, highlighting what is missing or lacking in students' reasoning, identifying misconceptions underlying students' reasoning, recognizing students' abilities and understandings, identifying productive aspects in students' reasoning, and describing students' doing or work in ways that cannot be categorized as deficit-based or strength-based.

It is important to note that participants may have identified more than one aspect in their attending, and a participant may exhibit more than one orientation in a single response.

Table 3 Aspects participants highlighted in attending to students' mathematical reasoning or work

Orientation	Aspect	Description	Example	No. of participants highlighting aspect	
				First session	Last session
Deficit-based				8	3
	Error or mistake	Referring to errors or mistakes in students' reasoning or work	"Chris mistook a function for a sequence."	2	0
	Failure	Attending to aspects students do not demonstrate in their work, including what students do not, cannot, or are unable to do	"Brian does not recognize 1 as the limit of the sequence."	4	2
	Lack or gap	Highlighting what is missing or lacking in students' reasoning or work	"Emma's understanding of the limit concept does not include the uniqueness of the limit, and the difference between limit and cluster point is not present."	4	1
	Misconception	Identifying misconceptions underlying students' reasoning	"Apparently, he [Brian] has the misconception that a sequence can never reach the limit, but can only approach it."	3	1
Strength-based				2	8
	Ability	Attending to aspects students are able and capable of doing, including recognizing students' abilities and understandings	"Brian is able to handle sequences, and by determining the terms of the sequence, … he is able to determine the limit."	2	6
	Strength	Identifying productive aspects in students' reasoning or work, including recognizing strengths in students' understanding	"It is this conception of limit that proved viable in his [Chris] reasoning."	0	4
Uncommitted				3	5
	Student doing	Describing students' doing or work in ways that cannot be categorized as deficit-based or strength-based	"She [Emma] places an epsilon strip around this point and thus comes to the result that 1 is the limit value of the sequence."	3	5

The numbers indicate how many participants highlighted each aspect in their attending unit. Since participants may have identified more than one aspect in their attending, the total number of aspects within a particular orientation (deficit-based, strength-based, or uncommitted) may exceed the total number of participants. A participant was considered to have a deficit-based, strength-based, or uncommitted orientation if they highlighted at least one aspect that corresponds to that orientation. Thus, a participant could exhibit more than one orientation in a single response

4.2 Stances participants adopted in interpreting students' mathematical understanding

Table 4 displays the stances that participants took when interpreting students' mathematical understanding, organized according to their orientation. The table presents the number of participants who used each stance in the first and last session of the course.

Table 4 Stances participants adopted in interpreting students' mathematical understanding

Orientation	Stance	Description	Example	No. of participants adopting stance	
				First session	Last session
Deficit-based				7	3
	Evaluative (negative)	Assessing students' understanding by providing negative qualifications	"Brian has a faulty conception of limit and has difficulty showing the correct course of the sequence."	6	2
	Expectation (conflict or contradiction)	Expecting students to experience cognitive conflict or contradiction	"Considering of the 'second limit'… did not lead to a cognitive conflict; so, Emma either looked at the two subsequences independently or, more likely, relied on her misunderstanding when she should have encountered a contradiction."	2	1
	Normative	Judging students' understanding against the teacher's own understanding or a pre-defined standard	"She [Emma] does not have sufficient knowledge about the properties of limits, especially in comparison to the curriculum standards."	2	0
Strength-based				2	8
	Evaluative (positive)	Assessing students' understanding by providing positive qualifications	"Emma's idea that a limit exists if there are infinitely many points [of the sequence] covered [by the epsilon strip] is adequate for converging sequences."	2	5
	Interpretative (asset-based or resource-based)	Interpreting students' understanding as an asset or resource	"Since she [Emma] knows the first part of the definition, that is, that there are infinitely many points within the epsilon environment, she is able to recognize the limit of sequences that have only one cluster point."	0	2
	Interpretative (in their own right)	Interpreting students' understanding as valuable and useful for the given context or task	"This idea [that a limit exists when infinitely many points are bounded by an epsilon strip] has its merits in itself."	0	4
Uncommitted				3	5
	Assumption or inference	Making assumptions or inferences about students' understanding	"At this point, it remains open whether Chris means a line consisting of individual points or a straight line. Given his statements about the sequence, however, it can be assumed that he understands a sequence consisting of individual points."	1	4

Table 4 (continued)

Orientation	Stance	Description	Example	No. of participants adopting stance	
				First session	Last session
	Interpretative (non-evaluative)	Interpreting students' understanding without evaluating it	"Brian understands the limit as a number and has the idea that a sequence converges to a value when its sequence terms approach that value monotonically."	2	2

The numbers indicate how many participants adopted each stance in their interpreting unit. Since participants may have adopted more than one stance in their interpreting, the total number of stances within a particular orientation (deficit-based, strength-based, or uncommitted) may exceed the total number of participants. A participant was considered to have a deficit-based, strength-based, or uncommitted orientation if they adopted at least one stance that corresponds to that orientation. Thus, a participant could exhibit more than one orientation in a single response

A total of eight stances were identified, which participants adopted when interpreting students' mathematical understanding. Of these stances, three were deficit-based, and seven of the nine participants used them in the first session, while only three participants used them in the last session. Three stances were strength-based, and only two participants used them in the first session, while eight participants used them in the last session. Two stances were uncommitted, and three participants used them in the first session, while five participants used them in the last session.

Table 4 includes brief descriptions and examples of the stances participants adopted, such as positively or negatively evaluating or assessing students' understanding, expecting cognitive conflicts or contradictions, judging students' understanding against a standard, interpreting students' understanding as an asset, interpreting students' understanding in their own right, making assumptions, and interpreting students' understanding without evaluating it.

It is important to note that participants may have adopted more than one stance in their interpreting, and a participant may exhibit more than one orientation in a single response.

4.3 Instructional moves participants proposed in responding to students' mathematical thinking

Table 5 summarizes the instructional moves that participants suggested when responding to students' mathematical thinking, organized according to their orientation. The table presents the number of participants who proposed each instructional move in the first and last session of the course.

A total of nine instructional moves were identified, with four being deficit-based, three being strength-based, and two being uncommitted. In the first session, eight out of nine participants proposed deficit-based moves, one participant proposed a strength-based move, and four participants proposed uncommitted moves. In the last session, two participants suggested deficit-based moves, seven participants proposed strength-based moves, and three participants suggested uncommitted moves in responding to students' mathematical thinking.

Table 5 Instructional moves participants proposed in responding to students' mathematical thinking

Orientation	Instructional move	Description	Example	No. of participants proposing instructional move	
				First session	Last session
Deficit-based				8	2
	Confronting or challenging misconceptions	Confronting or challenging students' misconceptions, including creating cognitive conflict to promote conceptual change	"I would present her [Emma] with a selection of sequences that would allow her to question her misconception."	2	0
	Flagging or correcting errors	Flagging or correcting students' mistakes or errors	"It is important to resolve Emma's erroneous idea that there can be two limit values for a sequence. This should be accomplished by going back to the definition of limit."	3	0
	Preventing obstacles or misunderstanding	Proposing instructional practices designed to prevent cognitive or conceptual obstacles and avoid the build-up of misunderstandings	"To avoid misunderstanding, I would introduce Emma to sequences for which, even if an infinite number of sequence terms lie within an epsilon strip, there is also an infinite number of terms outside the epsilon strip."	4	1
	Redirecting understanding	Redirecting students' understanding, including offering an alternative understanding	"I would offer Brian a different definition or representation of the limit because he doesn't seem to be coping with the representation used in class so far."	5	1
Strength-based				1	7
	Accessing understanding	Accessing students' understanding, including creating opportunities for continued noticing of students' thinking or for uncovering strengths in students' understanding	"From Chris' representation, it is not entirely clear how the sequence continues after the sequence elements are within the epsilon strip he drew. Therefore, I would have Chris determine the limit of a constant sequence to see if his ideas about limit allow it to be reached."	0	4

Table 5 (continued)

Orientation	Instructional move	Description	Example	No. of participants proposing instructional move	
				First session	Last session
	Extending or building upon understanding	Extending or expanding students' understanding or building on it	"I would go back to the second example and ask Brian to explain how he chose the epsilon environment. Since he has already applied this correctly, I would ask him about a sequence that has infinitely many points inside the epsilon environment and finitely many outside."	1	6
	Positive reinforcement	Providing positive reinforcement, such as praise	"From your drawing, we can see that you [Chris] already understand many different representations of the limit concept. I can see that very well in your drawing."	0	3
Uncommitted				4	3
	Clarifying student work or understanding	Asking clarifying questions to better understand students' work or ideas (rather than creating opportunities to further notice students' thinking)	"What are these two lines around the limit?"	1	3
	Giving a general response	Responding to students' work or understanding in general terms without making connections to specific understandings or areas of deficits or strengths	"He [Chris] should not only learn the definition of the limit concept and how to do the calculations, but also a visual understanding of the limit concept."	4	0

The numbers indicate how many participants proposed each instructional move in their responding unit. Since participants may have proposed more than one instructional move in their responding, the total number of instructional moves within a particular orientation (deficit-based, strength-based, or uncommitted) may exceed the total number of participants. A participant was considered to have a deficit-based, strength-based, or uncommitted orientation if they proposed at least one instructional move that corresponds to that orientation. Thus, a participant could exhibit more than one orientation in a single response

Table 5 includes examples of the instructional moves participants suggested, such as challenging students' misconceptions, correcting their errors, proposing instructional practices designed to prevent cognitive obstacles, redirecting their understanding, creating opportunities for accessing their understanding, extending their understanding, using positive reinforcement, clarifying their work, and giving a general response not necessarily connected to their thinking.

It is important to note that participants may have proposed more than one instructional move in their responding, and a participant may exhibit more than one orientation in a single response.

4.4 Participants' framings of students' mathematical thinking

To infer participants' framings based on their attention, interpretation, and response to students' thinking, the patterns of the identified aspects, stances, and instructional moves were analyzed at the level of individual participants' written noticing responses. To illustrate this approach, consider the written noticing responses provided by a participant, named Tasha, in the first and last sessions of the course (see Appendix Table 7).

In the first session, Tasha's written response demonstrated a deficit-based framing of the student's thinking. Firstly, Tasha identified a misconception underlying the student's argument about the existence of limits and explicitly highlighted it by stating, "Emma's argument is based on the misconception that a limit exists if there is an epsilon strip in which an infinite number of terms of the sequence lie." This suggests that Tasha considered the misconception as the root cause of the student's misunderstanding. Tasha also mentioned that the student "resorted to a misconception" and "misunderstood that if an infinite number of terms lie inside the epsilon strip, only a finite number lies outside," reinforcing Tasha's view that the student's reasoning was flawed. Secondly, Tasha commented on the absence of a cognitive conflict in the student's thinking ("did not lead to a cognitive conflict") and expected the student to encounter a contradiction ("should have encountered a contradiction"). Thus, Tasha implied that the student's reasoning should have led to a contradiction. Finally, Tasha proposed introducing sequences to the student that would challenge her current understanding and "lead to a contradiction of Emma's misconception." This suggests that Tasha considered the student's misconception as something that required to be actively addressed through targeted instruction. Tasha viewed herself as playing an active role in addressing the student's misconception, stating "To avoid misunderstanding, I would introduce Emma to sequences." Overall, Tasha's response reflected a deficit-based framing of the student's thinking, where she considered the misconception as the primary issue that needed to be addressed through targeted instruction.

In contrast to Tasha's deficit-based framing of the student's mathematical thinking during the first session, her written noticing response during the last session of the course reflected a more strength-based approach. Firstly, Tasha's response demonstrated careful attention to the substance and strengths of the student's mathematical reasoning. Specifically, Tasha acknowledged that the student had correctly identified a necessary condition for the existence of a limit and consistently applied her understanding, stating "Emma has already understood that the inclusion of an infinite number of terms by an epsilon strip is a necessary condition for the existence of a limit." This indicates that Tasha valued the student's reasoning as valuable in its own right. Secondly, Tasha interpreted the student's understanding of limits as coherent ("Emma's argument is very consistent with her understanding of the limit"), noting that the student had "applied her understanding from presumably familiar contexts in which this condition has already been shown to be sufficient," and had a clear grasp of the notion of infinity. This interpretation demonstrates Tasha's appreciation of the student's existing understanding as a resource to be built upon. Thirdly, Tasha suggested further exploration of other alternating sequences to "realize" the uniqueness of the limit, indicating that she saw the student's thinking as valuable and worthy of further cultivation. Tasha explicitly noted that the student's understanding of limits could be "expanded by exploring the meaning of her definition," suggesting that the student's thinking had potential for growth and development. Taken together, these pieces of evidence suggest that Tasha framed the student's thinking as valuable in its own right, to be cultivated. She paid attention to the student's strengths, interpreted the student's understanding as coherent, and encouraged the student to explore new ideas and expand her thinking.

By analyzing the patterns between various aspects, stances, and instructional moves, as well as by comparing individual participants' noticing responses, a spectrum of specific framings was developed. The spectrum includes three deficit-based framings, three strength-based framings, and one uncommitted framing (see Table 6).

The three deficit-based framings include viewing students' mathematical thinking as flawed and in need of correction, based on misconceptions that need to be challenged, or incomplete and in need of supplementation. Teachers who adopt these deficit-based framings tend to focus on identifying students' errors, misconceptions, or knowledge gaps, and treat them as obstacles to learning. They may attempt to correct students' mistakes, challenge their misconceptions, or provide instruction to fill gaps in their knowledge.

The three strength-based framings include viewing students' mathematical thinking as a capability to be fostered, valuable in its own right and to be cultivated, or a resource to build upon. Teachers who adopt these strength-based framings tend to concentrate on identifying students' abilities, substance, and strengths, and use them to enhance students' learning. They may provide opportunities for fostering students' abilities, demonstrating their strengths, and building upon them to nurture their mathematical thinking and learning.

The uncommitted framing views students' mathematical thinking as a state of becoming, without any judgment or evaluation. Teachers who adopt this framing tend to describe students' thinking as a process, without interpreting it as flawed or valuable. They may ask clarifying questions to gain a better understanding of students' thinking.

It is important to note that the spectrum of framings is not exhaustive, nor does it represent all possible variations in how teachers may frame students' thinking. However, it provides a useful representation of the variations in framings that emerged from the participants' written noticing responses, allowing for a nuanced understanding of each participant's perspective within a broader range of possibilities.

Table 6 Spectrum of specific framings on students' mathematical thinking

Orientation	Framing	Evidence
Deficit-based	Students' mathematical thinking is flawed and in need of correction	Attend to errors or mistakes in students' thinking, interpret them as flawed understanding, and correct them
	Students' mathematical thinking is based on misconceptions that need to be addressed or challenged	Attend to misconceptions in students' thinking, interpret their understanding as based on those misconceptions, and address or challenge them
	Students' mathematical thinking is incomplete and in need of supplementation	Attend to gaps in students' knowledge, interpret them as a lack of understanding, and supplement them
Strength-based	Students' mathematical thinking is a capability to be fostered	Attend to what students can do and are capable of, interpret it as evidence of their abilities, and foster it
	Students' mathematical thinking is valuable in its own right and to be cultivated	Attend to the substance of students' thinking, interpret their understanding on its own term, and cultivate it
	Students' mathematical thinking is a resource to build upon	Attend to students' mathematical strengths, interpret them as resources or assets, and build upon them
Uncommitted	Students' mathematical thinking is a state of becoming	Describe students' thinking as a state of being, interpret their understanding as a state of becoming, and ask clarifying questions if needed

For instance, Tasha's written noticing response during the first session reflects a framing of the student's thinking as based on misconceptions that need to be addressed or challenged, while also highlighting elements that suggest the student's thinking is incomplete and in need of supplementation. Similarly, Tasha's written noticing response during the last session frames the student's thinking as valuable in its own right and worthy of cultivation, but also acknowledges that the student's thinking serves as a resource to build upon. Thus, participants' framings may lean towards one or more of the seven identified framings.

In the first session, seven out of the nine participant's written noticing responses inclined towards deficit-based framings, while the remaining two manifested a tendency towards strength-based framings. In contrast, the written noticing responses provided by participants in the last session tended to showcase strength-based framings of students' mathematical thinking, except for two written noticing responses that exhibited more characteristics of the uncommitted framing.

5 Discussion

The analysis of the written noticing responses revealed that participants purposefully shifted their focus from deficit-based to strength-based orientations when noticing and framing students' mathematical thinking. This shift was apparent in four different ways.

Firstly, participants changed the aspects of students' thinking they attended to. They moved away from identifying students' shortcomings and towards highlighting their strengths, while paying less attention to the former and focusing more on the latter. For instance, in the first session, Tasha highlighted a "misconception" and problematic aspects evident in the student's reasoning about the limit of the given sequence (e.g., highlighting that the student uses "the inclusion of an infinite number of terms in an epsilon strip as the sole criterion"). However, in the last session, Tasha focused on the student's productive aspects in her reasoning, acknowledging the student's understanding that an infinite number of terms in an epsilon strip are a necessary condition for the existence of a limit. Tasha disregarded some of the problematic aspects in the student's reasoning that she had previously highlighted (e.g., that the student mistakenly believed that an infinite number of terms in an epsilon strip is sufficient for the existence of a limit), suggesting that she actively moved away from attention to deficits and towards attention to strengths. Furthermore, the participants no longer mentioned many of the shortcomings in the students' thinking that they had noted in the first session. This suggests that participants did not simply add a new focus to their existing attention, but instead, with the increased focus on students' mathematical strengths, they explicitly shifted away from the deficiencies in students' thinking.

Secondly, participants changed the stances they adopted when interpreting students' understanding. They moved away from their previous tendency to adopt deficit-based stances and towards adopting strength-based stances. However, they also went beyond simply assessing or judging students' thinking towards interpreting it in its own right, and focused on the context in which it was useful or viable. For instance, Tasha moved from negatively evaluating the student's understanding (e.g., "Emma misunderstood") and assuming contradictions in her reasoning (e.g., expecting a cognitive conflict), to interpreting the student's understanding on its own merit (e.g., "Emma's argument is very consistent with her understanding of the limit") and considering the contextual nature of her understanding. Interestingly, several participants continued to adopt an evaluative stance, even as they shifted from negative to positive qualifications of students' thinking.

Additionally, some participants were more cautious in their interpretations and made explicit their uncertainty about explaining students' understanding, which was evident in their increase in making assumptions or inferences.

Thirdly, participants changed the instructional moves they suggested in response to students' thinking. Initially, their suggestions aimed at addressing or overcoming deficits and weaknesses in students' thinking. However, in the end, they proposed instructional moves that aimed at enriching, extending, or building upon students' thinking. For instance, Tasha initially proposed challenging the student's misconception by introducing sequences that "contradict" it, leading to "reconsider" her understanding and promoting conceptual change. Later, she proposed providing the student with a learning opportunity to "expand" her understanding, such as exploring the meaning of the student's definition and discussing the difference between limit value and cluster point. Importantly, participants not only moved away from a general tendency to address or manage problematic aspects in students' thinking but also focused on creating opportunities to better access students' understanding, uncover strengths in their thinking, and further observe and explore their reasoning. For example, several participants suggested probing questions to understand students' thinking better or to elicit further strengths. Some also recommended collaborative learning activities that could help students learn from each other and build on their collective strengths.

Finally, participants also changed the ways in which they framed students' thinking. To visualize this shift in framing, their written noticing responses from the first and last sessions were located within the spectrum of framings, as depicted in Fig. 3. Notably, the results showed a discernable trend of participants moving from deficit-based to strength-based framings, with one exception. This exception was a participant who initially took an evaluative approach in interpreting students' understanding in terms of ability but later shifted towards a more descriptive approach with fewer assessments of the quality of students' thinking.

Four of the participants showed a clear shift towards strength-based framings in the last session, which contrasted with their deficit-based framings in the first session of the course. These prospective teachers displayed a response that was somewhat opposite to their initial one. For example, Tasha's later framing of students' mathematical thinking as valuable in their own right and worth cultivating contrasts with her initial tendency to frame students' mathematical thinking as based on misconceptions that needed to be addressed.

However, the other three participants who also shifted away from deficit-based framings demonstrated less of a corrective response in the last session but rather a transformative one. For instance, one of the participants repeatedly indicated that the student's thinking did not meet certain standards or expectations; however, the participant reinterpreted the deviation as a different way of thinking that could be valuable in its own right, instead of viewing it as a flaw that needed correction, as the participant initially did. Therefore, it appears that strength-based framings are not necessarily an antidote to deficits; they can also emerge from deficit orientation as a form of transformation.

The findings presented in this paper offer meaningful insights into how prospective teachers can shift their orientation from deficits to strengths in framing and noticing students' thinking. However, it is important to note several limitations. Firstly, due to time constraints in the teacher education course, data on only one noticing task could be collected from each prospective teacher, which may not account for variation in framing students' thinking within individual participants. It is not assumed participants have a fixed or single framing in noticing students' mathematical thinking. Framing is dynamic and context-sensitive, and may vary from context to context (Hammer et al., 2005). Although the present study identified specific framings that participants displayed in their written responses, an individual (prospective) teacher may use different framings or move

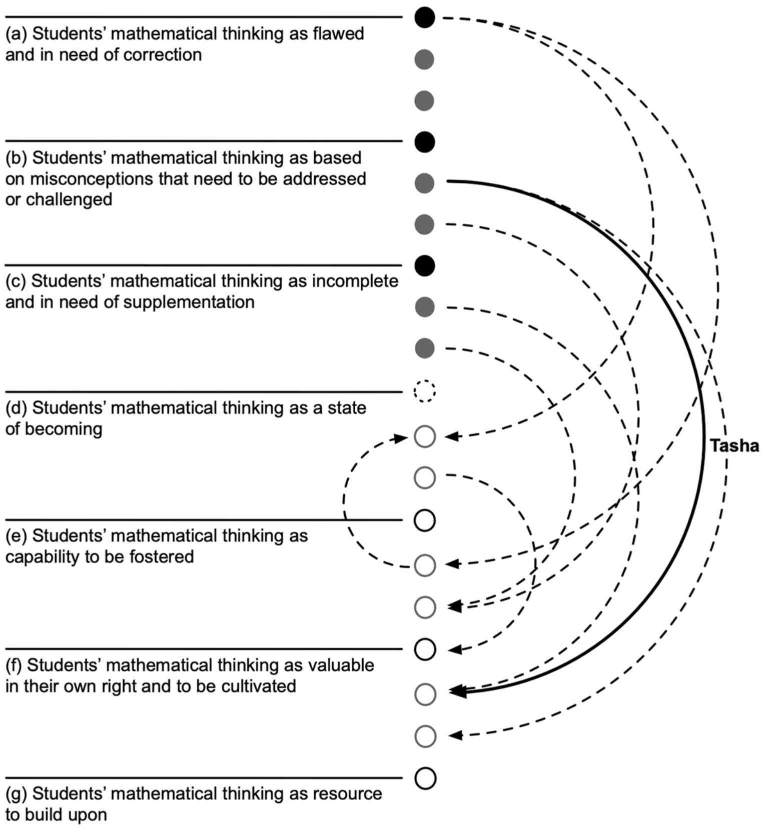


Fig. 3 Participants' shifts in framing students' mathematical thinking (adapted from Scheiner, 2022, p. 399). Note. Black circles represent specific framings, and gray circles show tendencies towards those framings. Filled circles denote deficit-based framings (**a**, **b**, and **c**), while the dotted circle represents the uncommitted framing (**d**), and unfilled circles indicate strength-based framings (**e**, **f**, and **g**). The dashed lines depict changes in participants' framings, with each dashed line referring to one participating and the arrow indicating the direction of the change. The bold line shows the shift in Tasha's framing

between them depending on context and purpose. Future research should investigate the variation in framing students' thinking within individual (prospective) teachers and the fluidity in which (prospective) teachers move between different framings, which would significantly advance the field.

Secondly, noticing students' thinking in written student work may differ from noticing it in a video or in the classroom (Goldsmith & Seago, 2011). In this study, participants had ample time to reflect on written student work, which contrasts with in-the-moment noticing and differs from most studies where participants respond directly to what they notice. This approach, however, allowed participants to explore and reflect more deeply on the substantive aspects of students' thinking, rather than merely addressing superficial aspects. The limited details available in the written student work likely shaped participants' framing and noticing of students' thinking. Nonetheless, these limitations could motivate future research into how (prospective) teachers frame and notice students' mathematical strengths in the moment and in relation to other mathematical subject matter.

Thirdly, it is important to recognize that teachers' noticing skills and profiles may differ based on their level of experience or expertise (see Bastian et al., 2022; Jacobs et al., 2023). While this study shows that prospective teachers can be highly competent at noticing the substance of students' thinking, it is likely that teachers with varying levels of experience or expertise use different aspects, stances, instructional moves, and framings. To better understand these nuances, future research could incorporate larger sample sizes or aggregate data from multiple studies, with a particular focus on encompassing teachers with varying levels of experience or expertise. This approach would enable a more thorough exploration of teacher framing and noticing, potentially leading to the identification of a wider spectrum of framings.

6 Conclusions

To promote positive learning experiences for all students, researchers and educators have increasingly advocated for strength-based approaches to mathematics teaching and learning, moving away from deficit-based perspectives (Boaler & Greeno, 2000). In this study, the aim was to support prospective teachers in shifting their orientation towards strengths when framing and noticing students' mathematical thinking. This transformational process required time and guidance for engaging prospective teachers in critical reflection on their individual and shared framings of students' thinking.

The analysis of written noticing responses from prospective teachers revealed notable shifts in their framing and noticing of students' mathematical thinking over the course of the study, reflecting a move towards a strength-based orientation. These shifts were evident through the markedly different aspects, stances, and instructional moves identified in the prospective teachers' written noticing responses. These findings offer valuable insights into the many ways in which prospective teachers attend to, interpret, and respond to students' thinking, and highlight important nuances in framing students' thinking.

The results from this study with nine prospective teachers demonstrate a proof of principle that a change of framing is possible within just one mathematics teacher education course, and how it was achieved. Additionally, the spectrum of framings that emerged from the study provided a valuable framework for placing each participant's written noticing responses within a range of possibilities, resulting in a more nuanced understanding of the changes in teachers' framing and noticing of students' thinking over time. By identifying and describing different ways of framing students' mathematical thinking, this spectrum of framings can serve as a useful tool for moving beyond deficit narratives and towards more strength-based approaches to mathematics teaching and learning. In combination with further research, this spectrum of framings has the potential to promote greater awareness and acknowledgment of students' strengths and unique perspectives, helping to create a more inclusive and supportive learning environment for all students.

Specifically, the coding scheme and spectrum of framings hold potential as valuable "epistemic forms" (Collins & Ferguson, 1993) for both research and practice. As a target structure that provides a framework with constraints and slots for filling information in a specific way, these forms can guide future inquiry and direct attention to particular aspects, stances, and instructional moves that may otherwise go unnoticed. By utilizing these forms, researchers and practitioners can adopt a structured approach to inquiry and teacher learning, refining and enhancing them over time to more effectively support teachers in adopting strength-based approaches in their framing and noticing of students' mathematical thinking.

Appendix

Table 7 Excerpts from Tasha's written noticing responses given in the first and last session of the course

	First session	Last session
Attending	"… Emma's argument is based on the misconception that a limit exists if there is an epsilon strip in which an infinite number of terms of the sequence lie. She found such a point, namely the value 1. She placed an epsilon strip around this point and found that an infinite number of terms of the sequence lie within this epsilon strip. From this, she concluded that 1 is the limit of the sequence. When asked if -1 is also a limit of the sequence, she seemed to think again for a moment. Then she applied her misconception again and determined that -1 must also be a limit, using the same argument as for the value 1."	"… She [Emma] directed her attention exactly to this subsequence [of even sequence terms] and concluded that consequently, an infinite number of sequence terms will lie in an epsilon strip around the value 1. From this, she concluded that 1 must be the limit, since, according to her understanding of limits, this is precisely the condition for there to be a limit. An analogous consideration of the odd series terms did not take place at first. Only when the teacher asked her if -1 was also a limit, she considered the subsequence of odd sequence terms in the same way and, consistent with her first consideration, also argued that -1 was (another) limit of the sequence."
Interpreting	"By using the inclusion of an infinite number of terms in an epsilon strip as the sole criterion, Emma resorted to a misconception. … Emma misunderstood that if an infinite number of terms lie inside the epsilon strip, only a finite number lies outside. Considering of the 'second limit', which also includes an infinite number of terms in another epsilon strip, did not lead to a cognitive conflict; so, Emma either looked at the two subsequences independently or, more likely, relied on her misunderstanding when she should have encountered a contradiction."	"Emma has already understood that the inclusion of an infinite number of terms by an epsilon strip is a necessary condition for the existence of a limit, and has applied her understanding from presumably familiar contexts in which this condition has already been shown to be sufficient. Since she is arguing in terms of 'infinitely many points,' she obviously understands the notion of infinity. Emma has not commented on the existence of sequence terms that lie on the 'other' side of the limit, so this does not seem to contradict Emma's understanding of the limit either. Emma's argument is very consistent with her understanding of the limit. She does not make the criterion for determining a limit dependent on the scenario but applies it consistently in determining the 'second limit -1."
Responding	"To avoid misunderstanding, I would introduce Emma to sequences for which, even if an infinite number of sequence terms lie within an epsilon strip, there is also an infinite number of terms outside the epsilon strip. I would then discuss the uniqueness of a limit, which should lead to a contradiction of Emma's misconception. So, Emma can reconsider that a limit exists if there is an infinite number of terms inside an epsilon strip and only a finite number of terms outside it."	"It would be important for her further learning to discuss the uniqueness of the limit. Other alternating sequences could be used for this purpose. Perhaps Emma will realize that -1 and 1 cannot both be the limit value of the sequence. Also, it should be discussed what the difference is between limit value and cluster point. Emma's conception of limit can then be expanded by exploring the meaning of her definition [that a limit exists if there is an infinite number of sequence terms within a given epsilon strip]."

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Data availability Data used in this study are not publicly available due to informed consent provided by participants. Requests to access the dataset should be directed to the author.

Declarations

Conflict of interest The author declares no competing interests.

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