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The Anatomy of Buyer-Seller Dynamics in Decentralized Markets

by Andrea Giovannetti*

Abstract

In this paper I investigate the nexus between buyer-seller dynamics, financial frictions and market efficiency in decentralized markets. To do so, I introduce financial frictions in a dynamic market with heterogeneous traders. Heterogeneously constrained buyers sequentially enter the market to acquire units of a generic good from heterogeneously endowed sellers. I characterize two closely related classes of equilibria, respectively called homogeneous equilibrium with no entry (HEWNE) and homogeneous equilibrium with entry (HEWE). Both equilibria prescribe a market where only the efficiently endowed type of seller exists in the limit. However, the two equilibria diverge in the specification of agents' behavior subsequent to trade. In HEWNE, sellers and buyers exit the market upon successful trading. In HEWE, like in supply chains, in every period certain types of buyers replace exiting sellers, thus becoming potential sellers for subsequent waves of buyers. First, I identify the critical role of frictions in steering the complex evolution of market heterogeneity for both classes of equilibria. Secondly, I operationalize the combined study of HEWNE and HEWE to obtain sharp predictions on market efficiency for a range of empirically-relevant situations in which buyer-seller dynamics are decoupled, for example when entry of new sellers is delayed or stopped. Third, I test the theoretical findings against a simulated artificial market.

Keywords: Bargaining, Frictions, Non-stationary markets, Evolutionary Markets, Bifurcation Theory, Over-The-Counter Markets, Heterogeneous agents
JEL Codes: C78, D40, G31, G32, G34

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1 Introduction

In this paper I study a dynamic decentralized market in which a (possibly *growing*) pool of heterogeneous sellers faces sequential *waves* of financially constrained buyers. Buyers bargain with available sellers to acquire units of a generic good. Buyers have a heterogeneous demand schedule that depends on the individual budget constraint. I show that the presence of a regulator controlling buyers' financial constraints bears direct implications on the long-run efficiency of the market. Specifically, I show the existence of two classes of equilibria which may sustain the dynamic market formation along a path in which an efficiently-endowed type of seller does emerge in the limit. I use these two equilibria to inspect the market allocation efficiency for a range of realistic classes of market dynamics in which buyer-seller dynamics are decoupled.

1.1 Background and Motivation

In decentralized markets[1], an investor who wishes to sell must search for a buyer, incurring opportunity or other costs until one is found. Often, traders must be approached sequentially. Hence, when two counter-parties meet, their bilateral relationship is inherently strategic. Prices are set through a bargaining process that reflects each investor's alternatives to immediate trade (Duffie et al., 2005). These search-and-bargaining features are empirically relevant in many markets, such as those for mortgage-backed securities (Glaeser and Kallal, 1997), corporate bonds (John and Nachman, 1985), emerging market debt (Arellano, 2008), bank loans (Diamond and Rajan, 2000) and derivatives[2] such as contract default swaps (Riggs et al., 2020) to name some of the most prominent examples[3]. In real-estate markets, for example, prices are influenced by imperfect search, the relative impatience of investors for liquidity, outside options for trade, and the role and profitability of brokers (Duffie et al., 2005). Markets are dynamical, and as such, the distribution of potential opportunities that traders face may evolve as markets unfold in time, possibly as a function of traders' own evolving expectations, heterogeneous tastes (Manea, 2017) and financial availability. In particular, financial constraints and corporate leverage critically shape the evolution of market structures around the world (see, for an empirical analysis, Lucey and Zhang, 2011). Under the lenses of allocative efficiency, the following question critically arises: *how do financial frictions affect allocation efficiency via market dynamics?*

[1] I thank one referee for the stimulating comments that led to the deep restructuring of Section 1.

[2] More in general, and even in very liquid markets, syndicates of buyers and sellers can form and engage in sequences of block trades (see Burdett and O'hara, 1987 for an application).

[3] Other examples of OTC markets include corporate and municipal bonds, foreign exchange swaps, and FED funds (Hugonnier et al., 2020). We refer the reader to Duffie (2011) for a an excellent methodological introduction to asset pricing in decentralized markets with several applications.

To contextualize the problem, consider for example the functioning of cross-divisional capital transfers in internal capital markets within conglomerates (see, for an example of financial conglomerates, Campello, 2002). These are *decentralized* markets in which complex flows of capital can take place across heterogeneous and financially-autonomous production units (PUs) via pairwise transactions[4]. In such context, capital can flow from established firms (i.e. the "sellers") via multiple market operations to newly acquired PUs (i.e. the "buyers"). The efficiency of internal capital markets in resource allocation is a highly disputed issue (see Khanna and Yafeh, 2007 for a review) as efficiency interacts with the degree of regulatory intervention on corporate leverage[5], hence it is critical to understand the theoretical link between centralized intervention, market dynamics and efficiency.

1.2 Methodology

I address the above question by constructing and studying a model of a dynamic decentralized market. The model allows to pin down the relationship between financial constraints, market dynamics and allocative efficiency. Formally, I embed the dynamic market with frictions in the frictionless framework of an infinite horizon bargaining game played in discrete time as introduced[6] by Manea (2017). The economy consists of a continuum of firms drawn from a finite set of types. Firms exogenously enter the market over time and exit upon trading. In this model, the market formation is made of an initial stage $t = 0$ and two intertwined processes taking place in each period $t = 1, 2, ...$. In the initial stage $t = 0$, every firm discovers her own endowment (potentially heterogeneous across firms), and induces an individual demand for the good as a function of technical conditions and the financial constraint. The first process takes place *within* each period $t = 1, 2, ...$. At the beginning of period t, a measure of buyers enters in the economy in order to satiate the individual demand for the capital good. This process is the *intra-period* purchase of units of capital: buyers purchase capital from the pool of sellers available

[4]See Giovannetti (2021) for a dynamic model of formation of internal capital markets in which firms have the option to rely on bank loans, internal capital transfers or a mixture of both to carry out production.

[5]For example, in the context of the Korean economy, characterized by a prevalence of conglomerates of highly independent firms, Lee et al. (2009) isolate the link between the intensity of cross-subsidization - the conglomerate's debt-to-equity ratio - and its market efficiency. They show that the paralyzing effect of liquidity regulations impacts the profitability of the whole conglomerate.

[6]Relevant elements of the framework of Manea (2017) as well as fundative results are described in detail in Section 2 and in Appendix B. In his work, traders bargain over the price of a heterogeneous good. The surplus that pairs of market participants can generate from trade may differ across traders. The distribution of bargaining opportunities that market participants face may change over time. The stock of potential trading partners and the amount of surplus available at any date depend on the inflows of new players into the market and the outflows of players who complete transactions. Players need to forecast the evolution of the macroeconomy, as determined by the endogenous volume of trade and the relative matching probabilities induced by inflows and outflows, and negotiations should reflect the anticipated market conditions.

in the market in period t. The distribution of available sellers determines the market composition at t. Each buyer is randomly matched with a seller and parties bargain over a single unit of the good. Importantly, the exchange incentive structure is determined by the market composition at t. Once all buyers have satiated their own demand (provided that there is a surviving positive measure of sellers in the market) the market moves one step ahead to $t+1$. This bring us to the second process characterizing market formation, that is the evolution of the market composition *across* periods. The market composition, determined by the outcome of the realized exchanges, is the key dimension of this model, linking intra-period and inter-period dynamics.

I study the model by means of two classes of equilibria that diverge in the specification of the intertemporal dynamics. The first equilibrium, which I call *Homogeneous Equilibrium with no Entry* (HEWNE) expands the canonical matching scenario, in which both the seller and the buyer leave the market upon successful trade. In this work, buyers' exit is conditional on satisfaction of an individual demand for good, therefore buyers are allowed to obtain multiple units of the good before leaving the market. The second equilibrium I characterize is the *Homogeneous Equilibrium With Entry* (HEWE). In this equilibrium, the subset of buyers that are endowed with *at least* the first-best investment level at the end of period t *settle* in the economy as potential sellers for waves of buyers entering at $t+1, t+2, ...$, thus replenishing the pool of sellers. In this equilibrium, the market converges to an expansionary path where in the limit a uniform type of seller exists, the first-best type seller[7]. I use these equilibria to explore the implications on market efficiency of various types of buyer-seller dynamics. Last, I construct an artificial market to validate in simulations the main theoretical results presented along the paper.

1.3 Contribution

My contribution is twofold. First, I contribute to the theoretical literature on search and bargaining[8] by introducing financial frictions in the seminal framework of Manea (2017). My paper expands his work by introducing heterogeneity along a dimension which is crucial for the evolution of real markets: the limited purchasing power of buyers. This is a systemic variable, a leverage parameter (uniform across traders) which may be controlled by regulators. By characterizing a relationship between frictions and market dynamics, I show that frictions can generate rich dynamics in the evolution of market composition. Second, I use the model to answer the question introduced in Section 1.1. More specifically, I assess allocation efficiency for several market protocols which are consistent with empirical OTC markets characterized by heterogeneous traders.

[7] As explained in Section 1.3, HEWNE and HEWE provide the tool to explore the allocative efficiency of markets by means of existence (or lack thereof) of these equilibria.

[8] See Chade et al. (2017) as well as the Introduction of Duffie et al. (2005) and Manea (2017) for an updated review of main results and contributions on search and bargaining literature.

I frame the efficiency problem in a decentralized market taking place between PUs with heterogeneous endowments and uniform first-best investment. PUs sell units of capital to other PUs against the promise of future payment[9]. To study efficiency, I ask whether the market can autonomously converge to a situation in which available sellers produce on the efficient frontier, or, in other words, I look at the conditions that enable a homogeneous type of seller to exist in the limit. I answer this crucial question by operationalizing the two classes of equilibria introduced above, HEWNE and HEWE. With respect to HEWNE, I show that the relationship between the *market composition* (i.e. the distribution of available sellers) and financial constraint is nonunivocal. The capability for a market to reach an efficient redistribution of inputs - whereby sellers are on the efficient production frontier - is subject to a *tipping point* which depends on the the specific level of financial frictions in place and the initial heterogeneity of sellers. When the allowed leverage is moderate, the exchange structure has a limited impact on sellers' distribution. This implies that sellers heterogeneity does not disappear in the limit. On the other hand, when the exchange structure is characterized by a high enough financial leverage, a *bifurcation* takes place and two extreme cases can emerge depending on the *initial* distribution: either the market will converge to an efficient equilibrium, where available sellers are endowed with the first-best investment level, or it will converge to a an equilibrium in which available sellers can be endowed with heterogeneous investment levels.

Relatively to HEWE, I show that entry of new sellers in every period imposes *stronger* conditions for the homogeneous type of seller to exist in the limit (i.e. the efficient outcome), hence there exists an ordered relationship between HEWNE and HEWE. The ordering is relevant from a normative point of view, as it allows to predict market evolution for empirically-relevant situations where exit of buyers is *decoupled* from entrance of sellers, possibly in non-trivial ways. I discuss allocative efficiency in the three following scenarios. In the first scenario, entrance of new sellers is *asynchronous* with respect to exit of buyers, as it can be the case, for example, of sellers requiring a certain number of periods to produce an exchangeable good and set-up exchange. In the second scenario, new sellers enter only for a limited number of periods, thus replicating a market undergoing a permanent negative supply shock. In the third scenario, existence of the homogeneous type of seller is discussed for a market that combines the above cases: delayed entry of sellers is interrupted altogether after a certain number of periods. For the three cases, I provided a taxonomy of bounds on financial constraint that allow the efficient outcome to survive.

[9] For example by means of financing-oriented trade-credit agreements (see for example Shenoy and Williams, 2017).

1.4 Literature Review

Some authors have recently integrated financial constraints in models of dynamic markets. In particular, Moll (2014), Liu and Wang (2014) and Mino (2015) studied the evolution of market economies in which firms are heterogeneous and financially constrained. Similarly to Mino (2015), in this paper I consider an endogenous market process. However, the setting draws on the granular non-stationary dynamic bargaining framework proposed by Manea (2017), which I further characterize in order to study two complementary classes of equilibria which lend sharp predictions on the asymptotic shape of the market. Other contributions introduced frictions in frameworks similar to Manea (2017). Lauermann and Noldeke (2015) have proven existence of a steady-state equilibrium for an economy with search friction. While their main contribution is to show that a dynamic equilibrium can be achieved with non-transferable utilities, steady states analysis is constrained by the fact that the bargaining game is treated as a "black box" (Lauermann and Noldeke, 2015).

The rest of the paper is organized as follows. In Section 2 I define the primitives of the dynamic market. In Section 3 I introduce two classes of equilibria which support the market formation as the outcome of the bargaining game with financial frictions. In Section 4 I explore market formation under three market structures in which buyer-seller dynamics are decoupled. In Section 5 I simulate an artificial market to validate the implications of the equilibria developed in the previous sections. Additional results and proofs are contained in the Appendix.

2 The Model

Consider an economy with a finite set of firm types, such that each firm's type corresponds to the endowment of a generic capital good measured in units $\omega \in \{0, 1, 2, ..., \bar{\omega}\} \equiv \Omega$. I refer to the amount of good available to firm i at time t as $\omega_{i,t}$. I consider a market with infinite horizon made of two stages: a *pre-market* stage, which I assume taking place at $t = 0$, and the *market formation* phase, which takes place at $t = 1, 2, ...$ In the pre-market phase, every firm i discovers her initial type $\omega_{i,0}$ and induces a (possibly zero) individual demand for capital q_{ω_i}, which I describe below. In period $t = 0$, a measure of sellers is available in the market. I represent the set of sellers by means of a profile of population size $\left\{\mu^S_{j,0} \in [0, 1]\right\}_{j=0}^{\bar{\omega}} \equiv \boldsymbol{\mu}^S_0$, such that every entry $\mu^S_{j,0}$ records the measure of available sellers endowed with type ω_j at $t = 0$. I further assume that $\sum_{\omega \in \Omega} \mu^S_{\omega 0} = 1$.

In every period $t > 0$, a fixed unit measure of buyers enters in the market. Importantly, buyers can have heterogeneous demand for the capital good. I represent the measure of buyers with a profile of population size $\left\{\mu^B_i \in [0, 1]\right\}_{i=0}^{\bar{\omega}} \equiv \boldsymbol{\mu}^B$ where every entry μ^B_i

is the measure of type ω_i buyers at time of entry. Each buyer i is randomly and pair-wise matched to available sellers. For simplicity, I also assume that types ω in $\boldsymbol{\mu}^B$ are distributed according to a rectangular distribution[10] such that $P(\omega_{i,0}) = 1/\bar{\omega}, \forall i \in \Omega$. I now discuss in details all the elements of the model.

Buyer's problem. Preferences $u(\omega)$ and cost for possessing the good are homogeneous across agents, with decreasing returns in the number of possessed units. I define $\omega^{**} \in \Omega$ such as the first-best endowment, or alternatively, the *efficient type*. The determination of ω^{**} depends on exogenous parameters (such as a market price, production cost and technology $u(\omega)$) which I assume uniform across firms. For example, ω^{**} can represent the optimal amount of working capital in a competitive market. Trade is against a promise of future payment[11]. For every buyer i endowed with initial type $\omega_{i,0}$ a leverage coefficient $\alpha \geq 0$ regulates the buyer's budget constraint such as

$$\bar{q}_i \leq \alpha \cdot \omega_{i,0},$$

where $\bar{q}_i$ is the maximum amount of capital a buyer is allowed to collect from the market. Given the unique first-best endowment ω^{**}, every buyer i endowed with $\omega_{i,0}$ units of capital at time of entry induces the following individual demand $q_{\omega_i} \equiv q_\omega$ for the good

$$q_\omega \equiv \max\left\{0,\ \min\left\{\bar{q}_{\omega_i},\ \omega^{**} - \omega_{i,0}\right\}\right\}.$$

Therefore, demand of the good for each type of buyer is completely pinned down by the leverage α and the exogenous preferences encapsulated in ω^{**}. The demand segmentation for this economy is given in the following

Definition 1 (Demand Segmentation)**.** *Let $D^* \equiv \{0, ..., \omega^{**}/(1+\alpha) - 1\}$ be the set of Fully Constrained Buyers, that is the set of buyer types for which the budget constraint is binding. Let*

$$D^{**} \equiv \begin{cases} \{\emptyset\} & \text{for } \alpha = 0 \\ \{\omega^{**}/(1+\alpha), ..., (\omega^{**} - 1)\} & \text{otherwise,} \end{cases}$$

be the set of Partially Constrained buyers, that is the set of buyers for which the budget constraint is only partially binding. Let $S \equiv \{\omega^{**}, ..., \bar{\omega}\}$ be the set of Pure Sellers, that is the set of buyer types characterized by zero demand for the good ℓ. I define $\mathcal{D}$ the set $\mathcal{D} \equiv D^* \bigcup D^{**}$. Lastly, define ω^* such as the smallest type capable of attaining the first-best endowment by using all the available leverage, $u(\omega^* + \alpha\omega^*) = u(\omega^{**})$.

Hence, the following proposition follows

[10] I can equivalently assume that in every period t one new buyer i draws an endowment $\omega_{i,0}$ from distribution P and enters in the market.

[11] For example, parties agree on a trade-credit contract in which the interest is computed on the retail value of the final good, so that bargaining is on shares of final good value.

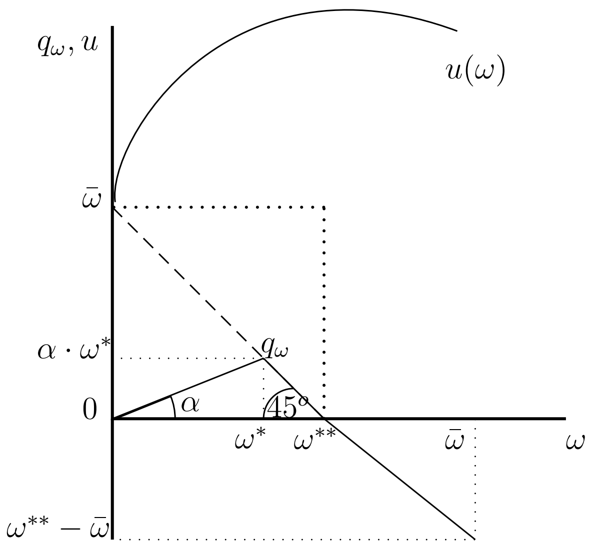

Figure 1: The market segmentation for a given α and $\omega^{**} \leq \bar{\omega}$.

Proposition 1. (*i*) *For every $\omega \in \Omega$, the constraint α and the first-best type ω^{**}, the continuous function $q : \Omega \to \Omega$ maps each type in its own demand for capital such as*

$$q_\omega = \begin{cases} \alpha \cdot \omega & \forall \omega \in D^* \\ (\omega^{**} - \omega) & \forall \omega \in D^{**} \\ 0 & \forall \omega \in S \end{cases}. \tag{1}$$

(*ii*) *The average demand for capital $D(\alpha, \omega^{**})$ is given by the quantity*

$$D(\alpha, \omega^{**}) = \frac{\alpha}{1+\alpha} \frac{(\omega^{**})^2}{2} \cdot \frac{1}{\bar{\omega}}. \tag{2}$$

Example 1 (Demand Segmentation)**.** Consider an economy with $\Omega = \{0, 1, ..., 12\}$, leverage $\alpha = 2$ and first-best investment $\omega^{**} = 6$. It follows that $\boldsymbol{\mu}^B = \{1/12\}_{i=0}^{12}$ and $\mathcal{D} = \{0, 1, 2, 3, 4, 5\}$. By computation of each individual demand schedule q_ω $\forall \omega \in \mathcal{D}$ I obtain $\{q_\omega\}_{i=0}^{5} = [0, 2, 4, 3, 2, 1]$. Therefore, the average demand induced by $\boldsymbol{\mu}^B$ is given by $D = \sum_{i=0}^{12} q_\omega = (12)/12 = 1$. By using (2), I confirm $D = (2/3) \cdot (36/2) \cdot (1/12) = 1$.

Market Composition. In every period $t \geq 0$, the set of buyers and sellers is represented by a profile of population sizes respectively given by $\boldsymbol{\mu}^B$ and $\boldsymbol{\mu}_t^S$. More precisely, $\left\{\mu_i^B = 1/\bar{\omega}\right\}_{i=1}^{\bar{\omega}} \equiv \boldsymbol{\mu}^B$, such that every entry $\mu_i^B = 1/\bar{\omega}$ contains the measure of buyers entering with type ω_i at t, and $\left\{\mu_{j,t}^S \in [0, \infty)\right\}_{j=0}^{\bar{\omega}} \equiv \boldsymbol{\mu}_t^S$, in which $\mu_{j,t}^S$ is the measure of available sellers endowed with type ω_j at t. I refer to $\boldsymbol{\mu}_t^S$ such as the *market composition* at time t.

Information Structure. Producer types $\omega \in \Omega$, as well as $\boldsymbol{\mu}_t^S$ and $\boldsymbol{\mu}^B$ are publicly observed.

Matching Technology. Within each period $t > 0$, agents are randomly matched in pairs[12] and bargain upon one unit of capital ℓ. Every producer encounters a trading partner with probability p and has a probability equal to $1/2$ to submit the offer. I characterize the matching via a *soft linear search* (see for instance Gale, 1987) technology. Hence, the probability for the buyer i endowed with $\omega_{i,t} \equiv \omega$ when she enters the market at t to meet a seller j endowed with $\omega_{j,t} \equiv \omega'$ at t and submit an offer is given by

$$\pi^S_{\omega' t}(\mu^S_{\omega' t}) = \frac{p}{2} \frac{\mu^S_{\omega' t}}{\sum_{k \in \Omega} \mu^S_{kt}}.$$

The (time-invariant) probability $\pi^B_\omega(\mu^B_\omega)$ for a seller to find a type ω buyer is similarly defined. For simplicity, in the rest of the analysis I will assume $p = 1/2$. As in other models of search-and-bargaining (see, for example, Duffie et al., 2005), the random matching technology is a metaphor consistent with a large variety of *quote-driven* markets such as specialist-based equity and Contract for Differences (CFDs) markets (Brown et al., 2010).

Now, define the space[13] $\mathcal{P}$ of the buyers' matching probabilities such as

$$\mathcal{P} = \left\{ (\pi^S_{\omega t})_{\omega \in \Omega, t \geq 0} | \pi^S_{\omega t} \in [0,1] \, \forall \omega \in \Omega, t \geq 0 \right\}.$$

Surplus Function and Space of Payoffs. The payoff structure I adopt directly descends from the assumption of decreasing returns to the utility of possessing the good and does not depend on the *identity* of traders but on their *type*. For every ordered pair (ω, ω'), where $\omega \equiv \omega_{i,t}$ is the type of a generic buyer i at t and $\omega' \equiv \omega_{j,t}$ is the type of the seller j at t, the surplus function $s_{\omega\omega'} : \Omega^2 \to \mathbb{R}$ is the linear mapping

$$s_{\omega\omega'} \equiv \max \left\{ a \cdot (\omega' - \omega), \; 0 \right\}, \tag{3}$$

with a being a positive scalar. For every possible pair (ω, ω'), the surplus function orders the possible matches in terms of the maximum gains of pairwise trades. Albeit simple, the function can capture the fact that when the double-coincidence window is determined by decreasing returns, the pairwise surplus is monotonically increasing (respectively, decreasing) in the type of the seller (respectively, buyer).

[12] This is a one-to-one matching process where every player interacts only with a measure zero of other traders. The matching process is measure preserving, that is for any measurable set of proposers engaged in a match, the corresponding set of respondents is measurable and has the same measure. A positive measure of players is left unmatched every period.

[13] As in Manea (2017), $\mathcal{V}$ and the other sets defined in this section can be regarded as topological vector spaces via a natural embedding in the space $\mathbb{R}^{|\Omega|}$ endowed with product topology. Because the product topology in $\mathbb{R}^{|\Omega|}$ is metrizable, the characterizations of closed sets and continuous functions in terms of convergent sequences apply for the sets defined here.

For every pair of matched agents i and j at t, define $v^B_{\omega_i t}$ and $v^S_{\omega_j t}$ as elements of the space of the payoffs $\mathcal{V} = \mathcal{V}^B + \mathcal{V}^S$, where $\mathcal{V}^B$ (respectively, $\mathcal{V}^S$) is the set of the seller (respectively, buyer) types' payoffs such that[14]

$$\mathcal{V}^S = \left\{(v^S_{\omega' t})_{\omega'\in\Omega, t\geq 0} | v_{\omega' t} \in \left[0, \max_{\omega\in\Omega} s_{\omega\omega'}\right], \forall \omega' \in \Omega, t \geq 0\right\}$$

$$\mathcal{V}^B = \left\{(v^B_{\omega t})_{\omega\in\Omega, t\geq 0} | v_{\omega t} \in \left[0, \max_{\omega'\in\Omega} s_{\omega\omega'}\right], \forall \omega \in \Omega, t \geq 0\right\}.$$

Time Preferences. Traders discount time according to a homogeneous discount factor $\delta \in [0, 1)$.

Agreements. Since agents of the same class and endowment have equal payoff, I will identify each player i endowed with ω_i directly by means of her type ω_i. Hence, given a matched pair of types (ω, ω'), with slight abuse of notation I define $\alpha_{\omega\omega' t}$ as the fraction of types ω and ω' that reach an agreement in period t, with $\omega' > \omega$ being the seller. The fraction of agreements may be 0, 1 or a value in $[0, 1]$ depending on $s_{\omega\omega'}$ and each type's continuation value. Therefore, I construct the correspondence $\alpha_{\omega\omega' t}$ such as

$$\alpha_{\omega\omega' t} = \begin{cases} 0 & \delta v^B_\omega + \delta v^S_{\omega'} > s_{\omega\omega'} \\ [0,1] & \delta v^B_\omega + \delta v^S_{\omega'} = s_{\omega\omega'} \\ 1 & \delta v^B_\omega + \delta v^S_{\omega'} < s_{\omega\omega'} \end{cases}. \tag{4}$$

Across periods, the space of path of agreement rates is defined as follows

$$\mathcal{A} = \left\{(\alpha_{(\omega\omega' t)} \;_{\omega,\omega'\in\Omega, t\geq 0} \mid \alpha_{\omega\omega' t} \in [0, 1], \forall \omega, \omega' \in \Omega, t \geq 0\right\}.$$

Evolution of Market Composition. For every seller type $\omega' \in \Omega$, given an initial measure $\mu_{\omega' 0}$ of potential sellers of type ω' available in the market at period $t = 0$, the across-period transition from $\mu^S_{\omega' t}$ to $\mu^S_{\omega'(t+1)}$, $\forall t > 0$ is dictated by

$$\mu^S_{\omega'(t+1)} = \phi\left(\boldsymbol{\mu}^S_t\right), \tag{5}$$

where I refer to $\phi(\cdot)$ as the *transition protocol* of the economy. The main contribution of this paper is the analysis of the economy under specific transition protocols (Section 3 and 4). Across periods, the space of market composition is given by

$$\mathcal{M} = \left\{(\mu^S_{\omega t})_{\omega\in\Omega, t\geq 0} | \mu^S_{\omega t} \in [0, 1], \forall \omega \in \Omega\right\}.$$

Strategies. I study pure strategies. I restrict the analysis to pure strategies because the

[14]One can think of the set $\mathcal{V}^S$ and $\mathcal{V}^B$ as a function of the set of ask and bid, respectively.

intertemporal equilibrium will depend on the evolution of the market composition $\boldsymbol{\mu}_t^S$, which in turn is determined by the rate of *actual* matches. Given a matched pair of type (ω, ω'), the proponent submits to the receiver a division $(s_{\omega\omega'} - x,\ x)$ of the surplus. If the offer is accepted, the receiver obtains $(s_{\omega\omega'} - h)$ and the proponent obtains h. Otherwise, the receiver rejects the offer and the match dissolves. The intra-period game ends when all the buyers collect q_ω. In the intra-period bargaining game, matching probabilities are determined on the basis of $\boldsymbol{\mu}^B$ and $\boldsymbol{\mu}_t^S$. Therefore, buyers' (respectively, sellers') reservation values will depend on the composition of the sellers (respectively, buyers) available in the market at t. This formulation is consistent with the bargaining structure of real-world quote-driven markets, where traders understand the time-evolving structure of bid-ask spreads and may set the intensity of their search by comparing the offer they receive from their current counter-part against the composite market structure[15].

Solution Concept of the Intra-Period Game. Following Manea (2017), I restrict the solutions of the game to Subgame Perfect Equilibria (SPE) which are robust in the sense that no trader can affect the equilibrium path $\boldsymbol{\mu}_t^S$ or $\boldsymbol{\mu}^B$ by unilaterally deviating from the prescribed strategy[16] The intra-period game is solved by means of iterated deletion of dominated strategies. In Appendix B I report results on equilibrium existence and characterize the equilibrium payoff structure for the general class of intertemporal bargaining games which this dynamic market belongs to. Given the above, I can define a dynamic market with frictions such as

Definition 2 (Dynamic Market with Frictions)**.** *Given a leverage $\alpha \geq 0$, a first-best type $\omega^{**} \in \Omega$, an initial market composition $\boldsymbol{\mu}_0^S$, a buyer inflow $\boldsymbol{\mu}^B$, a dynamic market is given by the bundle $M \equiv \left\{\left\{\Omega, \omega^{**}, \alpha, a, \delta, \left\{\boldsymbol{\mu}_t^S\right\}_{t=0}^{\infty}, \boldsymbol{\mu}^B\right\}, \phi(\cdot)\right\}$, where $\phi(\cdot)$ is the associated transition protocol defined in* (5)*.*

I can eventually describe the evolution of the dynamic market with frictions.

Market Evolution. The inter-temporal evolution of market M is dictated by means of the following (see also Figure 8)

1. *Pre-Market Stage.* Every agent i discovers her own endowment $\omega_{i,0}$ and induces a (possibly zero) individual demand q_{ω_i} for the good that depends on preferences and the financial leverage α. An initial measure of sellers $\boldsymbol{\mu}_0^S \in [0,1]^{|\Omega|}$ is available in the market.

[15] See Section 4 of Duffie et al. (2007) for several examples of motivated search frictions in empirical OTC markets.

[16] This solution notion rules out situations where noise traders can alter the market dynamics and create profitable bubbles. This allows to isolate the contribution of the paper. In fact, a main result of this work is to provide a taxonomy under which even under full rationality of traders and one single first-best allocation, the market can fail to reach allocative efficiency as a joint effect of financial constraints and buyer-seller dynamics. For alternative models where traders are heterogeneous with respect to their degree of rationality, I refer the reader to Brock and Hommes (1998) and Anufriev et al. (2021).

2. *Entry Protocol.* entry is explicitly regulated by an exogenous mechanism. At the beginning of every period t, a fixed measure of *buyers* $\boldsymbol{\mu}^B$ enters in the market.

3. *Intra-Period Matching.* Every buyer (respectively, seller) is randomly matched to a counterpart of type ω with a probability $\pi^S_{\omega t}$ (respectively, π^B_{ω}) which depends on $\boldsymbol{\mu}^S_t$ (respectively, $\boldsymbol{\mu}^B$). Each party has a probability p to submit the offer. The receiver's (respectively, the sender's) outside option is endogenously determined on the basis of $\pi^S_{\omega t}$ (respectively, π^B_{ω}). Either the traders agree on the exchange, in which case the seller transfers one unit of capital to the buyer, or they fail to. In either case, the match dissolves. The intra-period bargaining phase terminates when every buyer has cleared the (possibly zero) demand for the good q_ω.

4. *Intertemporal Market Evolution* At the end of (3) the market composition updates from $\boldsymbol{\mu}^S_t$ to $\boldsymbol{\mu}^S_{t+1}$ according to the selected transition protocol $\phi(\cdot)$. Eventually, the market moves to $t+1$, with a new wave of buyers $\boldsymbol{\mu}^B$ entering into the market and the process repeats.

3 Homogeneous Equilibria

I now introduce[17] two classes of equilibria which can sustain the formation of market M in presence of financial frictions. In Section 4 I will show that these two equilibria are intimately linked and that can be operationalized to study complex market formation protocols. Each equilibrium depends on a corresponding transition protocol, defined in the following

Definition 3 (Transition Protocols). *Given any seller j endowed with $\omega_{j,t} \equiv \omega'$ and buyer i endowed with $\omega_{i,t} \equiv \omega$ present in the market at time $t > 0$, the transition protocol $\phi(\cdot)$ is characterized by either of the following:*

T.1 (No entry of new sellers). Each seller j (respectively, buyer i) leaves the market upon successful trade of one unit of the good (respectively, of satisfaction of demand schedule q_{ω_i}).

T.2 (Entry of new sellers). Each seller j leaves the market upon successful trade of one unit of the good. Upon satisfaction of demand schedule q_{ω_i},

(a) If $q_{\omega_i} + \omega_i < \omega^{**}$, buyer i leaves the market

(b) If $q_{\omega_i} + \omega_i \geq \omega^{**}$, buyer i settles in the market, that is i becomes potential seller for buyers entering at periods $t+1, t+2, \ldots$.

[17] In Appendix B I recall equilibrium existence for the general class of intertemporal bargaining games sustaining the specific market formation processes discussed in Section 3.

For analyzing the proposed classes of equilibria, I will focus on the evolution of ω^{**}-type sellers relative to the aggregate of other types. For such purpose, I define the *market index* x_t for a dynamic market at time t as

Definition 4 (Market Index)**.** *For any period t, given the market composition $\boldsymbol{\mu}_t^S \equiv \left\{\mu_{j,t}^S\right\}_{j=0}^{\bar{\omega}}$, the market index x_t is*

$$x_t \equiv \frac{\mu^S_{\omega^{**}t}}{\sum_\omega \mu^S_{\omega t}} \qquad x_t \in [0,1]\,, \tag{6}$$

whereby the long-run dynamics of the market index are captured by x^* defined as $x^* \equiv \lim_{t\to\infty} x_t$.

Lastly, in order to make the analysis tractable, I will impose some structure to the dynamic market by means of the following assumption

Assumption 1. *The dynamic market with constraints of Definition 2 is such that $\mu^S_{\omega 0} \geq 0$ for $\omega \in \{\omega^{**}, \bar{\omega}\}$ and $\mu^S_{\omega t} = 0$ otherwise.*

Assumption 1 simply implies that the set of sellers in every period t is made at most by two distinct types, given by the first-best type ω^{**} and the upper type $\bar{\omega}$, with $\omega^{**} \leq \bar{\omega}$. The assumption allows tractability without sacrificing sufficient heterogeneity in the population structure.

3.1 Homogeneous Equilibria with no Entry (HEWNE)

I start the analysis with the case in which sellers' dynamics are encapsulated by the simple monotonic transition protocol $T.1$ (Definition 3). This prescribes that each type ω' seller active in the market at time t is *removed* from the market at the end of period t if a successful matching involving j (either as receiver or proponent) and any buyer i is realized within t. An empirical counter-part of such transition mechanism can be given for example by the short-run reallocation of capital taking place in conglomerates and internal capital markets (see Giovannetti, 2021 and Khanna and Yafeh, 2007), where capital is allocated from more mature firms (corresponding, in our case, to the pool of available sellers in any period $t > 0$) to newly acquired production units (i.e. the buyers entering the market at $t+1, t+2, \ldots.$)[18]. Under $T.1$, the law of motion for the market composition (5) is given by the system of equations

[18] See also Buchuk et al., 2014 and Almeida et al., 2015.

$$\mu^S_{\omega'(t+1)} = \mu^S_{\omega' t} - \underbrace{\sum_{\{\omega \in \Omega | \omega \leq \omega^{**}\}} \left(\alpha_{\omega\omega' t} \pi^S_{\omega' t} \mu^B_{\omega} + \alpha_{\omega'\omega t} \pi^B_{\omega} \mu^S_{\omega' t} \right)}_{type\ \omega'\ out-flow} \quad \forall \omega' \in \Omega, \tag{7}$$

where the elements in bracket correspond to successful matching involving sellers either as receivers of the offer or as proponents. While seller-type dynamics in (7) exhibit a clearly monotonic pattern, it is unclear how the *heterogeneity* of seller population x_t evolves as the market process unfolds. To understand this, I introduce the following class of equilibria

Definition 5 (**Homogeneous Equilibria with no entry (HEWNE)**). *Given the dynamic market with frictions from Definition 2 with the transition protocol $T.1$ from Definition 3, I define Homogeneous Equilibria with no entry the class of equilibria of the bargaining game such that:*

(i) *The market follows the law of motion in (7).*

(ii) *In the limit $t \to \infty$ there is no seller-type dispersion and it holds that $x^* = 1$.*

Hence, the main result follows

Theorem 1. *For a dynamic market with frictions following law of motion in (7) fulfilling the following*

(i) $x_0 \in \left(\frac{1}{2} \cdot \left(1 + \sqrt{1 - 4D^2} \right),\ 1 \right]$ *for* $\alpha < \alpha^*$,

$x_0 > \frac{1}{2}$ *for* $\alpha \geq \alpha^*$ *where* $\alpha^* \equiv \frac{\bar{\omega}/(\omega^{**})^2}{1 - \bar{\omega}/(\omega^{**})^2}$,

(ii) $\delta \leq \bar{\delta} \equiv \frac{7}{4} - \frac{\sqrt{17}}{4}$,

Homogeneous Equilibrium with no entry (HEWNE) is the unique stable equilibrium of the economy and as such the market index x_t converges to it for $t \to \infty$.

The equilibrium expands the scenario considered in Manea (2017), in which both the seller and the buyer leave the market upon successful trade. Differently from that work, buyers are allowed to obtain multiple units of the good before leaving the market. In the equilibrium, whether the economy converges into expressing a homogeneous type of seller ω^{**} (as long as a positive measure of sellers is available in the market) depends on agents' time discount δ, together with the interaction between the initial composition of the sellers' vector $\boldsymbol{\mu}^S_0$ and the level of financial friction α currently in place. Importantly, in the model a tipping point $\alpha = \alpha^*$ determines the evolution of the market composition $\boldsymbol{\mu}^S_t$.

I now analyze the relationship between the initial heterogeneity of the sellers' vector $\boldsymbol{\mu}^S_0$

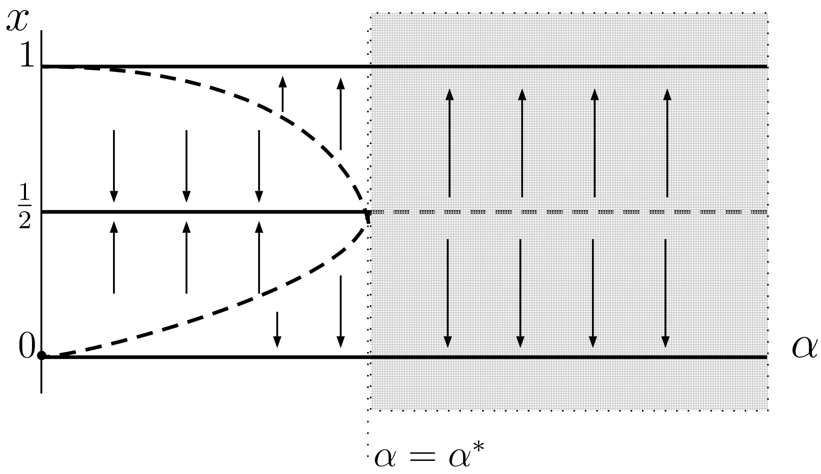

Figure 2: Phase diagram of market index x_t under the Homogeneous Equilibrium with no Entry. For any couple (α, x) in the graph, the solid-lines (respectively, dashed lines) indicate the stable (respectively, unstable) equilibrium x^*, with arrow pointing at the associated stable equilibrium. The threshold α^* indicates a regime switch for which the heterogeneous equilibrium $x^* = 1/2$ loses stability.

and leverage α in determining the long-run market dynamics. To do so, let us assume that δ is small enough so that (ii) of Theorem 1 holds. From Theorem (1) I notice that the population dynamics of the Homogeneous Equilibrium with no entry are qualitatively dependent on the value of the financial leverage α. Let us consider the evolution of type ω^{**} sellers relative to other types. Therefore, consider the evolution of x_t from Definition 4. From the proof of Theorem 1, the heterogeneous equilibria x^* associated with the law of motion (7) are given by

$$x_0^* \equiv \frac{1}{2} \quad x_+^* \equiv \frac{1}{2}\left(1 + \sqrt{1 - 4D(\alpha, \omega^{**})^2}\right) \quad x_-^* \equiv \frac{1}{2}\left(1 - \sqrt{1 - 4D(\alpha, \omega^{**})^2}\right), \tag{8}$$

together with the equilibrium $\bar{x}_0^* \equiv \mu_{0,\omega^{**}}$ for $\alpha = 0$ and homogeneous equilibria $\bar{x}_+ = 1$, $\bar{x}_- = 0$. With no financial leverage ($\alpha = 0$), the system is at rest of any point $x \in [0, 1]$. Intuitively, from Proposition 1, for $\alpha = 0$, $D(0, \omega^{**}) = 0$, therefore the initial market index x_0 is an equilibrium of the system. For $\alpha < \alpha^*$, the heterogeneous equilibrium $x_0^* = 1/2$ is reached by any index with initial composition $x_0 \in \left(x_-^*, x_+^*\right)$. Rest point $x = 0$ (respectively, $x = 1$) is a stable fixed point for $x_0 < x_-^*$ (respectively, $x_0 > x_+^*$). For $\alpha > \alpha^*$, rest point $\bar{x}_-$ (respectively, $\bar{x}_+$) is reached by all orbits between 1/2 and 0 (respectively, between 1/2 and 1). In other words, for a sufficiently high financial leverage $\alpha > \alpha^*$, the equilibrium $x = 1/2$ loses stability and the system converges to $x^* = 1$ (respectively, to $x^* = 0$) for any x_0 above (below) 1/2.

Therefore, the relationship between the market composition and financial constraint is nonunivocal. The capability for a market to express in the limit of $t \to \infty$ a homogeneous population of sellers that are on the efficient production frontier is subject to a tipping point which depends on the level of financial frictions and the initial heterogeneity of sellers. When the allowed leverage is moderate, the exchange structure has a limited impact on sellers' distribution. This implies that sellers heterogeneity does not disappear in the limit. On the other hand, when the exchange structure is characterized by a high enough financial leverage, two extreme cases emerge depending on the *initial* distribution. Either the exchange dynamics will bring the market to an efficient equilibrium (with all sellers endowed with the first-best investment ω^{**}), or it will converge to an equilibrium in which surviving sellers fail to attain the first-best investment level.

From a regulatory perspective, this result shows that when entry of new sellers is limited (or restricted), a regulator can not assess the effect of financial constraints on allocative efficiency in isolation from the initial composition of the market, thus explaining why incremental regulations to corporate leverage have diverse effects in conglomerates around the world (see Lee et al., 2009 and Buchuk et al., 2014 for two alternative examples).

3.2 Homogeneous Equilibria with Entry (HEWE)

In the previous section I showed that a market endowed with simple monotonic rules of motion such that in the limit all sellers leave the market can display a surprising degree of richness in the dynamics of sellers' *composition*. Indeed, such framework can effectively depict short run dynamics of capital transfers, for example within conglomerates (see for example Campello, 2002). However, it fails to capture more complex dynamics. From a longer-run perspective, capital can be *re*-allocated within conglomerates. Similarly, input-exchanges along value chains undergoe multiple rounds of transformation, with receivers of inputs transforming and selling to subsequent buyers acquired units of capital. In other words, many real-market environments feature a possibly *expanding* population of sellers, with potential sellers entering and quitting as the market unfolds.

To capture this, I now explore the dynamics of a *growing* economy. I begin by adopting the simple expansionary transition protocol $T.2$ of Definition 3. For every period t, I will allow a subset of buyers to settle in the economy at the end of the period in order to become potential sellers for waves of buyers entering in the market in period $t+1, t+2, \ldots$. Given Assumption 1, the subset of buyers who settle in the market is given by Partially Constrained buyers in D^{**} and Pure Sellers in S as per Definition 1. To define the (possibly growing) measure of sellers, let us recall Definition 11 from Appendix A

Definition 6 (Settlements). *For every type $\omega \in S$ and any time period $t > 0$, let*

$$\hat{\mu}^B_{\omega'} = \mu^B_{\omega'} + \sum_{\omega < \omega'} \mu^B_{\omega} \cdot \mathbb{I}_{\omega',\omega} \left[\omega + q_{\omega}(\alpha, \omega^{**}) = \omega'\right],$$

be the cumulative measure of buyers endowed with $\omega = \omega'$ at the end of the trading period t. In the above, $\mathbb{I}$ is the indicator function and takes value 1 when the condition in square *brackets is matched and zero otherwise.*

Measure $\hat{\mu}^B_{\omega'}$ encapsulates the fraction of entrants with initial type $\omega \in \Omega$ that possess an amount of capital $\omega' \in \{D^{**}+S\}$ at the end of period t[19]. Given Transition protocol $T.2$, $\hat{\mu}^B_{\omega^{**}}$ and $\hat{\mu}^B_{\bar{\omega}}$ measure the fixed inflow of sellers of type ω^{**} and $\bar{\omega}$ respectively. Therefore, under transition protocol $T.2$, the system of equations dictating the evolution of the market composition (5) becomes

$$\mu^S_{\omega'(t+1)} = \mu^S_{\omega' t} - \sum_{\{\omega\in\Omega|\omega\le\omega^{**}\}} \left(\alpha_{\omega\omega' t}\pi^S_{\omega' t}\mu^B_{\omega} + \alpha_{\omega'\omega t}\pi^B_{\omega}\mu^S_{\omega' t}\right) \qquad \forall\omega' < \omega^{**} \tag{9}$$

$$\mu^S_{\omega'(t+1)} = \mu^S_{\omega' t} + \underbrace{\hat{\mu}^B_{\omega'} - \sum_{\{\omega\in\Omega|\omega\le\omega'\}} \left(\alpha_{\omega\omega' t}\pi^S_{\omega' t}\mu^B_{\omega} + \alpha_{\omega'\omega t}\pi^B_{\omega}\mu^S_{\omega' t}\right)}_{net\ equilibrium\ in-flow\ of\ \omega'-type\ sellers} \ \forall\omega' \ge \omega^{**}. \tag{10}$$

From (9), I intuitively notice that sellers characterized by type $\omega' < \omega^{**}$ will eventually disappear from the market. On the other hand, the apparent symmetry of dynamics characterizing types $\omega' \ge \omega^{**}$ in (10), implies that further analysis is required to understand the evolving composition of sellers. To do so, I introduce the following class of equilibria

Definition 7 (**Homogeneous Equilibria with entry (HEWE)**)**.** *Given the dynamic market with frictions in Definition 2 with the transition protocol $T.2$ from Definition 3, I define Homogeneous Equilibria with entry the class of equilibria of the bargaining game such that*

(*i*) *The market follows the law of motion in* (9)-(10) *where* $\mu_{\omega^{**}0} = 1$,

(*ii*) *Demand D clears in every period $t > 0$,*

(*iii*) *In the limit $t \to \infty$ there is no seller-type dispersion and it holds that $x^* = 1$.*

And state the second main result.

Lemma 1. *Let $A \equiv \alpha/(1+\alpha)$ and suppose a dynamic market with frictions following law of motion in* (9)-(10) *that fulfills the following*

(*i*) $\sum_{\omega=0}^{\bar{\omega}} \mu^S_{\omega 0} \ge \sum_{\omega=0}^{\omega^{**}-1} \tilde{\mu}^B_{\omega} = \dfrac{\omega^{**}(\omega^{**}-1)}{2\bar{\omega}} \cdot A,$

(*ii*) $A \le \bar{A} \equiv \dfrac{2(2+\bar{\omega}-\omega^{**})}{(\omega^{**}-2)\omega^{**}},$

(*iii*) $A \ge \tilde{A}$, *where $\tilde{A}$ is the solution of* $\tilde{A}\left(1+\sqrt{1+\tilde{A}\omega^{**}}\right) = \dfrac{2\bar{\omega}}{(\omega^{**})^2}$

(*iv*) $\delta \le \bar{\delta}$,

[19] In Appendix A I provide additional intuition and formal results related to measure $\hat{\mu}$ together with two computed example.

where $\bar{\delta}$ is as defined in Theorem 1. Then, the Homogeneous Equilibrium with entry (HEWE) is the unique stable equilibrium of the economy and as such the market index x_t converges to it for $t \to \infty$.

Therefore, a dynamic market characterized by sufficient degree of initial homogeneity in sellers' types and a compatible degree of financial leverage α will converge to expressing only first-best endowment sellers ω^{**}. Financial leverage in the present context has two roles. On one hand, it positively affects the size of demand D induced by buyers entering in every period. As such, market clearing puts an upper bound on financial leverage. On the other hand, the larger average demand may allow a larger measure of first-best type agents to settle in the market as potential sellers, thus allowing the first-best seller measure to grow faster across time relative to the measure of other seller types available in the market. The combination of the two effects determines in the limit the degree of heterogeneity of the sellers' vector x^*. In particular, from the proof of Proposition 1, I know that if financial leverage is such that $A < \tilde{A}$, heterogeneity will not disappear in the limit. In fact, the *growth rate* of the first-best endowed agents at $t = 1$, defined as $m_{\omega^{**}1} \equiv \mu^S_{\omega^{**}1} - \mu^S_{\omega^{**}0}$, will be negative, thus implying that the market index x_t will converge to x^* where $0 \leq x^* < 1$. I consider this situation in the last set of simulations in Section 5.

In the next Section I show that a surprising link between HEWNE and HEWE exists which can be used to study the role of x^* in determining the dynamical properties of markets in which more complex transition protocols are enforced, in particular, when the entrance of new sellers is decoupled from the exit of buyers.

4 Analysis: Decoupling buyer-seller dynamics

In this section I focus on the link between buyer-seller flow dynamics and market heterogeneity. Therefore, I will assume fully-impatient agents ($\delta = 0$) and ask the following question: what market dynamics can I expect when entry of new sellers is decoupled from buyers' exit? As example, suppose an economy in which buyers require $\tilde{t}$ periods to transform the purchased input into an output that can be sold to other buyers. In such context, the entry of new sellers begins at period $t = \tilde{t} > 0$. In other words, this is equivalent to assuming that the market is characterized by transition protocol $T.1$ for $t = 1, 2, ..\tilde{t}$ and by transition protocol $T.2$ for $t > \tilde{t}$. I consider such scenario in Section 4.1 As alternative example, consider an economy where entry of new sellers ceases after $\tilde{t} > 0$ periods. The interruption can either be temporary[20] or permanent. In particular,

[20]This can be the case for a market hit by a large exogenous shock. For example, Barrot and Sauvagnat (2016) find that sales growth of Japanese firms directly affected by the Tohoku Earthquake recovered after six quarters from the occurrence of the natural disaster.

in the framework a permanent halt of sellers' entry at $t = \tilde{t} > 0$ can be studied by introducing a market following transition protocol $T.2$ for $t = 1, ...\tilde{t}$ and protocol $T.1$ for period $t = \tilde{t}+1, \tilde{t}+2, ...$ I consider that situation in Section 4.2 Indeed, more complex scenarios can be encompassed. For example, a market characterized by transformation delay $\tilde{t}$ such the one described in the first example may undergo an exogenous shock in period $\kappa > \tilde{t}$ which interrupts the sellers inflow from $t = \kappa$ onward. Such situation, where new sellers enter only for a *limited* number of periods, can be addressed by introducing an additional switch from transition protocol $T.2$ to protocol $T.1$ at time $t = \kappa > \tilde{t}$. This case will be considered in Section 4.3

I now show how the equilibria introduced in the previous sections allow us to explore market dynamics in the above environments. From Theorem 1, let us define

$$A^* \equiv \frac{\alpha^*}{1+\alpha^*} = \frac{\bar{\omega}}{(\omega^{**})^2},$$

I can then construct the following taxonomy

Lemma 2. *Consider the dynamic market with frictions M in Definition 2. Assume $A < \bar{A}$. The following hold*

1. $A \geq A^*$ *or* $\tilde{A} \leq A \leq A^*$. *If HEWE exists and is stable under transition protocol* $T.2$*, HEWNE always exists and is stable under transition protocol* $T.1$.
2. $A < \tilde{A}$ *HEWE does not exists under transition protocol* $T.2$. *HEWNE exists under transition protocol* $T.1$ *if* $x_0 \geq x_+^*$.

Hence, conditions for existence and stability of HEWE are more stringent than the ones of HEWNE. The results in Lemma 2 allow us to study the asymptotic behavior of market index x_t for a variety of complex transition protocols, discussed below.

4.1 Asynchronous entry of new sellers

Consider a situation in which sellers *delay* entrance in the market for $\tilde{t}$ periods, as it would be the case, for example, if buyers require a certain number of periods $\tilde{t}$ to transform the purchased input into an exchangeable output. For the purpose, let us introduce a *delay* $\tilde{t} > 0$, which I define below, and modify transition protocol $T.2$ in the following

Definition 8 (Transition Protocol $T.3$)**.** *Each seller j leaves the market upon successful trade of one unit of the good. Upon satisfaction of demand schedule q_{ω_i}, if $q_{\omega_i} + \omega_i < \omega^{**}$, buyer i leaves the market. Otherwise, if $q_{\omega_i} + \omega_i \geq \omega^{**}$, buyer i leaves the market as buyer, and settles in the market as potential seller in period $T = t + \tilde{t}$, with $\tilde{t}$ sufficiently large. That is i becomes potential seller for buyers entering at periods $T, T+1, ...$*

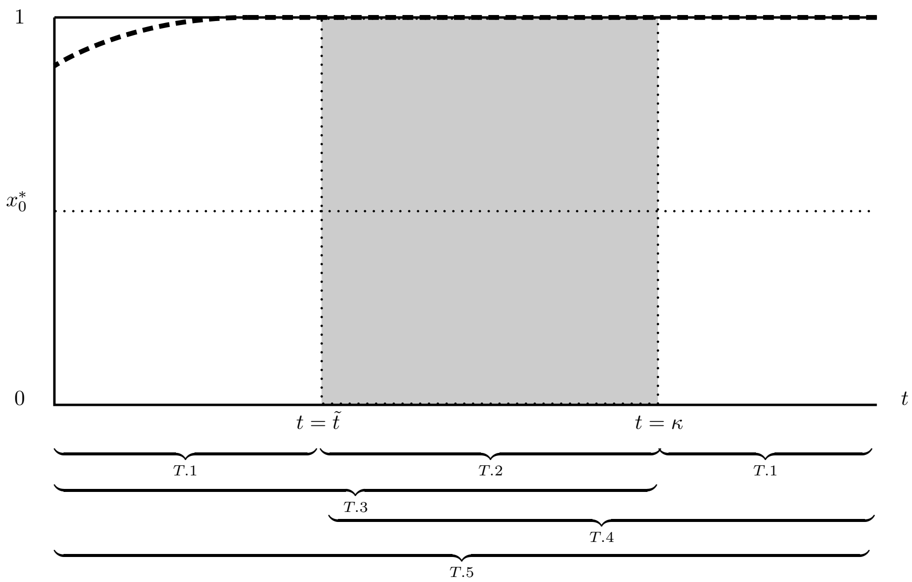

Figure 3: Dynamics for a market in which $A > A^*$ and $x_0 > x_0^*$ with respect to the possible transition protocols $T.1 - T.5$. Transparent (respectively, colored) area contains the dynamics for market compatible with transition protocol $T.1$ (respectively, $T.2$).

Indeed, the *asynchronous* entry of new sellers implied by protocol $T.3$ affects the evolution of market composition $\boldsymbol{\mu}_t^S$ as follows: for $t < \tilde{t}$ (respectively, for $t \geq \tilde{t}$), the market composition x_t evolves according to protocol $T.1$ (respectively, $T.2$), with law of motions given by (7) (respectively, (9)-(10)). The following result holds

Lemma 3 (Market behavior under asynchronous entry)**.** *Suppose two dynamic markets with frictions, M_1 and M_3, where, everything equal, the associated transition protocol is respectively given by $T.1$ and $T.3$. Assume that HEWNE exists and is the stable solution of M_1. Either of the following is then the case*

1. *$A \geq A^*$ or $\tilde{A} < A < A^*$. HEWE is the unique stable equilibrium of market M_3.*
2. *$A < \tilde{A}$. In M_3, HEWE does not exist. Financial leverage α relative to the size of ω^{**} and $\bar{\omega}$ is such that the fraction of ω^{**} type buyers settling in the market at $t = \tilde{t}+1, \tilde{t}+2, \ldots$ is not sufficiently high, and as such the market will leave the orbit dictated by HEWNE.*

In Figure 3 (respectively, Figure 4) the orbits corresponding to interval $[0, \kappa]$, with $\kappa \to \infty$, depict the evolution of market composition x_t for the case in which $A > A^*$ (respectively, $A < A^*$). In particular, with respect to Figure 4, it is clear that actual market dynamics in the case $A < \tilde{A}$ (orbits $B - E$) will depend on the size of A.

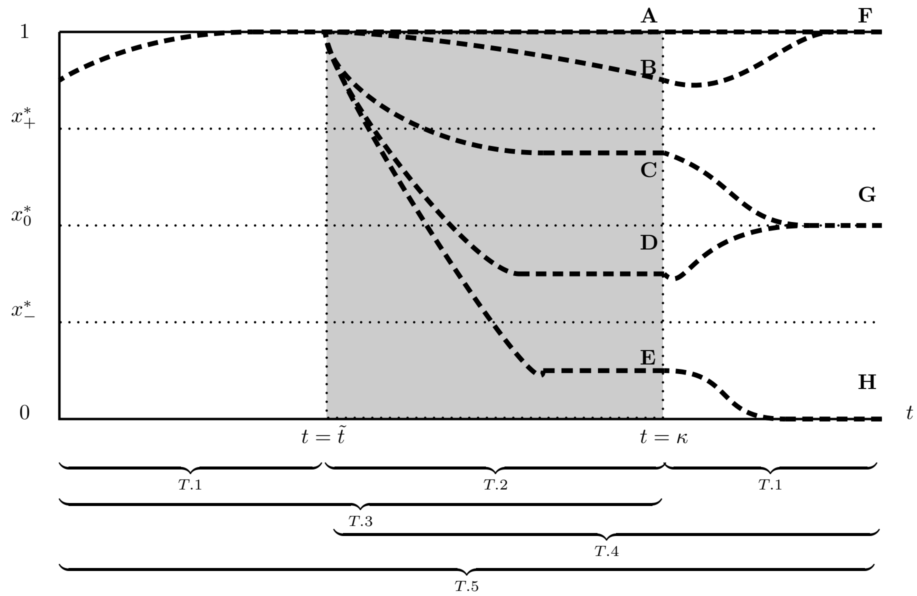

Figure 4: Dynamics for a market in which $A < A^*$ and $x_0 > x_+^*$ with respect to the possible transition protocols $T.1 - T.5$. Transparent (respectively, colored) area contains the dynamics for market compatible with transition protocol $T.1$ (respectively, $T.2$).

4.2 Permanent Interruption of Seller Entry

I now consider the case of a market in which entry of new sellers permanently ceases after a certain number of periods $\kappa > 0$. For this purpose, I introduce the following transition protocol

Definition 9 (Transition Protocol $T.4$)**.** *For $t < \kappa$ (respectively, $t \geq \kappa$), market M behaves according to ransition Protocol $T.2$ (respectively, transition protocol $T.1$).*

Therefore, a permanent interruption of the sellers' inflow can be incorporated in the framework by introducing an exogenous switch in the market dynamics in period κ such that the market composition x_t evolves according to law of motions given by (9)-(10) (respectively, (7)) for $t = 0, 1, ..., \kappa - 1$ (respectively, for $t \geq \kappa$). I then have the following result

Lemma 4 (Market behavior under permanent interruption of seller entry)**.** *Suppose two dynamic markets with frictions, M_2 and M_4, where, everything else equal, the associated transition protocol is respectively given by $T.2$ and $T.4$. Assume that HEWE exists and is the stable solution of M_2. Then, HEWNE is the unique stable equilibrium of market M_4.*

Lemma 4 simply reinstates that HEWE requires stronger existence conditions when compared to HEWNE. Notice however that under transition protocol $T.4$, HEWE can emerge as market equilibrium even if $A < \tilde{A}$. This situation is captured by orbit F in Figure 4, where HEWNE is obtained for all orbits with initial market composition $x_0 \in (x_+^*, 1)$.

The equilibrium characterization obtained for transition protocol $T.1$ in the proof of Theorem 1, reported in (8) and ensuing discussion allow us to get sharp predictions on the market composition in the limit of $t \to \infty$.

4.3 Temporary Entry of New Sellers

Results in Theorem 1, Proposition 1 and Lemma 3 allow us to explore the effect on market composition x_t of more elaborate transition protocols. For example, I briefly consider a market where entry of new sellers is allowed for a subset of periods $t = 1, ..., \kappa$ *and* such that capital reallocation is characterized by some transformation delay $\tilde{t}$. In other words, only a limited number of buyers, those belonging to waves $t = 1, ..., \kappa - \tilde{t}$ are allowed to settle in the market as sellers for subsequent waves of buyers. This can be captured by the following transition protocol

Definition 10 (Transition Protocol $T.5$)**.** *Given $\tilde{t}$ and κ such that $0 < \tilde{t} < \kappa$, for $0 < t < \tilde{t}$, and for $t > \kappa$, market M behaves according to transition Protocol $T.1$. For $\tilde{t} \leq t \leq \kappa$ market M behaves according to transition Protocol $T.2$.*

With a sufficiently high financial leverage $A > A^*$, following an argument similar to the one of the previous two sections, it is easy to show that market composition x_t will converge to the homogeneous case $x = 1$ for $t \to \infty$ regardless to actual determination of $\tilde{t}$ and κ. More importantly, I notice that for all cases in which $A < A^*$, in the limit of $t \to \infty$, the market composition will converge to either of the following $x \in \{0, 1/2, 1\}$ (see Figure 4). This follows immediately from the discussion of (8).

5 Simulations of Market Dynamics

Parameter:	Value:
Type set Ω:	$\{1, 2, 3, 4, 5\}$
first-best endowment ω^{**}:	4
Discount factor δ:	$\delta < \bar{\delta}$
Financial Leverage α:	variable
Initial measure of first-best sellers $\mu_{\omega^{**}0}$:	variable

Table 1: Simulation parameters (Section 5).

I validate the results obtained in Section 3 by simulating an artificial market which allows us to study market evolution under either transition protocol $T.1$ or $T.2$ (see Definition 3). The common parameter configuration is reported in Table 1. Given the common

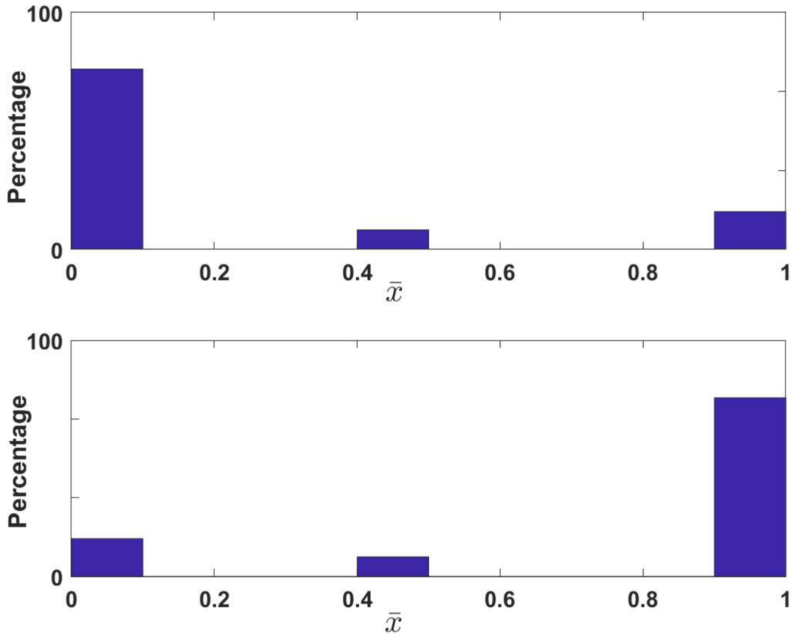

Figure 5: Distribution of average asymptotic index value $\bar{x}$ for an artificial market under transition protocol $T.1$, financial leverage $\alpha = 0.8$ and initial first-best seller distribution respectively given by $\mu^S_{\omega^{**}0} = 0.3$ (*Upper Panel*) and $\mu^S_{\omega^{**}0} = 0.6$ (*Bottom Panel*).

configuration, I can compute $\alpha^* = 0.45$ from Theorem 1. Furthermore, from Proposition 1 I compute $\bar{A} = 0.75$ and $\tilde{A} \approx 0.26$ respectively corresponding to $\alpha = 3$ and $\alpha \approx 0.36$. The artificial market adheres to the program described in Section 2 and in Figure 8 with one minor adjustment. I will assume that one single buyer endowed with ω, distributed according to $P(\omega) = 1/\bar{\omega}\ \forall \omega \in \Omega$ enters the market in every period t, rather than a fixed measure $\left\{\mu^B_i \in [0,1]\right\}_{i=0}^{\bar{\omega}} \equiv \boldsymbol{\mu}^B$ entering the market every period t.

In the first set of simulations I fix $\alpha = 0.8 > \alpha^*$ and consider an economy where transition protocol $T.1$ dictates market formation. Each economy k runs for $t = 1, 2, ..., 1,000$ periods (corresponding to $N = 1,000$ sellers available in the market). For each economy k, I store $x_t = x^{(k)}_{1000}$. I simulate a total of $s = 10,000$ economies and compute the *average asymptotic index value* $\bar{x} \equiv \mathbb{E}[x_{1000}] = \frac{1}{s}\sum_{k=1}^{s} x^{(k)}_{1000}$. From Theorem 1, this configuration implies that three possible equilibria can exist, respectively given by x^*_0, $\bar{x}_+$ and $\bar{x}_-$, depending on the initial heterogeneity of sellers given by $\mu_{\omega^{**}0}$. In the first simulation, I assume $\mu_{\omega^{**}0} = 0.3$. As predicted by Theorem 1, the artificial market converges for more than 80% in the predicted equilibrium, $\bar{x} = 0$, where first-best producers are completely displaced from the market before other available types (see Figure 5, top panel). In the second simulation, everything equal, I assume an initial heterogeneity given by $\mu_{\omega^{**}0} = 0.6$. As expected, more than 90% of market simulations converge to scenario where first-best type sellers leave the market after all other types, that is $\bar{x} = 1$ (see Figure 5, bottom panel).

In the second set of simulations, I consider an economy where market evolution follows transition protocol $T.2$. First, I impose $\mu_{\omega^{**}0} = 1$ and $\alpha = 2.5$, so that conditions $(i)-(iv)$

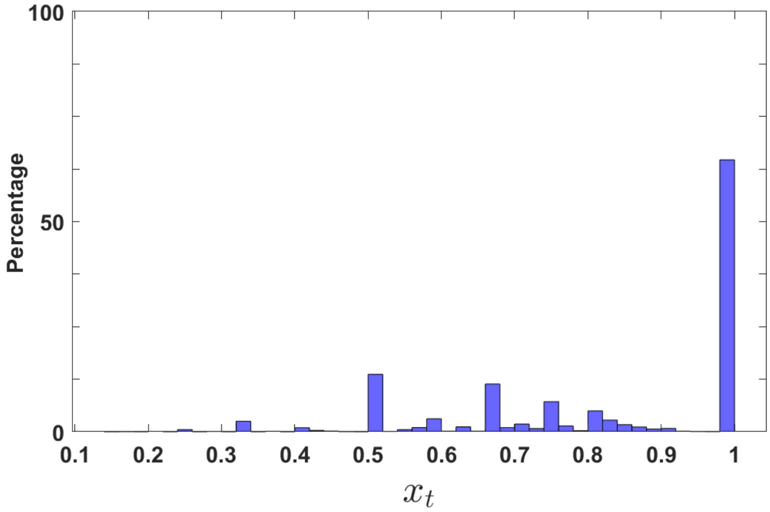

Figure 6: Distribution of index value x_t for an artificial market under transition protocol $T.2$ with financial leverage $\alpha = 2$ and initial first-best seller distribution $\mu_{\omega^{**}0} = 1$.

of Proposition 1 are satisfied. Each economy k runs for $t = 1, 2, ..., 50,000$ periods. For each economy k, I store $x_t^{(k)}$, $t > 0$. I simulate a total of $s = 100$ economies and compute $P(x)$, the empirical distribution of realized index values x_t, across all periods and simulations. Results are reported in Figure 6. Given the selected parametrization, it is easy to see that the market rapidly converges to gravitating around the expected equilibrium $x^* = 1$ with a limited dispersion in the upper side of the distribution.

Secondly, I modify the above setup by imposing $\alpha = 0.35$ (corresponding to $A < \tilde{A}$), thus violating condition (iii) for existence of HEWE (see Proposition 1). My aim is to explore the steady state behavior of the market outside HEWE. The result is reported in Figure 7. Clearly, as $A < \tilde{A}$, the mass of first-best endowed agents settling in the market as potential sellers is not sufficiently large to overcome the in-flow of alternative sellers. As a result, the market index converges to a heterogeneous steady-state $x^* \approx 0.65 < 1$.

6 Conclusions

In this paper I explored the complex relationship between financial frictions and allocative efficiency in decentralized markets. To this purpose, I constructed a model of dynamic market formation in which financially constrained buyers purchase multiple items from a pool of sellers whose composition evolves on the basis of realized trades. I posed the efficiency problem in terms of existence (or lack thereof) of two alternative classes of equilibria, respectively named Homogeneous Equilibria with no Entry (HEWNE) and Homogeneous Equilibrium with Entry (HEWE) of new sellers, respectively. In these equilibria, available sellers produce on the efficient frontier, or, in other words, only a homogeneous type of seller exists in the limit. For HEWNE, the capability for a market

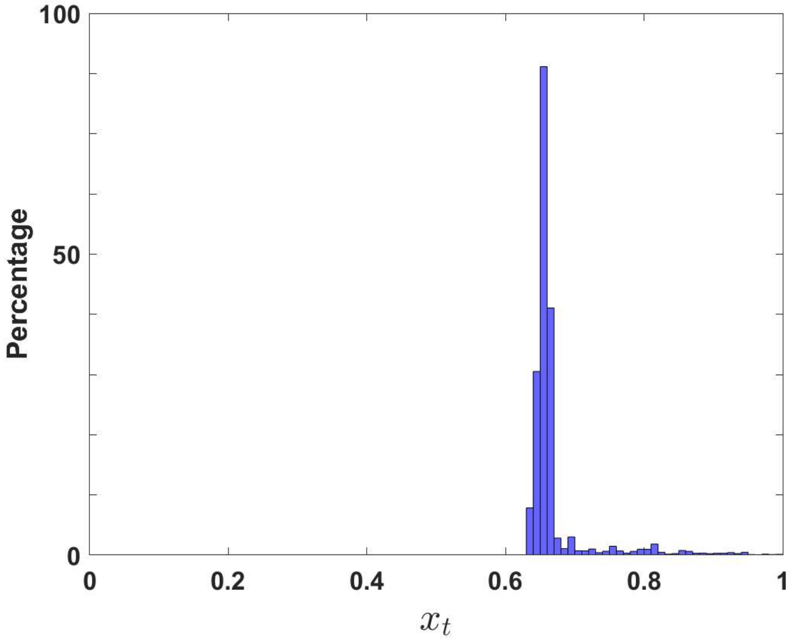

Figure 7: Distribution of index value x_t for an artificial market under transition protocol $T.2$ with financial leverage $\alpha = 0.3$ and initial first-best seller distribution $\mu_{\omega^{**}0} = 1$.

to reach an efficient redistribution of inputs is subject to a *tipping point* which depends on the the specific level of financial frictions in place and the initial heterogeneity of sellers. From a regulatory and operative perspective, this result shows that when entry of new sellers is limited (or restricted), a regulator can not assess the effect of financial constraints on allocative efficiency in isolation from the initial composition of the market. This result may explain why incremental regulations to financial leverage may induce diverse effects in financial conglomerates and OTC markets around the world.

Relatively to HEWE, I formally showed that entry of new sellers in every period imposes *stronger* conditions for the homogeneous type of seller to exist in the limit (i.e. the efficient outcome), hence an ordered relationship exists between the two equilibria. The ordering is relevant from a normative point of view, as it allows to deliver sharp predictions on the market evolution for realistic situations where exit of buyers is *decoupled* from entrance of sellers in non-trivial ways. I discussed allocative efficiency in the three following scenarios. In the first scenario, entrance of new sellers is *asynchronous* with respect to exit of buyers, as it can be the case, for example, of sellers requiring a certain number of periods to produce an exchangeable good or to set-up exchange. In the second scenario, new sellers enter only for a limited number of periods, thus replicating a market undergoing a permanent negative supply shock. In the third scenario, existence of the homogeneous type of seller is discussed for a market that combines the above cases: delayed entry of sellers is interrupted altogether after a certain number of periods. For the three cases, I provided a taxonomy of bounds on financial constraint that allow the efficient outcome to survive.

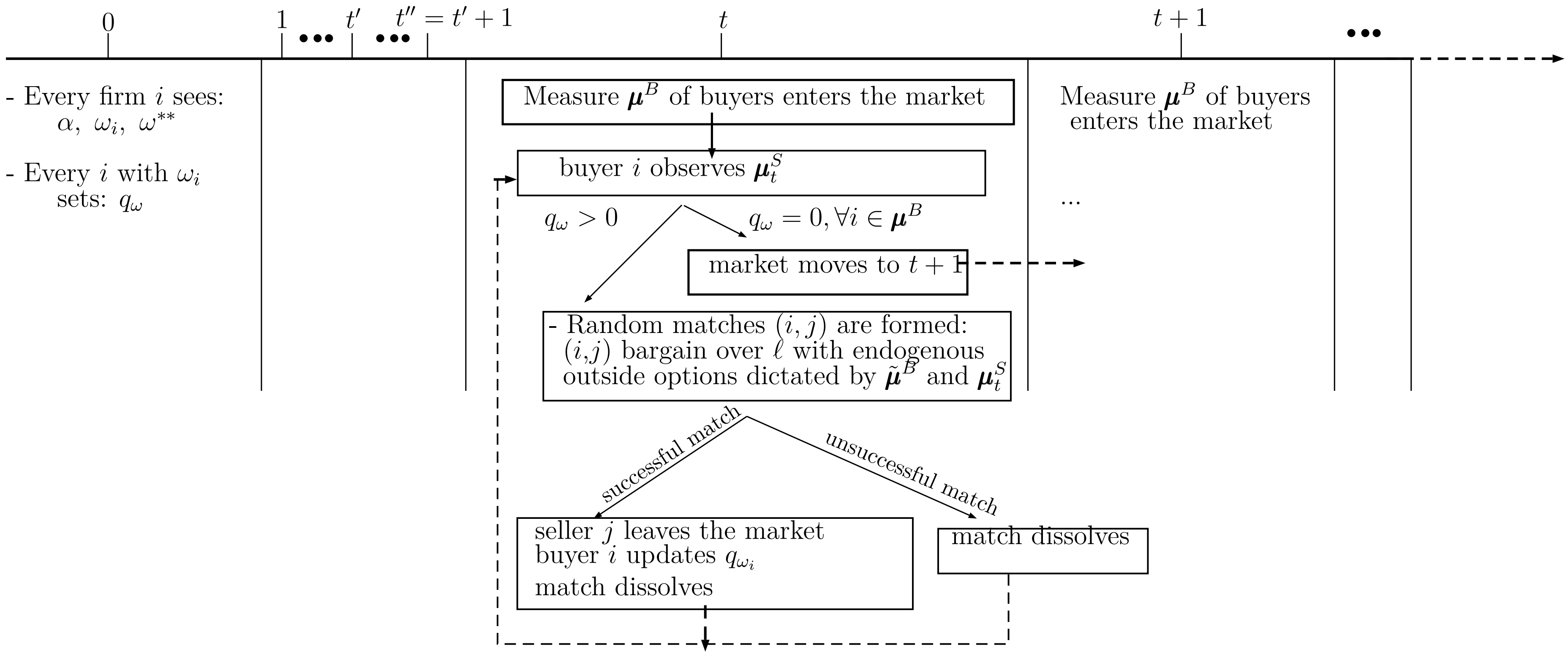

Figure 8: The model time-line under transition protocol $T.1$ (Definition 3).

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Appendix

A Construction of cumulative buyer in-flow

The intra-period matching incurs in the following complexity[21]. It implicitly assumes a further temporal dimension within every period t in which sequences of bargaining take place. Because for any of the sub-period matches the pairwise surplus is a function of both the seller's and the buyer's type at the time of match, a buyer's payoff would depend on the entire sequence of trades that may take place within the period. I overcome this problem by exploiting the fact that the good is traded in fixed units. This allows us to construct a *cumulative* measure of types $\hat{\mu}^B_{\omega'}$ tracking how many times a type ω' is expressed in the population of buyers in the course of the intra-period bargaining process. Crucially, for all buyers of type $\omega' \geq \omega^{**}$, $\hat{\mu}^B_{\omega'}$ coincides with the measure of buyers endowed with ω' at the end of period t. I first provide an example to clarify ideas and then formalize measure $\hat{\mu}^B_{\omega'}$ with a definition.

Example 2 (Cumulative buyer in-flow)**.** Consider an economy such that $\Omega \equiv \{1, 2, ..., 20\}$, $\omega^{**} = 18$, and $\alpha = 2$. I assess how many times type $\omega = 12$ will be expressed in the population of buyers across all the transactions taking place in one single period t. In other words, I compute the size of the set $\bar{\Omega}_{12} \equiv \{\omega \in \Omega : \omega \leq 12, q_\omega \geq 12\}$. For example, consider $\omega = 4$. Agents endowed at entry with $\omega = 4$ are allowed to collect up to 8 units and will end up with $4 + 8 = 12$ units. Now consider type $\omega = 10$. Agents with type $\omega = 10$ can collect up to 20 units of the good. However, as $\omega^{**} = 18$, they will collect only $18 - 10 = 8$ units of the good. The same rationale applies to the remaining types in $\bar{\Omega}_{12}$. Given the above parameters, it is easy to verify that such set contains types $\omega \in \{4, 5, 6, 7, 8, 9, 10, 11, 12\} \equiv \bar{\Omega}_{12}$, so that its size is $|\bar{\Omega}| = 9$ so that in expectations every period type ω_{12} is observed with frequency $9/20$.

I can now introduce the following definition

Definition 11 (Cumulative buyer in-flow)**.** *For every buyer type $\omega \in \mathcal{D}$, define the cumulative measure of type ω buyer such as*

$$\hat{\mu}^B_\omega(q_\omega) \equiv \mu^B_\omega + \sum_{\omega' < \omega} \mu^B_{\omega'} \cdot \mathbb{I}_{\omega',\omega}\left[\omega' + q_{\omega'} \geq \omega\right], \tag{11}$$

where $\mathbb{I}$ is the indicator function taking value $\mathbb{I}_{\omega',\omega} = 1$ if conditions in square brackets

[21]Results in this sections are required for the proof of Theorem 1 and Proposition 1.

are matched and zero otherwise. In particular

$$\hat{\mu}^B_{\omega^{**}}(q_\omega) = \mu^B_{\omega^{**}} + \sum_{\omega'<\omega^{**}} \mu^B_{\omega'} \cdot \mathbb{I}_{\omega',\omega^{**}}\left[\omega' + q_\omega = \omega^{**}\right] \equiv \Delta, \tag{12}$$

is the cumulative measure of ω^{**} type buyers induced by the set of buyers belonging to D^{**} (see Definition 1). Define $\tilde{\mu}^B_\omega(q_\omega)$ such as the the cumulative measure of buyer ω available for trading in any period t

$$\tilde{\mu}^B_\omega(q_\omega) \equiv \sum_{\omega'<\omega} \mu^B_{\omega'} \mathbb{I}_{\omega',\omega}\left[\omega' + q_{\omega'} \geq \omega\right]. \tag{13}$$

Furthermore, define $\tilde{\pi}^B_\omega(\tilde{\mu}^B_\omega)$ such as the probability for any seller j active in market to meet an available buyer ω along the $|q_\omega|$ successful matchings of that buyer.

Given (11) and (13), in the proof of Theorem 1 I make use of the following

Proposition 2. *For every type $\omega \in \mathcal{D}$ and the cumulative buyer measure $\tilde{\mu}^B_\omega(q_\omega)$ defined above, the following properties hold:*

(i) Given a buyer's type ω and two measures $\hat{\mu}^B_\omega(\alpha')$, $\hat{\mu}^B_\omega(\alpha'')$ with $\alpha'' > \alpha'$, $\hat{\mu}^B_\omega(\alpha')$ First Order Stochastically Dominates $\hat{\mu}^B_\omega(\alpha'')$ or, equivalently, $\hat{\mu}^B_\omega(\alpha') \succ_{FO} \hat{\mu}^B_\omega(\alpha'')$. Moreover, it holds that

$$\begin{aligned}\hat{\mu}^B_\omega(q_\omega) &= \frac{1}{\bar{\omega}} \cdot \left\lfloor 1 + \frac{\alpha}{1+\alpha}\omega \right\rfloor \\ &\approx \frac{1}{\bar{\omega}} \cdot \left[1 + \frac{\alpha}{1+\alpha}\omega\right],\end{aligned} \tag{14}$$

where $\lfloor \cdot \rfloor$ represents the floor function.

(ii) The measure $\tilde{\mu}^B_\omega(q_\omega)$ from Definition 11 corresponds to

$$\tilde{\mu}^B_\omega(q_\omega) = \frac{1}{\bar{\omega}} \cdot \left\lfloor \frac{\alpha}{1+\alpha}\omega \right\rfloor, \tag{15}$$

I also compute the following associated items used in Theorem 1 and Proposition 1

(iii) $\sum_{\omega=0}^{\omega^{**}-1} \tilde{\mu}^B_\omega(q_\omega) = \dfrac{\alpha\omega^{**}(\omega^{**}-1)}{2(1+\alpha)\bar{\omega}}$;

(iv) $\sum_{\omega=0}^{\omega^{**}-1} s_{\omega\omega^{**}} \tilde{\pi}^B_\omega \geq a \cdot \underbrace{\left[\dfrac{\omega^{**}(\omega^{**}+1)(3+\alpha(2+\omega^{**}))}{6(1+\alpha)\bar{\omega}}\right]}_{\sigma} = a \cdot \sigma$,

where $a \cdot \sigma$ bounds the expected surplus generated by matches with a ω^{**}-type seller.

Example 3 (Cumulative buyer in-flow (continued))**.** Consider again Example 2. Equation (14) verifies that $|\tilde{\Omega}_{12}| = 9$ types will be "crossing" type $\omega = 12$ at some point of the capital collection during period t, so that in expectations every period type ω_{12} is observed with frequency $9/20$.

B Equilibrium Existence

I first introduce the equilibrium of the intra-period bargaining game from Manea (2017), where the market composition $\boldsymbol{\mu}_t^S$ is exogenously fixed and then I discuss the equilibrium of the intertemporal bargaining game, where $\boldsymbol{\mu}_t^S$ is allowed to vary across periods. The following result characterizes the strategies which are selected through iterated deletion of dominated strategies, identifies the unique payoff vector for all the types of agents and establishes the existence of the equilibrium in the intra-period game for a general class of dynamic markets which this model belongs to. I then adapt the result to the present setting in proof of Theorem 1 by making use of the comulative measures obtained in Appendix A.

Theorem 2 (p. 63 Manea (2017))**.** *For every fixed pair $\boldsymbol{\mu}_t^S$ and $\boldsymbol{\mu}^B$, there exists a unique pair of payoff vectors $\left\{v_{\omega' t}^{S^*}(\boldsymbol{\mu}_t^S)\right\}_{\omega'\in\Omega}$ and $\left\{v_{\omega t}^{B^*}(\boldsymbol{\mu}_t^S)\right\}_{\omega\in\Omega}$ such that:*

(i) The only date t actions which would survive iterated dominance specify that all buyers (respectively, sellers) of type ω (respectively, ω') reject any offer $x < \delta v_{\omega(t+1)}^{B^*}(\boldsymbol{\mu}_t^S)$ (respectively, $\delta v_{\omega'(t+1)}^{S^*}(\boldsymbol{\mu}_t^S)$) and accept any offer $x > \delta v_{\omega(t+1)}^{B^*}(\boldsymbol{\mu}_t^S)$ (respectively, $\delta v_{\omega'(t+1)}^{S^*}(\boldsymbol{\mu}_t^S)$).

(ii) In every equilibrium, the expected payoff of any active buyer (respectively, seller) of type ω (respectively, ω') at time t is given by $v_{\omega t}^{B^*}(\boldsymbol{\mu}_t^S)$ (respectively, $v_{\omega' t}^{S^*}(\boldsymbol{\mu}_t^S)$).

(iii) The equilibrium payoffs $\left\{v_{\omega' t}^{S^*}(\boldsymbol{\mu}_t^S)\right\}_{\omega'\in\Omega}$ and $\left\{v_{\omega t}^{B^*}(\boldsymbol{\mu}_t^S)\right\}_{\omega\in\Omega}$ constitute the unique solution to the system of equations

$$v_{\omega t}^{B^*}(\boldsymbol{\mu}_t^S) = \sum_{\omega'\in\Omega} \pi_{\omega' t}^S(\mu_{\omega' t}^S)\left(s_{\omega\omega'} - \delta v_{\omega'(t+1)}^{S^*}\right) + \left(1 - \sum_{\omega'\in\Omega} \pi_{\omega' t}^S(\mu_{\omega' t}^S)\right)\delta v_{\omega(t+1)}^{B^*}(\boldsymbol{\mu}_t^S)$$

$$v_{\omega' t}^{S^*}(\boldsymbol{\mu}_t^S) = \sum_{\omega\in\Omega} \pi_{\omega}^B(\mu_{\omega}^B)\left(s_{\omega\omega'} - \delta v_{\omega(t+1)}^{B^*}\right) + \left(1 - \sum_{\omega\in\Omega} \pi_{\omega}^B(\mu_{\omega}^B)\right)\delta v_{\omega'(t+1)}^{S^*}(\boldsymbol{\mu}_{t+1}^S)\ ,$$

such that $\left\{v_{\omega' t}^{S^*}(\boldsymbol{\mu}_t^S)\right\}_{\omega'\in\Omega}$ and $\left\{v_{\omega t}^{B^*}(\boldsymbol{\mu}_t^S)\right\}_{\omega\in\Omega}$ represents the payoff respectively of all the buyers and sellers in the market in period $t \geq 0$.

(iv) An equilibrium of intra-period bargaining game exists.

(v) For every type $\omega \in \Omega$, the payoffs $\left\{v_{\omega' t}^{S^*}(\boldsymbol{\mu}_t^S)\right\}_{\omega'\in\Omega}$ and $\left\{v_{\omega t}^{B^*}(\boldsymbol{\mu}_t^S)\right\}_{\omega\in\Omega}$ vary continuously in $\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B$.

Theorem 2 establishes that the structure of the expected payoffs is determined by $\boldsymbol{\mu}_t^S$ via the matching probability $\pi_{\omega' t}^S(\boldsymbol{\mu}_t^S), \forall \omega \in \Omega$ and that the economy is well defined along all the possible paths of $\boldsymbol{\mu}_t^S$. Therefore, for any exogenously $\boldsymbol{\mu}_t^S$ and $\boldsymbol{\mu}_{t+1}^S$, all the dimensions of the game are determined. Furthermore, the Theorem states that sellers (respectively, buyers) of same type have equal payoff. Importantly, only sellers have time-varying payoffs. This specification is instrumental for characterizing two classes of equilibria in Section 3.

To clarify the above result, let us consider the payoff of any buyer k endowed with $\omega_{k,0} \equiv \omega$ units of capital when entering the market at some period t. As in Manea (2017), the expected equilibrium payoff $v_{\omega t}^{B*}(\boldsymbol{\mu}_t^S)$ is the same for matched and unmatched agents. Therefore, the identifier k is dropped from the equation. For any buyer endowed with ω and participating to the market at t, the equilibrium element $v_{\omega t}^{B*} \in \mathcal{V}^B$ is the period t expected payoff computed on three possible matching outcomes. The left prospect identifies the expected payoff of matching with a counterpart and be the offer-maker. In this case, the agent proposes the counterpart her outside option. The second prospect captures the two remaining cases: either the agent is matched to a counterpart as offer-receiver or no match takes place. In either case, the agent's payoff is her continuation value. Sellers equilibrium payoff $v_{\omega' t}^{S*}(\boldsymbol{\mu}_t^S)$ deviates from the buyer's payoff for the fact that continuation value depends on next period market population vector $\boldsymbol{\mu}_{t+1}^S$. A potential difference with the framework of Manea (2017) in this regards is that in this model, depending on their type, buyers do not necessarily exit after a trade, but may engage in multiple trades during the same round. I reconcile the model with the more general class of dynamic markets of Manea (2017) in two steps. First, the pairwise surplus structure I adopt in the model provides an incentive-compatibility condition from which I derive a sufficient condition for trades between sellers and any type of buyers to take place (see the discussion after (26) in the proof of Theorem 2). Secondly, I construct buyers' cumulative measure which accounts for all trades occurring in equilibrium in every period (see Appendix A).

I now look at the intertemporal bargaining game, where the market composition $\boldsymbol{\mu}_t^S$ is endogenously determined. The characterization of the spaces $\mathcal{A}, \mathcal{M}, \mathcal{P}, \mathcal{V}$ defined in Section 2, and their relationships are compatible with the general non-stationary market structure introduced by Manea (2017). Therefore I construct the correspondence $f : \mathcal{A} \rightrightarrows \mathcal{A}$ by composing the correspondence α and the functions v^S, π^S, μ^S, so that

$$\mathcal{A} \overset{\mu^S}{\to} \mathcal{M} \overset{\pi^S}{\to} \mathcal{P} \overset{v^S}{\to} \mathcal{V} \overset{\alpha}{\rightrightarrows} \mathcal{A},$$

and use the following result

Theorem 3 (p.64, Manea, 2017)**.** *An equilibrium exists for the inter-temporal Bargaining Game.*

Theorem 2 and Theorem 3 support the formation of the dynamic market as the equilibrium of an infinite horizon bargaining game. Moreover, the formation process is essentially pinned down to the evolution of the market composition $\boldsymbol{\mu}_t^S$, which determines the matching probabilities, the structure of payoffs and the agreements.

C Proofs

Proof of Proposition 1. (i) Define $D^{**} \subset D$ such as the set of *fully-constrained* buyers (buyers for which given α and ω^{**} the constraint holds tightly) and consider $\omega^{**} \in D^{**}$, defined as the endowment such that due to the strictly decreasing returns in possessing the goods, the utility function is a bijection and therefore I can define ω^* such as the smallest type capable of reaching the first-best endowment by using all the available leverage, or, equivalently, $u(\omega^* + \alpha\omega^*) = u(\omega^{**})$. Using again the fact that $u(\cdot)$ is a bijection, it must hold that $\omega^* = \omega^{**}/(1+\alpha)$. The construction of q_ω easily follows. (ii) The result follows immediately from

$$\int_0^{\omega^{**}} q_\omega P(\omega)d\omega = \int_0^{\omega^*} q_\omega P(\omega)d\omega + \int_{\omega^*}^{\omega^{**}} q_\omega P(\omega)d\omega$$

■

Proof of Theorem 2. (i), (ii), (iv) and (v) correspond to Theorem 1, Manea (2017). (iii) Given the sequential entry assumption I imposed upon the general framework of Manea (2017), I refine the system of equation in (iii), Theorem 1 of Manea (2017) by adopting a stationary *within-period* payoff structure for the buyers. Buyers entering at t will trade only with the measure of sellers which is available at $t, t \geq 0$. Therefore, the matching probability $\pi^S_{\omega' t}(\mu^S_{\omega' t})$ determining the buyers' within-period outside option *is fixed* in every period. Consequently, within every period t $v^{B*}_{\omega' t}$ is fixed. On the other hand, sellers' expected payoff $v^{S*}_{\omega t}$ evolves *across* periods.

■

Proof of Theorem 1. I split the proof in two steps. In the first step, I focus on market dynamics. In the second step, I implement conditions guaranteeing incentive compatibility.

Step 1. Let us study the details of the market evolution. I revisit the population law of motion for type ω^{**} sellers in light of Definition 5 and (15). From (7), I write

$$\mu^S_{\omega^{**}(t+1)} = \mu^S_{\omega^{**}t} - \underbrace{\sum_{\omega \in \mathcal{D}} \left(\alpha_{\omega\omega^{**}t} \pi^S_{\omega^{**}t} \tilde{\mu}^B_\omega + \alpha_{\omega^{**}\omega t} \pi^B_\omega \mu^S_{\omega^{**}t} \right)}_{type\ \omega^{**}\ out-flow}. \tag{16}$$

The out-flow of ω^{**}-type sellers is due to the measure of type ω^{**} sellers who agreed on selling capital at t either as recipients or as proponents of an offer. Combining (15) with definitions of $\pi^S_{\omega^{**}}$ and the probability $p = 1/2$ of being active in every trade I obtain

$$\mu^S_{\omega^{**}(t+1)} = \mu^S_{\omega^{**}t} - \frac{1}{2}\sum_{m=1}^{\tilde{Q}} \left(\mu^S_{\omega^{**}t(m)} + x_{t(m)}\right) \sum_{\omega \in \mathcal{D}} (\alpha_{\omega\omega^{**}t(m)} + \alpha_{\omega^{**}\omega t(m)})\tilde{\mu}^B_\omega,$$

where $\tilde{Q}$ is the number of realized intra-period exchanges for a ω^{**}-type seller in equilibrium. First, notice that due to the structure of payoffs in (3), for ω^{**} type seller, $\alpha_{\omega^{**}\omega t(m)} = 1 \geq \alpha_{\omega'\omega(m)}$ and $\alpha_{\omega\omega^{**}t(m)} = 1 > \alpha_{\omega\omega' t(m)}$, $\forall m \leq \tilde{Q}, \omega \in \Omega$. Therefore, I can rewrite

$$\mu^S_{\omega^{**}(t+1)} \approx \mu^S_{\omega^{**}t} - \sum_{m=1}^{\tilde{Q}} \left(\mu^S_{\omega^{**}t(m)} + x_{t(m)}\right) D, \tag{17}$$

where I used the fact that $\sum_\omega \tilde{\mu}^B_\omega = D + (\alpha\omega^{**})/(2(1+\alpha)\bar{\omega}) \approx D$ to simplify computations[22]. To compute the expected number of exchanges Q, first, I notice that in equilibrium, the sum of all exchanges implies that

$$\sum_{m=1}^{\tilde{Q}} \left(\mu^S_{\omega^{**}t(m)} + x_{t(m)}\right) = \mu^S_{\omega^{**}t} - D + x_t - \frac{\mu^S_{\omega^{**}t} - D}{\sum_{\omega\in\Omega}\mu^S_{\omega t}} < x_t - D.$$

Second, given that the measure of non-matched ω^{**}-type sellers at t is

$$\mu^S_{\omega^{**}t} - D\mu^S_{\omega^{**}} > \mu^S_{\omega^{**}t} - D\mu^S_{\omega^{**}} - D\sum_{\omega\neq\omega^{**}} \mu^S_\omega = \mu^S_{\omega^{**}} - D\sum_{\omega\in\Omega}\mu^S_\omega,$$

the dynamics of $\mu^S_{\omega^{**}(t+1)}$ become

$$\mu^S_{\omega^{**}(t+1)} \approx \mu^S_{\omega^{**}t} - (x_t - D)\left(\mu^S_{\omega^{**}} - D\sum_{\omega\in\Omega}\mu^S_\omega\right)\cdot D.$$

In order to study the evolution of ω^{**}-type sellers as opposed to other types, let us adopt the *market index* x from Definition 4. To make the analysis tractable, for the moment let us assume that the population of sellers is made by two types, so that μ^S_ω, $\omega \in \{\omega^{**}, \bar{\omega}\}$. Then, I can write

$$\tau(x) \equiv \frac{\mu^S_{\omega^{**}(t+1)}}{\mu^S_{\omega^{**}(t+1)} + \sum_{\omega\neq\omega^{**}}\mu^S_{\omega(t+1)}} = \frac{x - (x-D)^2\cdot D}{x - (x-D)^2\cdot D + [(1-x) - (1-x-D)^2\cdot D]}, \tag{18}$$

[22]In Section 5 I verify in simulations that there is no qualitative discrepancy between the perceived law of motions (16) (followed by agents) and the approximation (17) I adopt to uncover the market dynamics. If we instead assume that agents perform the approximation, we would be required to introduce additional conditions on the sizes of ω^{**} and $\bar{\omega}$ to make the Equilibrium robust to such discrepancy. I thank an anonymous referee for raising this point.

where the mapping $\tau(x)$ tracks the market evolution when exchanges are as conceived in the equilibrium definition. I proceed as follows. First, I disregard the fact that x must be bounded in the $[0, 1]$ interval and derive the (unconstrained) rest points of the system. Second, I re-introduce the bound in order to refine the properties of the system and refine the effect of the financial constraints on the qualitative behavior of the market's evolution. I impose:

$$\tau(x) = x$$

From Proposition 2, the measure of buyers ending up with ω^{**} in equilibrium is given by Δ. Hence, by exploiting the fact that $D = (\Delta - (1/\bar{\omega})) \cdot (\omega^{**}/2)$, I compute

$$\left(\Delta - \frac{1}{\bar{\omega}}\right) \cdot \frac{\omega^{**}}{2} \cdot (2x - 1) \cdot (D^2 + x(x-1)) = 0,$$

which delivers the following (unconstrained) rest points

$$x_0^* = \frac{1}{2} \qquad x_+^* = \frac{1}{2}\left(1 + \sqrt{1 - 4D^2}\right) \qquad x_-^* = \frac{1}{2}\left(1 - \sqrt{1 - 4D^2}\right),$$

together with the trivial equilibrium $\bar{x}_0 = \mu_{0,\omega^{**}}$ for $\alpha = D = 0$. First, we are interested in the stability of x_0^*. By differentiating (18) with respect to x at point $x = 1/2$, I find that

$$\left.\frac{d\tau(x)}{x}\right|_{x=x_0^*} = \frac{2(4 - D + 2D^2)(2 + D^2 - D^3 - \frac{1}{4}D)}{(D - 4 - 2D^2 + 2D^3 - \frac{1}{2}D)^2}.$$

The stability of x_0^* requires the following two conditions to hold jointly

$$\left.\frac{d\tau(x)}{x}\right|_{x=x_0^*} < 1 \rightarrow D \cdot (D - 8 + 28D^2 + 16D^4 - 16D^5) \leq 0 \tag{19}$$

$$\left.\frac{d\tau(x)}{x}\right|_{x=x_0^*} > -1 \rightarrow 128 - 40D + 131D^2 - 116D^3 + 48D^4 - 48D^5 + 16D^6 \geq 0. \tag{20}$$

It is immediate to see that the first equation in (19) satisfies the inequality in the interval $[0, 1]$. From differentiation, I find that the second equation is increasing in the interval $[0, 1]$, and it breaks the inequality for $D > \bar{D}$, with $\bar{D} = 1/2$. This verifies that at $D = \bar{D}$, the equilibrium point x_0^* loses stability. Now, let us re-introduce the bound $x \in [0, 1]$ and focus on the behavior of the system in the neighborhood of $\bar{x}_-^* \equiv 0$ and $\bar{x}_+^* \equiv 1$. In particular, I will evaluate the iterations near $\tau(\bar{x}_+^*)$, which is the relevant bound for the homogeneous equilibria. By substitution I have that:

$$\tau(x)|_{x=\bar{x}_+^*} = \frac{1 - (1-D)^2 \cdot D}{1 - (1-D)^2 D - (-D)^2 \dot{D}}$$

It is easy to see that for $D \in [0,\ 1/2]$ the index function is increasing at $\bar{x}_+^*$, hence $\bar{x}_+^*$ is a stable fixed point by construction. Because x_+^* for $D < 1/2$ is bounded between the two stable equilibria x_0^* and $\bar{x}_+^*$, it must be that x_+^* is unstable. Therefore, for $D < 1/2$, the index orbit x_t will converge to a homogeneous equilibrium whenever $\mu^S_{\omega^{**}0} \in \left(x_+^*,\ \bar{x}_+^*\right]$. On the other hand, for $D > 1/2$, the equilibrium x_0^* loses stability and the remaining two unbounded equilibria become complex. Consequently, for $D > 1/2$ the surviving stable equilibria are $\bar{x}_+^*$ and $\bar{x}_-^*$ and x_t will follow the equilibrium path whenever $\mu^S_{\omega^{**}0} > 1/2$. A symmetric argument can be invoked to show that for an initial index $\mu^S_{\omega^{**}0} < 1/2$ the system would converge to $x = 0$, that is a no ω^{**}-type equilibrium $\bar{x}_-^*$. Now, I may re-express the above argument in terms of the relationship between the leverage α and types' measure as follows

$$D \leq \bar{D} \ \rightarrow \ \alpha < \frac{\frac{\bar{\omega}}{(\omega^{**})^2}}{1 - \frac{\bar{\omega}}{(\omega^{**})^2}},$$

from which I can retrieve $\alpha = \alpha^*$ as stated in the Theorem's body.

Step 2. I now derive the conditions which allow the above dynamics to be incentive compatible. From Theorem 2 and Definition 11, for every period $t \geq 0$ and buyer $\omega \in \mathcal{D}$, the system of payoffs reads

$$v^B_\omega = \pi^S_{\omega^{**}t}\left(s_{\omega\omega^{**}} - \delta v^S_{\omega^{**}(t+1)}\right) + \left(1 - \pi^S_{\omega^{**}t}\right)\delta v^B_\omega \qquad \forall \omega \in \mathcal{D} \tag{21}$$

$$v^S_{\omega^{**}t} = \sum_{\omega' \in \mathcal{D}} \tilde{\pi}^B_{\omega'}\left(s_{\omega'\omega^{**}} - \delta v^B_{\omega'}\right) + \left(1 - \sum_{\omega' \in \mathcal{D}} \tilde{\pi}^B_{\omega'}\right)\delta v^S_{\omega^{**}(t+1)}\ , \tag{22}$$

in which I suppressed the time notation in the buyer's payoff as I work out an incentive-compatible condition which holds for any value of the index x along the intertemporal market evolution. As I noted along the main text, within every period t, the payoff of buyers entering at t is stationary due to the transition protocol I imposed (the system moves to $t+1$ only after every buyer has cleared her own demand) and completely depends upon the current market index x_t. I rewrite the buyers' payoff by accounting for the results stated in Proposition 2 and Step 1 of the proof of Theorem 1

$$v^B_\omega(x) = \pi^S_{\omega^{**}}\frac{a(\omega^{**} - \omega) - \delta v^S_{\omega^{**}(t+1)}}{1 - \delta(1 - \pi^S_{\omega^{**}t})} = \frac{\tau(x)^t}{4}\frac{a(\omega^{**} - \omega) - \delta v^S_{\omega^{**}(t+1)}}{1 - \delta(1 - \frac{\tau(x)^t}{4})}.$$

By substituting v^B_ω in $v^S_{\omega^{**}t}$ I find that

$$v^S_{\omega^{**}t}(x) = a\sigma - \delta\cdot\frac{\frac{\tau^t(x)}{4}}{1 - \delta(1 - \frac{\tau^t(x)}{4})}\left(\sum_{\omega \in \mathcal{D}} \tilde{\pi}^B_\omega a(\omega^{**} - \omega) - \delta v^S_{\omega^{**}(t+1)}\right) + \left(1 - \left(3\sigma - \underbrace{\frac{\omega^{**} + 1}{2\bar{\omega}(1+\alpha)}}_{\rho}\right)\right)\delta v^S_{\omega^{**}(t+1)}.$$

Let us define $\gamma(x)$ such as

$$\gamma(x) \equiv \frac{\frac{\tau^t(x)}{4}}{1 - \delta(1 - \frac{\tau^t(x)}{4})}.$$

Then, by applying point (iii) from Proposition 2 the system of payoffs reads

$$\begin{aligned} v^B_{\omega^{**}t} &= \gamma(x)\left(a(\omega^{**}-\omega) - \delta v^S_{\omega^{**}(t+1)}\right) \\ v^S_{\omega^{**}t} &= a\sigma(1-\delta\gamma(x)) + (1+\delta\gamma(x) - 3\sigma + \rho)\delta v^S_{\omega^{**}(t+1)}, \end{aligned}$$

and the one-step ahead expansion of $v^S_{\omega^{**}t}$ gives us

$$\begin{aligned} v^S_{\omega^{**}t} = a\sigma - a\sigma\delta\gamma^t(x) + \delta(a\sigma - a\sigma\delta\gamma^{t+1}(x))(1+\delta\gamma(x)^t - 3\sigma+\rho) + \quad (23) \\ + \delta^2 v^S_{\omega^{**}(t+2)}(1+\delta\gamma^{t+1}(x) - 3\sigma+\rho)(1+\delta\gamma^t(x) - 3\sigma+\rho). \quad (24) \end{aligned}$$

I discard the decaying component and further expand the series to obtain

$$v^S_{\omega^{**}t} = a\sigma(1 - \sum_{t\geq 0}\delta^t\left(1+\delta x - 3\sigma+\rho\right)\left(1+\delta\gamma(x) - 3\sigma+\rho\right)\cdot\ldots\cdot(1+\delta\gamma^{t-1}(x) - 3\sigma+\rho)(\delta\gamma^t(x)-1)),$$

so that I rewrite the above equation

$$v^S_{\omega^{**}t} = a\sigma(1 + \sum_{t\geq 0}\delta^t\left(1+\delta x - 3\sigma+\rho\right)\left(1+\delta\gamma(x) - 3\sigma+\rho\right)\cdot\ldots\cdot(1+\delta\gamma(x)^{t-1} - 3\sigma+\rho)(1-\delta\gamma^t(x))).$$

From the definition of γ, I know that it can move along the following interval

$$\bar{\gamma} \in \left[0, \frac{\frac{1}{4}}{1-\delta\left(\frac{3}{4}\right)}\right].$$

Now, I bound the series by fixing $\gamma(x)^t = \bar{\gamma}, \forall t \geq 0$ and explore the incentive-compatible index orbits. Therefore, I rewrite the above equation

$$v^S_{\omega^{**}t} \leq a\sigma\left(1 + (1-\delta)\sum_{t\geq 0}\delta^t(1+\delta\bar{\gamma} - 3\sigma+\rho)^t\right), \quad (25)$$

so that a sufficient condition for the series in (25) to converge is

$$\bar{\gamma}\delta^2 - \delta(3\sigma - \rho - 1) \leq 1.$$

I rewrite

$$\bar{\gamma}\delta^2 - \delta\left(\frac{(\omega^{**}+1)(\omega^{**}(3+\alpha(2+\omega^{**})-1)-1)}{2\bar{\omega}(1+\alpha)} - 1\right) - 1 \leq 0,$$

and, for $\bar{\omega}$ large enough, I obtain that

$$\bar{\gamma} \leq \frac{1-\delta}{\delta^2}.$$

I re-express the condition in terms of the index x and get a bound on δ which guarantees that all the orbits $x \in [0,1]$ are incentive compatible. From the definition of γ, the condition implies

$$\frac{\frac{x}{4}}{1-\delta\left(1-\frac{x}{4}\right)} \leq \frac{1-\delta}{\delta^2}.$$

Therefore, for $\delta \leq \bar{\delta} \equiv \frac{7}{4} - \frac{\sqrt{17}}{4}$ and $\bar{\omega}$ large enough, the series defined above converges for all the orbits $x \in [0,1]$ and I have that

$$v^S_{\omega^{**}t} \leq a\sigma\left(1 + \frac{1-\delta}{1-\delta(1+\delta\bar{\gamma}-3\sigma+\rho)}\right). \tag{26}$$

I substitute (26) in the expression for v^B_ω

$$v^B_{\omega t} = \bar{\gamma}\left[a(\omega^{**}-\omega) - \delta a\sigma\left(1+\frac{1-\delta}{1-\delta(1+\delta\bar{\gamma}-3\sigma+\rho)}\right)\right],$$

and work out the matching incentive-compatible condition of Equation (4)

$$\begin{aligned} \delta v^B_\omega + \delta v^S_{\omega^{**}} &\leq s_{\omega\omega^{**}} \\ \Rightarrow a\delta\sigma\left[1+\frac{1-\delta}{1-\delta(1+\delta\bar{\gamma}-3\sigma+\rho)}\right] &\leq a(\omega^{**}-\omega). \end{aligned}$$

In order to obtain an exchange which is incentive-compatible for ω^{**}-type sellers and every $\omega \in \mathcal{D}$, I look at type $\hat{\omega}$ buyer defined as $\hat{\omega} \equiv \omega^{**} - 1$. Since the surplus function is decreasing in the buyer's type, if the exchange is incentive-compatible for type $\hat{\omega}$ buyer, it will also be for the types $\omega < \hat{\omega}$. I rewrite Equation (4) such as

$$\delta\sigma(2-\delta(\delta\bar{x}_+ - 3\sigma+\rho)-2\delta) \leq 1-\delta(\delta\bar{\gamma}+1-3\sigma+\rho),$$

from which I get

$$\bar{\gamma} \leq \frac{1-\delta(1-3\sigma+\rho)-\delta\sigma(2-2\delta-\delta(\rho-3\sigma))}{\delta^2(1-\delta\sigma)}.$$

again, by following the rationale I adopted above, I find out that for $\bar{\omega}$ large enough, the last condition stated in the Theorem's body is verified for all the values of δ such that

$$X^* = \bar{x}_+ \;\leftrightarrow\; \delta \leq \frac{7}{4} - \frac{\sqrt{17}}{4} \approx 0.72.$$

■

Proof of Proposition 1. I have to expand the previous proof in the following dimensions. First, I have to add a further step which guarantees that the market formation can be sustained across periods with a positive net inflow of settled buyers. Secondly, I need to refine the market law of motions and the related rest points.

Step 1. I derive (i) and (ii) as follows. First, define the cumulative measure of Pure Sellers belonging to S (see Definition 1) entering in the market at every period t as

$$\Sigma \equiv \sum_{\omega \in S} \mu_{\omega}^{B} = 1 - \frac{\omega^{**}}{\bar{\omega}}. \tag{27}$$

Then, a set of sufficient conditions such that the economy can clear every period according to the protocol specified in the equilibrium is given by

$$\min \left\{ \sum_{\omega=0}^{\bar{\omega}} \mu_{\omega 0}^{S} \ , \ \Sigma + \Delta \right\} \geq \sum_{\omega=0}^{\omega^{**}-1} \tilde{\mu}_{\omega}^{B} = \frac{\alpha \omega^{**}(\omega^{**}-1)}{2(1+\alpha)\bar{\omega}},$$

where I made use of Definition 11 and (27). the first (respectively, second) argument on the LhS of the inequality corresponds to (i) (respectively, to (ii)) and guarantees that the market clears in period $t = 0$ (respectively, in every period $t > 0$). In particular, the second argument follows from the fact that in equilibrium under transition protocol $T.2$ (Definition 3), for every period t and $t+1$ the set of partially constrained buyers D^{**} entering in period t as buyers, at $t+1$ are incorporated in the set of sellers.

Step 2. To make the analysis tractable, I assume again that the population of sellers is made by two types, so that μ_{ω}^{S}, $\omega \in \{\omega^{**}, \bar{\omega}\}$, with $\bar{\omega} = \omega^{**} + 1$. Using Step 1 in the proof of Theorem 1, system (9)-(10) becomes

$$\mu_{\omega^{**}(t+1)}^{S} = \mu_{\omega^{**}t}^{S} + \Delta - (x_t - D) \left(\mu_{\omega^{**}}^{S} - D \sum_{\omega \in \Omega} \mu_{\omega}^{S} \right) \cdot D \tag{28}$$

$$\mu_{\bar{\omega}(t+1)}^{S} = \mu_{\bar{\omega}t}^{S} + \frac{1}{\bar{\omega}} - (1 - x - D) \left(\mu_{\bar{\omega}}^{S} - D \sum_{\omega \in \Omega} \mu_{\omega}^{S} \right) \cdot D. \tag{29}$$

In (28) (respectively, (29)) I accounted for the fact that in every period $t+1$, the measure of available sellers encompasses the measure $\mu_{\omega^{**}t}^{S}$ (respectively, $\mu_{\bar{\omega}t}^{S}$) and the measure Δ (respectively, $1/\bar{\omega}$), given by agents that entered the market as buyers at $t-1$ and become sellers at t.

I now obtain sufficient conditions for equilibrium existence. Let us define $m_{\omega,t+1} \equiv \mu_{\omega(t+1)}^{S} - \mu_{\omega t}^{S}$ such as the *growth rate* of seller type ω, for $\omega \in \{\omega^{**}, \bar{\omega}\}$. I can then construct the *growth index* X_t defined as

$$X_t \equiv \frac{m_{\omega^{**}t}}{m_{\omega^{**}t} + m_{\bar{\omega}t}},$$

and study the evolution of X_t by means of a one-step iterator $T(X)$ defined as

$$T(X) \equiv \frac{m_{\omega^{**}(t+1)}}{m_{\omega^{**}(t+1)} + m_{\bar{\omega}(t+1)}} = \frac{\frac{\Delta}{\mu^S_{\omega^{**}t}+\mu^S_{\bar{\omega}t}} - (x-D)^2 \cdot D}{\frac{\Delta}{\mu^S_{\omega^{**}t}+\mu^S_{\bar{\omega}t}} - (x-D)^2 \cdot D + \left(\frac{1/\bar{\omega}}{\mu^S_{\omega^{**}t}+\mu^S_{\bar{\omega}t}} - (1-x-D)^2 \cdot D\right)}. \tag{30}$$

First, differentiate (30) with respect to $\mu^S_{\omega^{**}t}$ and evaluate it at $t = 1$ around $\mu^S_{\omega^{**}t} = 1$

$$X_1' \equiv \frac{dX_t(\mu^S_{\omega^{**}t}, \mu^S_{\bar{\omega}t})}{d\mu^S_{\omega^{**}t}}\Big|_{t=1,\ \mu^S_{\omega^{**}0}=1} = \frac{D(D^2\Delta - D(D-1)^2\frac{1}{\bar{\omega}})}{(-D(1+2D(D-1)) + \Delta + \frac{1}{\bar{\omega}})^2},$$

where I made use of the fact that $\sum_{\omega\in\Omega}\mu^S_{\omega,0} = 1$. To guarantee equilibrium existence, it then suffices to show that $X_1' > 0$ or equivalently that

$$\frac{1-D}{D} \leq \sqrt{\bar{\omega}\Delta}$$

from which point (iii) immediately follows.

Step 3. Given that the entry condition does not affect the qualitative behavior of $\gamma(x)$ as characterized in the proof of Theorem 1, exchanges here follow the incentive-compatibility structure obtained for the homogeneous equilibria with no entry. ■

Proof of Lemma 2. Using the definition of A^* in the main text, rewrite condition (iii) from Proposition 1

$$A(1 + \sqrt{1 + A\omega^{**}}) \geq 2A^*.$$

Given $\tilde{A}$ as defined in Proposition 1, it is immediate to see that $A^* > \tilde{A}$. Therefore, for $A = A^*$ the above inequality is always true. It follows that a sufficient condition for HEWE to exist, if the market fulfils conditions $(i), (ii)$ and (iv) is that

$$A \geq A^*. \tag{31}$$

The taxonomy follows as direct consequence of (31), Theorem 1 and Proposition 1.

■

Proof of Lemma 3-4. Proof is omitted as it follows from the direct application of Theorem 1, Proposition 1 and (31).

■

Proof of Proposition 2. (i) I provide the derivation of $\hat{\boldsymbol{\mu}}^B$ from which it is immediate to recover (14) and the first part of the Proposition statement. Consider a generic type $\omega' < \omega^{**}$ and any type $\omega'' \in \bar{\Omega}_{\omega'}$, such that

$$\bar{\Omega}_{\omega'} \equiv \{\omega'' : \omega''(1+\alpha) \geq \omega'\} \cap \{\omega'' : \omega'' < \omega'\}$$

From Definition 11 it holds that $\mathbb{I}_{\omega'',\omega'}[\cdot] = 1 \; \forall \omega'' \in \bar{\Omega}_{\omega'}$. It is easy to quantify the size of set $\bar{\Omega}_{\omega'}$ such as

$$|\bar{\Omega}_{\omega'}| = (\omega' - 1) - \frac{\omega'}{1+\alpha} + 1 = \frac{\alpha}{1+\alpha}\omega'$$

From which the statement follows.

$(iii) - (iv)$ Results respectively follow from a straight application to (15) of the series $\sum_0^{\omega^{**}-1} = 1 + 2 + 3 + ... = (\omega^{**} - 1)((\omega^{**}))/2$ and of series $\sum_0^{\omega^{**}} = 1 + 4 + 9... = (1/6)(\omega^{**} - 1)\omega^{**}(2\omega^{**} - 1)$ combined to the definition of $s_{\omega\omega^{**}}$ in (3).

■