LISTENING TO CHILDREN'S EXPLANATIONS OF FRACTION PAIR TASKS: WHEN MORE THAN AN ANSWER AND AN INITIAL EXPLANATION ARE NEEDED[1]

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Research has shown that children can offer the right answer but have mathematically incorrect reasoning (Clements & Ellerton, 2005). One-to-one task-based interviews enabled the researchers to engage in observational listening (Empson & Jacobs, 2008) and uncover the mathematical strategies used by Grade 6 students in fraction pair tasks. Some students' answers and initial explanations were similar, but different strategies were revealed by further questioning: the correct strategy of benchmarking or the misconception of gap thinking.

Introduction

Careful listening is essential for good teaching. If we want to know what to teach next, we need to know the mathematical thinking of individual students. Teachers cannot assume that a correct answer indicates misconception-free thinking. To teach within the framework of constructivism, we need

- specialised content knowledge
- observational listening skills, and
- classroom norms that value mathematical explanations

The analysis of specific strategies for the comparison of the relative size of two fractions, $^4/_5$ and $^4/_7$, illustrates the complexity of assessing children's fraction understanding. Two strategies, the gap thinking misconception and the mathematically correct strategy of benchmarking both gave the same answer ($^4/_5$ is larger) and had similar initial explanations. Responsive teachers know that the different strategies exist, have the listening skills to determine which is being used, and create classroom norms which value explanations from students that enable peers and the teacher to engage with their mathematical thinking.

Review of the literature

A co-ordinated fraction understanding encompasses several contexts. Kieren's model for understanding rational number knowledge identified *four* sub-constructs (measure, quotient, operator, and ratio) and *three* underlying concepts (partitioning, equivalence,

[1] The authors wish to thank Anne Roche and Doug Clarke for their help in developing tasks and record sheets, and double coding the data.

and unit-forming) (Kieren, 1988, 1992). These could be engaged with on *four* levels, ethnomathematic, intuitive, technical-symbolic, and axiomatic-deductive. We use the term the *four-three-four* model to distinguish it from Kieren's five-part model (1980). The concepts that concern us in this paper are the measure sub-construct, the concept of equivalence, and intuitive understandings.

Fraction tasks in the primary school considered part of the measure sub-construct include, number lines (Lamon, 1999), area and length diagrams (Kieren, 1992), and fraction pair comparisons (Ni, 2000). The relative size of fractions has been called *order* in order and equivalence studies. *As many as* equivalence, (multiplicative) partitioning, is used to generate equivalent fractions. For example, $^4/_8$ is equivalent to $^1/_2$. *As much a,s* equivalence (additive) unit-forming, is used to combine fractions (Kieren, 1992). For example, $^5/_8$ equals $^1/_2$ plus $^1/_8$. *Intuitive* approaches were planned mathematical activity, firmly located in a context developed from schooled or taught knowledge (Kieren, 1988).

Strategies for comparing the relative size of fractions include both correct strategies and misconceptions. Correct strategies include residual thinking, benchmarking and common denominators. Misconceptions, often inappropriate generalisations, include gap thinking, higher or larger numbers, and bigger denominator indicates bigger fraction thinking. Knowledge of these strategies forms part of a teacher's specialised content knowledge. This knowledge of mathematics and knowledge of students is necessary for pedagogical content knowledge (Hill, Ball, & Shilling, 2008).

The *residual thinking* strategy has been observed in the comparisons of fractions, such as $^5/_6$ and $^7/_8$ that are both one piece away from the whole (see, for example, Clarke & Roche, 2009; Cramer & Wyberg, 2009; Post, Behr, & Lesh, 1986). Students reason correctly that an eighth away from the whole is closer than one sixth away from the whole and so $^7/_8$ is the larger fraction.

Using half as a benchmark was a strategy that children could use when they combined the (additive) unit-forming aspect of equivalence, $^5/_8$ is as much as $^1/_2$ and another piece, and the (multiplicative) partitioning aspect of equivalence, $^4/_8$ is as many as $^1/_2$. Benchmarking had been reported in Australia (Clarke & Roche, 2009), and had been called the transitive or reference point strategy in the United States (Behr & Post, 1986; Post et al. 1986; Post & Cramer, 1987). For example, $^5/_8$ is larger than $^3/_7$ because $^3/_7$ is less than a half and $^5/_8$ is more than a half.

Gap thinking has been observed in Australia (Clarke & Roche, 2009; Gould, 2011; Mitchell & Horne, 2010; Pearn & Stephens, 2004) and was one of four whole number dominance strategies described by Post & Cramer, (1987) and observed in recent studies (Cramer & Wyberg, 2009). Children with this misconception looked at the numerical difference between the numerator and denominator and chose the fraction with the smallest gap as the largest fraction. For example, in a study of 323 Grade 6 students, 35.6% of the incorrect answers comparing $^3/_4$ and $^7/_9$ demonstrated gap thinking: $^3/_4$ was larger because it had a gap of 1 while $^7/_9$ had a gap of 2 (Clarke & Roche, 2009). Nearly 30% of Grade 6 students incorrectly said that $^5/_6$ and $^7/_8$ were equivalent because two fractions, both with a "gap" of one, were the same, instead of using a correct strategy such as residual thinking (Clarke & Roche, 2009). In a separate study, 50% of Grade 6 students used gap thinking on this same pair to conclude that the fractions were the same, and the misconception was shown to emerge at the same time as early equivalence understanding (Mitchell & Horne, 2010).

In these examples, $^3/_4$ and $^7/_9$, and $^5/_6$ and $^7/_8$, gap thinking gives the wrong answer with incorrect reasoning. However, in fraction pairs such as $^4/_5$ and $^4/_{7,}$ gap thinking gives the right answer for the wrong answer: $^4/_5$ is larger because the gap of 1 is less than the gap of 3 in $^4/_{7.}$ A matrix of answer and explanation types (Clements & Ellerton, 1995, 2005) has been elaborated as:

- correct answer, correct mathematical thinking;
- correct answer, incorrect reasoning;
- incorrect answer, mathematically correct not fully executed/partially correct reasoning; and
- incorrect answer, incorrect reasoning.

Directive listening by teachers focussed on whether a child's answer matched an expected response (Empson & Jacob, 2008). The term *directive listening* corresponds to the term *evaluative listening* used by Davis (1997). Teachers who used this type of listening in classroom contexts were *listening for* something, not *listening to* the students (Even, 2005) and this could result in teachers overestimating what students knew (Empson & Jacobs, 2008) by assigning understanding to correct answers with vague explanations (Even, 2005).

Observational listening (Empson & Jacobs, 2008), on the other hand, was a term used to describe teachers listening to students and trying to work out what the students were actually thinking. Davis had described this as *interpretive listening* (1997). Empson and Jacobs (2008) specified one-to-one task-based interviews as contexts for the use (and practise) of observational listening.

Responsive listening (Empson & Jacobs, 2008) by teachers encompassed trying to understand individual students' approaches and responding to them individually and instantaneously, whilst keeping 25 children engaged and included, in the group dynamic of a single lesson. Davis had termed this *hermeneutical listening* (1997).

Calculation explanations described the calculation steps of a strategy rather than communicated the purpose of the calculations (Cobb, Yackel, & Wood, 1992). For example, when adding three 19s, Grade 2 children used calculation explanations in their initial peer conversation "Nine and nine is ... 18 ... and nine more..." (p. 104). They assumed they were all using the same strategy (adding ones and then tens). However, they did not have *equivalent* strategies (the same), they had *parallel* (assumed the same when not) strategies because one was adding ones and tens, 27 plus 30, whereas the other was adding ones before adding three more ones (incorrectly treating tens as ones), 27 plus 3. The children did not explain what they were doing mathematically; they described the calculation steps that they were using to execute their mathematical thinking. In some classrooms, calculation explanations counted as an acceptable mathematical argument despite the fact that calculation explanations made it difficult for students to recognise whether they had equivalent strategies or parallel strategies (Cobb, 2011).

Methodology

One-to-one task-based interviews were conducted with 88 Grade 6 students, offering 65 tasks that assessed their understanding of length and area measurement, dynamic imagery, multiplication, and fraction understanding. Each student was interviewed for up to three hours over several sessions. Observational listening and non-directive prompts were used to elaborate further explanations. The students' responses were noted on record sheets during the interview, and as two thirds of the interviews were video-taped (and all audio-taped) transcripts enabled the classification of their answers and explanations. Pseudonyms have been used when quoting the students' explanations.

One question will be examined in detail in this paper. The Fraction Pair task assessed students' understanding of the relative size of fractions. The eight fraction pairs were the same as used by Clarke and Roche (2009): $^3/_8$ and $^7/_8$, $^2/_4$ and $^4/_8$, $^1/_2$ and $^5/_8$, $^2/_4$ and $^4/_2$, $^4/_5$ and $^4/_7$, $^3/_7$ and $^5/_8$, $^5/_6$ and $^7/_8$, $^3/_4$ and $^7/_9$. The children were shown a card with the two fractions (symbolic inscriptions) and were asked, please point to the larger fraction or tell me if they're the same. After they stated or pointed to their answer they were asked, and how did you work that out? Two of the fraction pairs are discussed in this paper, $^4/_5$ and $^4/_{7,}$ and $^5/_6$ and $^7/_8$.

Results

In the present study, Sarah provided an example of residual reasoning when comparing the fraction pair $^5/_6$ and $^7/_{8.}$ She chose $^7/_8$ as larger, "because if I imagine a pie cut into sixths and you do five of them. And I imagine a pie cut into eight and there's seven of them; that's a little more." When prompted, "How do you know?" she elaborated correctly, "Because eighths are smaller, and like seven of them would be closer to a whole than five sixths." In contrast, Meg used gap thinking to conclude incorrectly that "They're the same because five sixths has got one more to become a whole. And seven eighths it also has one more to become a whole." In the present study, gap thinking was used by 50% of the students for this fraction pair.

Table 1. Answers, initial and further explanations for the comparison of the fraction pair $^4/_5$ and $^4/_{7.}$

Strategy		
Gap thinking	Lara:	This one [points to $^4/_5$]
	I:	And how did you decide?
	Lara:	'Cause it's only one away from being a whole.
	I:	Mmm?
	Lara:	And this is three away from being a whole
Benchmarking	Chris:	[points to $^4/_5$]
	I:	How did you decide?
	Chris:	Well, five, ff; four fifths is almost a whole
	I:	Mmm?
	Chris:	And four sevenths is um, a bit higher than half
Benchmarking	Adam:	This one. [points to $^4/_5$]
	I:	And how did you decide?
	Adam:	Um four is closer to five.
	I:	Can you tell me a bit more about that?
	Adam:	Um. Four. The four and the seven, there's more less, like, um close to a half, but this one's like almost a whole.

The fraction pair $^4/_5$ and $^4/_7$ lent itself to the correct strategy of benchmarking because $^4/_5$ was close to one and $^4/_7$ was just over a half. However, it was difficult to hear the difference between benchmarking (a correct strategy) and gap thinking (a misconception) in the students' explanations (see Table 1).

Discussion

In the terms of Kieren's four-three-four model (1988), Sarah's explanation using residual thinking "Because if I imagine a pie cut into sixths and you do five of them", illustrated her engagement at an intuitive level. The difference between residual thinking (a correct strategy) and gap thinking (a misconception) was easiest to hear in the explanations of the comparison of the fraction pair $^5/_6$ and $^7/_8$ because residual thinking gave the correct answer with correct thinking and gap thinking gave an incorrect answer with mathematically incorrect reasoning. This was because the gap answer was distinctive: "They're the same".

In contrast, the correct answer, $^4/_5$ was given by Sarah, Chris, and Adam when comparing the fraction pair $^4/_5$ and $^4/_7$ and their initial explanations sounded similar:

- " 'cause it's only one away from being a whole."
- "Four fifths is almost a whole."
- "Four is closer to five."

However, in response to the non-directive prompting, "Mmm?", Lara elaborated, "And this is three away from being a whole" (see Table 1). Lara was using the gap thinking misconception, calculating the complement to one for each fraction, by working out the numerical difference between numerator and denominator, and choosing the fraction with the smaller gap. Lara had the right answer for the wrong reason.

In contrast, when prompted, "Mmm?" Chris added, "And four sevenths is um, a bit higher than a half." In Adam's case, after being prompted "Can you tell me a bit more about that?", he explained that, "The four and the seven, there's more less, like um close to a half, but this one's like almost a whole." These further explanations revealed that both Chris and Adam had been benchmarking and so had the correct answer with correct mathematical reasoning.

Implications

The similarity of the initial explanations with the correct answer for the responses by students who were benchmarking or were using gap thinking has implications for how teachers talk to students and how students explain their thinking to each other.

It has been observed that teachers using directive listening interpreted vague explanations as correct mathematical reasoning if the answer was also correct (Even, 2005). Prompting for further elaboration of the students explanations was needed to determine whether the students were correct (correct answer and mathematically correct strategy) or incorrect (correct answer and mathematically incorrect strategy). The relationship of observational listening had to be maintained, without cueing the student into a directive listening exchange. Teachers with high specialized content knowledge should be alert to this possible confusion and prompt for further elaboration of the strategy to specifically establish which strategy is being used by the student.

If a teacher were explaining the benchmarking strategy for the fraction pair $^4/_5$ and $^4/_7$ and said, "Four fifths is nearly a whole", Adam might hear his benchmarking strategy confirmed (four is closer to five) but Lara would also hear her gap thinking strategy confirmed (it's only one away from being a whole). Lara might not experience cognitive conflict between the teacher's strategy and her own. The difference between the two strategies (benchmarking and gap thinking) could not be distinguished by the researchers using the students' answers and initial explanations, so it would also be difficult for Lara to hear the distinction between the mathematically correct reasoning of the teacher and her own mathematically incorrect reasoning if only an initial explanation was offered.

It is possible that students participating in peer conversations could react in the same way: if the answer was the same as their own and the explanations were similar, they would assume their strategy was the same as the other student. For example, let us imagine that Lara, Adam, and Chris were working together to solve the fraction comparison task $^4/_5$ and $^4/_7$. The terminology of peer conversation would enable us to describe their initial explanations as calculational: "'Cause it's only one away from being a whole", "four fifths is almost a whole" and "four is closer to five." All three children describe a difference calculation and none explain why they are doing this. At this point they might imagine that they are all agreeing on the strategy (that they have equivalent strategies). Even if Lara added "And this is three away from being a whole", Adam might not realise that she was not benchmarking like he was, unless he knew to listen for gap thinking. Parallel interpretations have the same answer and the same initial calculational explanation, but are actually different strategies. Lara and Adam have parallel strategies. Adam and Chris who are both benchmarking have equivalent strategies.

Cobb, Yackel, and Wood's (1992) examples of calculational, parallel, and equivalent explanations were of addition by Grade 2 children. Excellent teaching by the classroom teacher in their study enabled students to increase their knowledge of addition strategies. Grade 6 teachers have access to a repertoire of descriptions of strategies and misconceptions to draw on when responding to student explanations. However, for students to recognise parallel explanations, they may need to acquire a similar sophisticated repertoire of possible strategies in order to make sense of other students' explanations.

Conclusion

A highly detailed knowledge of gap thinking is needed in teachers' pedagogical content knowledge. Observational listening by teachers may require interpretations not only of answers and initial explanations but also prompting for further explanations.

The students' responses when comparing the fraction pair $^4/_5$ and $^4/_7$ demonstrated that these initial answers were considered acceptable mathematical answers by the students in the interview context. If students are to learn through peer conversation then teachers must establish the classroom norm that calculational answers are only partly acceptable mathematical answers. Acceptable mathematical answers include

- an answer,
- an explanation describing the strategy, and
- a description of the calculational steps used to execute that strategy.

This means that students will also have to develop their own knowledge of strategies, such as gap thinking and benchmarking so that they recognise when they have equivalent explanations or parallel explanations.

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