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Developing a task design and implementation framework for fostering mathematical modelling competencies

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Abstract
In this article, we describe the generation of a Design and Implementation Framework for Mathematical Modelling Tasks (DIFMT) through a researcher-teacher collaboration. The purpose of the framework is to support holistic approaches to instructional modelling competency. This framework is underpinned by principles drawn from theory and praxis which are informed by the anticipatory capabilities that teachers require for the design and effective implementation of quality modelling tasks in secondary classrooms. A draft DIFMT was developed from a synthesis of research literature and was refined through an iterative process of task development, implementation and observation, reflection through teacher/student interviews, and revision of the framework. Each iteration made use of the most recent refinement of the co-constructed DIFMT, building theory while simultaneously addressing a problem in educational practice, consistent with a design-based methodology. Thus, the DIMFT developed organically throughout the project. While initial modelling exemplars were researcher-designed, the locus of responsibility moved to teachers as the project progressed. The DIFMT consists of two major components—principles for modelling task design and pedagogical architecture—each of which is structured around dimensions that include elaborations which detail the knowledge required for modelling as well as teacher and student capabilities.

Keywords Mathematical modelling · Competencies · Competency · Task design · Task implementation · Framework

1 Introduction

The importance of mathematical modelling within school education has been recognized in areas such as national economic prosperity (e.g., STEM) and critically informed citizenship

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(e.g., Maaß, Geiger, Ariza, & Goos, 2019). This recognition is reflected in the inclusion of modelling in school mathematics curriculum in a growing number of countries, including Australia.

Research into mathematical modelling competency has provided insight into a range of influential factors including teachers' and students' personal knowledge (e.g., Blum, 2011), dispositions and beliefs (e.g., Jankvist & Niss, 2019), blockages between phase transition (e.g., Galbraith & Stillman, 2006), and use of digital tools (e.g., Geiger, 2011). Despite such findings, the development of students' modelling competency and teachers' instructional modelling competency continues to present unresolved issues in educational research and practice (e.g., Maaß, 2010).

The purpose of this article is to report on the development of an evidence-based Design and Implementation Framework for Mathematical Modelling Tasks (DIFMT) aimed at supporting holistic approaches to instructional modelling competency. Development of the DIFMT was informed by a critical review of research literature related to factors that are key to the development of mathematical modelling competency. As the DIFMT was designed to be first and foremost a tool for teachers' reflective practice, our goal was that it be directly relevant to mathematics classrooms. Thus, DIFMT development was undertaken in close collaboration with teachers, an approach consistent with research in task design and implementation (e.g., Geiger, 2019; Jones & Pepin, 2016), in which teachers' knowledge of student needs and classroom conditions is recognised as central.

In addressing this purpose, we will first review and synthesise relevant literature related to the nature of mathematical modelling, the notions of competency and sub-competencies, implemented anticipation, and task design and implementation. Second, the DIFMT generation process will be described and illustrated via data drawn from field trials including selected examples of researcher/teacher discussion. Finally, we discuss the framework in the light of knowledge related to effective instruction in mathematical modelling.

2 Theoretical perspectives

In this section, we provide a critical review of the research literature that informed the theoretical foundations of the DIFMT.

2.1 The nature of the mathematical modelling process

The purpose of modelling a real-world situation is typically to understand key features and properties, explain or predict phenomena and processes, and to inform related decision-making. At its core, mathematical modelling consists of identifying a problem within a real-world context, developing a relevant mathematical representation, determining a subsequent mathematical solution, interpreting the solution within the original context, and evaluating the solution's validity for resolving the problem. The capacity to undertake all aspects of mathematical modelling in a holistic manner is what we understand as *modelling competency*.

As it is often necessary to engage in the iterative refinement of solutions to real-world problems, modelling is typically depicted as cyclic in nature (e.g., Blomhøj & Jensen, 2003). While different representations of the modelling process have been proposed, none describes the actual sequentially ordered itineraries followed by any given modeller. Rather, they are analytic reconstructions of the components that are identifiably present in the complete

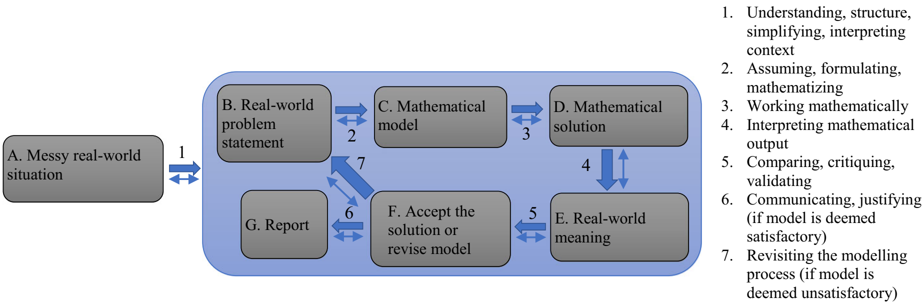

Fig. 1 The process of mathematical modelling (adapted from Stillman et al., 2007)

solution to any modelling problem (Niss & Blum, 2020). Thus, we make use of a representation adapted from Stillman, Galbraith, Brown, and Edwards (2007), presented in Fig. 1, that represents both the key phases and transitions related to the modelling process. Entries A–G denote the phases in the modelling process, with thicker arrows indicating transitions in activity as noted in descriptors 1–7. The double-headed arrows indicate forwards and/or backwards looking reflective activity between phases. Transitions may also occur in a non-sequential manner across the cycle, but additional arrows have been omitted for the sake of simplicity.

Student *sub-competencies* therefore include the capacity to undertake work within each of the phases of the modelling process as well as the management of the transitions between and across phases—cognitive and metacognitive capabilities (see Stillman et al., 2007). The ultimate goal of instruction is to develop a holistic competency that enables a modeller to address a problem in its totality within a real-world context.

2.2 Developing modelling competency and (sub-)competencies

Teacher competence is a subject of increasing research interest. Competence refers to real-world performance within a specific domain, a combination of knowledge, cognition, purpose, affect, and motivation (Blömeke, Gustafsson, & Shavelson, 2015). The nature of competence is a concern within most trades, and many professions—such as law, engineering, medicine, and teaching. Teacher competence involves knowledge, skills, problem-solving capacities, and affective factors that must be orchestrated to optimise performance (e.g., Shulman, 1987). How to promote competence remains an issue of ongoing debate focused on two principal perspectives: (1) an analytic view associated with distinct traits of performance that can be measured separately, and (2) a holistic view that recognises the complexity of "live" performance and gives primacy to the complete execution of a role. Both perspectives are seen to have inherent strengths and weaknesses with some researchers arguing that they should be seen as complementary—both needed for a complete picture of performance quality (e.g., Blömeke et al., 2015). The tension between these two perspectives, however, remains unresolved.

Mathematics teacher competence has been the focus of significant research studies including TEDS-M (e.g., Blömeke, Hsieh, Kaiser, & Schmidt, 2014) and TEDS-FU study (e.g., Santagata & Yeh, 2016). While most studies have investigated cognitive aspects of competence, more recently, recognition of the situated nature of teaching performance has led to a complementary focus on its affective dimensions (e.g., Schoenfeld, 2011). The findings of such projects indicate that both cognitive and situated aspects influence the quality of teaching practice (Kaiser et al., 2017).

The notion of modelling competency can be traced back to research related to the development of activities designed to promote a learner's capacity to carry out the modelling process—competency as a modeller (e.g., Treilibs, 1979). Other early developments in this area include investigations into the quality of mathematical processes learners employed during modelling activity (e.g., Haines, Crouch, & Davies, 2001) and the role of metacognitive processes in the coordination of modelling capacities (e.g., Stillman, 1998).

Current discussion about developing competency in mathematical modelling is centred on two primary themes, holistic and atomistic approaches (Blomhøj & Jensen, 2007). The holistic or "top-down" approach acknowledges an overarching integrated competency necessary to undertake mathematical modelling in a broad variety of contexts. From this perspective, it is possible to identify several crucial components as modelling sub-competencies. These are not perceived as independent entities, however, but as facets of an overarching modelling competency (e.g., Kaiser & Brand, 2015). The atomistic or "bottom-up" approach operates with a set of separate and independent modelling sub-competencies typically aligned with phases of the modelling cycle, for example, making assumptions, mathematizing, and validating a model. In this approach, the aggregation of individual (sub-) modelling competencies is seen as resulting in modelling competency, although Niss and Blum (2020) argue this conglomerate has no further properties in and of itself.

How teachers can assist learners to develop modelling competency has been the focus of a number of studies. Borromeo Ferri (2018) and Borromeo Ferri and Blum (2010), for example, have developed theoretically initiated dimensions from a sub-competency perspective, presented in Fig. 2. In this model, they defined the cognitive demands of task creation, quality instruction, and assessment of modelling activity. Taken as a whole, however, studies into the effectiveness of holistic and atomistic approaches to instruction aimed at developing modelling competency have yielded mixed results, both with strengths and weaknesses. Kaiser and Brand (2015), for example, claimed that while a holistic approach appeared to be more effective in promoting, interpreting, and validating capacities, an atomistic approach better

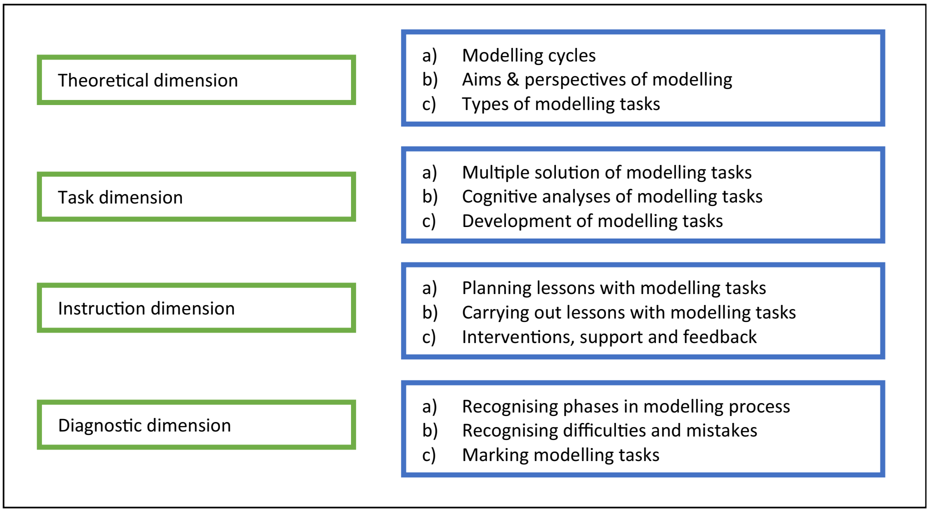

Fig. 2 Model for competencies needed in teaching mathematical modelling, Borromeo Ferri (2018)

supported the development of working mathematically. Thus, there remain unresolved questions about how to best promote competency in mathematical modelling, in part because "there has not yet been a comprehensive description of modelling competencies" (Maaß, 2010, p. 289). This has implications for the achievement of overall competence. It is in this context we seek to contribute a new perspective on teacher instructional modelling competency.

Thus, our current research is focused on fostering teachers' instructional competency in mathematical modelling with the goal of supporting upper secondary students' holistic modelling competency. Consistent with this goal, we have developed the DIFMT, a research informed framework aimed at supporting the teaching of modelling through attention to task design and the demands of implementation. In developing the DIFMT, we draw on modelling sub-competencies (embedded in the DIFMT), while maintaining a focus on holistic modelling competency and related pedagogical action.

A focus of this study is the role of *anticipation* in enabling modelling competency (see Section 2.3). As such, modellers must be aware of the specific role of sub-competencies and how they interrelate in parallel and in sequence. That is, they must be capable of looking forward and backwards, to determine how current activity informs decision-making about consequent action. This means that modellers must be able to both enact modelling sub-competencies and synthesise these into a global competency. For this reason, we adopt the overarching, holistic modelling competency (top-down) approach (Niss, Blum, & Galbraith, 2007), which used the singular Modelling Competency consistent with the singular holistic perspective of Blomhøj and Jensen (2003) and what Kaiser and Brand (2015) call "the holistic dimension" of the competency classification.

2.3 The role of implemented anticipation in modelling

As observed in the previous section, in developing their competency, modellers must also appropriate forward-looking metacognitive processes related to what is likely to be useful mathematically in subsequent steps, which act as enablers of successful decision-making and consequent action (Niss & Blum, 2020). Niss (2010) used the term *implemented anticipation* to identify such activity and proposed three constituent processes:

1. Structuring of an extra-mathematical situation, to prepare it for mathematization, must be focused on features that are anticipated as essential in addressing a mathematically feasible problem.
2. Anticipation of mathematical representations that are suitable for capturing the situation as structured in (1). Such representations must be familiar to the modeller and, ideally, the modeller would have had experience with their use in mathematizing similar situations.
3. Anticipation of how the mathematization and resulting model will provide a mathematical solution to the questions posed. This means that the outcomes of applying selected mathematical procedures and problem-solving strategies must also be anticipated after mathematization is complete.

These processes are qualified by a presumption that modellers must have the following attributes:

1. Mathematical knowledge relevant to the modelling context.
2. Capability with relevant mathematical knowledge to engage with modelling.
3. Positive disposition towards the use of mathematics to address real-world problems.

4. Confidence in their mathematical capacities and preparedness to persevere when challenged.

We have added a fifth requirement that students need to be familiar with the mathematical modelling process in order to activate 2 (Geiger, Galbraith, & Niss, accepted for publication). The key role of implemented anticipation in mathematization has been identified in other studies (e.g., Czocher, 2018; Jankvist & Niss, 2019; Stillman & Brown, 2014); however, a modeller must also anticipate potential mathematical representations, relationships, procedures, and problem-solving strategies (e.g., Geiger, Stillman, Brown, Galbraith, & Niss, 2018). This also requires the instantiation of implemented anticipation—to strategize a solution to the mathematical problem, and to think forward about how a chosen pathway will lead to an effective solution.

In this study, we have focused on the essential role of teacher's own anticipation in developing their instructional competency in modelling. We argue that is key to both the design and implementation of modelling tasks, as teachers must anticipate how students will respond, what scaffolds should be prepared, and where challenges are likely to emerge. Accordingly, the notion of anticipation is embedded, implicitly and explicitly, in both the design and implementation components of the DIFMT.

2.4 Principles for task design and implementation in mathematical modelling

In developing the DIFMT, we found the notions of both *design as intention* and *design as implementation* (Czocher, 2017) useful in framing discussion around their development. The former is concerned with the initial formulation of the design, while the latter is related to the processes by which a task is delivered into a classroom environment and then progressively refined. In this construction, task design and implementation are different sides of the same coin—offering a more coherent perspective than when considered separately.

The coupling of design and implementation has implications for the relationship between expert designers (mathematical modelling in this context) and those who implement tasks. In our case, it was apparent that teachers' intimate knowledge of key factors, such as local curriculum requirements and students' previous experience in mathematics, was essential if tasks were to be successfully implemented in classrooms. Furthermore, teachers must understand the principles that underpin a task in order that it be successfully implemented. Jones and Pepin (2016) argue that the most effective way of enabling this level of understanding is collaboration between teachers and others with expertise in task design. Hence, we saw the most effective approach to generating the DIFMT was through researcher-teacher collaboration—identified as key across a range of studies concerned with task design (e.g., Johnson, Coles, & Clarke, 2017).

A number of researchers have offered advice on the nature of modelling tasks and how these can be generated. Maaß (2010), for example, argues the quality of a task is key to promoting modelling competency. In her view, the first step in promoting quality is to identify the objectives of a task, which include focus of the modelling activity, nature of the relationship to reality, type of model used, type of representation, openness of a task, cognitive demand, and mathematical content. Clarity about objectives informed the selection of tasks for specific cohorts of students. While this scheme can inform the selection of appropriate tasks for particular groups of students, it does not specifically address the issue of how to design and implement modelling tasks—an area in which there is limited advice within the literature.

Table 1 Galbraith's (2006) principles of design for real-world tasks

Principle 1:	There is some genuine link with the real world of the students.
Principle 2:	There is opportunity to identify and specify mathematically tractable questions from a general problem statement.
Principle 3:	Formulation of a solution process is feasible, involving the use of mathematics available to students, the making necessary assumptions, and the assembly of necessary data.
Principle 4:	Solution of the mathematics for the basic problem is possible for the students, together with interpretation.
Principle 5:	An evaluation procedure is available that enables checking for mathematical accuracy, and for the appropriateness of the solution with respect to the contextual setting.
Didactical principle:	The problem may be structured into sequential questions that retain the integrity of the real situation. (These may be given as occasional hints or used to provide organized assistance by scaffolding a line of investigation.)

One of the few frameworks that address this gap is Galbraith's (2006) principles of design for real-world problems (Table 1)—generated by identifying structural components needed to scaffold the development of tasks aimed at promoting secondary students' successful modelling performance. It should be noted, however, that these principles provide necessary but not sufficient conditions for an effective task. Accordingly, these principles were used as a starting point for the design component of the DIFMT (Table 2).

Consistent with our commitment to the coupling of the design and the implementation of tasks, we drew on Geiger's (2019) notion of *pedagogical architecture* (Geiger, 2019; Goos, Geiger, Dole, Forgasz, & Bennison, 2019)—a frame for identifying and structuring teachers' activities both prior to and when actualising tasks in classrooms. The *pedagogical architecture* of a lesson is concerned with teacher capabilities related to:

- Initial setup of a lesson
- Initial selection of pedagogy (or pedagogies) that are appropriate for a task
- Flexible use of a repertoire of pedagogical strategies to align with classroom circumstances
- Adapting tasks *in situ* when students provide unanticipated responses
- Utilizing measured responsiveness to provide just enough information for students to make progress
- Orchestrating progress on critical aspects of a task
  (Geiger, 2019)

These capabilities informed the implementation component of the DIFMT which is focused on structured planning, knowledge of resources (e.g., materials), anticipatory capability, and pedagogical flexibility. The refinement of pedagogical architecture was further informed by other known factors impacting on the teaching and learning of modelling including teachers' tendency to intervene in ways which reduced cognitive challenge (e.g., de Oliveira & Barbosa, 2010); propensity of teachers to channel students' efforts toward pre-determined solutions (Tan & Ang, 2016) in contrast to advice about openness to multiple possible solutions (Schukajlow & Krug, 2014; Schukajlow, Krug, & Rakoczy, 2015); opportunity for diagnosing students' difficulties with the development of modelling competency (Jankvist & Niss, 2019); and the importance of task authenticity (Hernandez-Martinez & Vos, 2018).

We also drew on educational research from outside modelling, including Brousseau's (1984) notion of *didactical contract* which relates to students' understanding of teacher expectations. What kinds of task, for example, will the teacher give students to work on? In the same way, what can the teacher *expect* students to do, and not do? From this perspective, teachers' expectations of new or different work require the overt renegotiation of an existing didactical contract. Additionally, the influence of *socio-mathematical norms* (Yackel & Cobb, 1996) should not be underestimated when considering the optimal level of intervention. Socio-mathematical norms govern acceptable modes of reasoning and ways of working, as well as judgements about the quality of mathematical ideas and the appropriateness of solutions to problems. Socio-mathematical norms are established and promulgated in a context governed by the didactical contract, which in turn is partly shaped by students' and teachers' enactment of those norms. This circularity engenders reinforcing feedback whereby the constituents tend to confirm rather than challenge each other. If it is desired to instigate sustained change (e.g., through a modelling initiative), it is necessary to identify, confront, and alter elements responsible for resisting change.

In operationalizing this component when working with teachers, we took the position that the teachers themselves must first undertake open real-world mathematical modelling in order to develop a holistic appreciation for, and understanding of, relevant constructs and processes—a pre-requisite for anticipating and understanding issues surrounding modelling competence.

3 Research questions

The literature reviewed in the previous section points to two theory-practice gaps that are embodied in the following research questions:

1. What are the principles for the design and implementation of tasks that support instructional competency in mathematical modelling of practicing secondary teachers?
2. What role can teachers play in developing principles for the design and implementation of mathematical modelling tasks that support their own instructional competency?

These questions were used to guide the development of the DIFMT in collaboration with experienced teachers, the goal of which was to support the development of their instructional practice in modelling. Our response to the first question was initially informed by a review of existing research literature, with the DIFMT refined through fieldwork trials and teachers' reflections on its effectiveness for guiding instructional competence. The second question was addressed via the methodological approach and teacher input that contributed to the refinement of the DIFMT.

4 Research design and instrumentation

Our research design was derived in response to the research questions, with selection of data sources, interpretations, analyses, and syntheses following as consequences. Our approach can be recognized as related to the design-based methodology advanced by Cobb, Confrey, diSessa, Lehrer, and Schauble (2003). This methodology is suited to applied research that sets out to develop contextualised theories of learning and teaching in tandem and geared towards educational problems that feature variable and multiple settings.

4.1 Participants

Participants included six teachers and their year 9–11 classes drawn from schools in Queensland (Australia). Targeted year levels allowed for the deployment of tasks with sufficient challenge to provoke sophisticated modelling practices. Teachers were selected purposively (Burns, 2000), on the basis of their interest in promoting student modelling competency within their classrooms. Their experience with teaching modelling varied from novice to highly experienced. Groups of students within these classes were also selected purposively for classroom observation and interviewed on the basis of teacher advice, with criteria including the capacity to articulate their approaches to a task clearly and a willingness to share their thoughts with both fellow students and researchers. While mathematical modelling is a mandatory element of the curriculum context in which the study was conducted, there is no advice within relevant documents related to its instruction. What the syllabus sets out are criteria by which modelling is to be characterised and assessed. These include formulation, solution (that includes interpretation), evaluation and verification, and communication. Together, these embody the complete modelling process. Consequently, the notions of holistic or atomistic modelling are not part of the discourse surrounding modelling implementation in school classrooms. Consistent with how modelling is characterised in curriculum documents and assessed, its practice is aimed at total competency and is therefore aligned with holistic rather than atomistic approaches. Of course, modelling subskills are dealt with directly, but always in the context of the complete modelling process. There is no necessary one-on-one alignment between their articulation and the nomenclature commonly surrounding sub-competencies in the atomistic approach.

4.2 Design

The research program was based on an iterative process of design-implement-reflect as the basis for researcher/teacher collaboration in developing the DIFMT. This process was effected through three whole-day researcher/teacher meetings and two classroom observation visits per year. Classroom visits took place between researcher/teacher meetings. This cycle of researcher/teacher meetings and classroom observation visits (Fig. 3) took place in each year of the project.

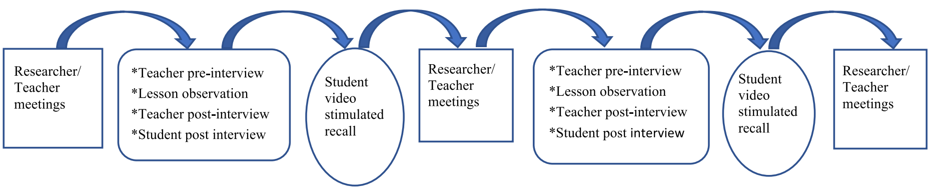

Fig. 3 Yearly cycle of researcher/teacher meetings and classroom observation visits

The purpose of researcher/teacher meetings was to develop tasks, plan for their implementation, and iteratively refine the DIFMT through collaborative reflection on data gathered during classroom observation visits. The developing framework was used progressively as a lens to code transcripts of data collected during school visits. Ongoing reflection on the DIFMT as a tool supporting coding led to its further refinement. Active teacher involvement throughout the research-development process was central, in contrast to research approaches in which teachers are treated as objects of research (Jung & Brady, 2016).

Data (classroom observations, teacher and student interviews, researcher/teacher meetings) were collected relevant to teachers' efforts to support students' modelling competency as follows: mapping progress on modelling tasks; gathering information on what supported or hindered progress; the extent to which students consciously used a modelling process; ways in which students indicated forward thinking in planning and executing sub-tasks; how collaborative groups functioned; how students saw modelling activity in terms of importance and interest. Student data sources included post-lesson interviews with target students after modelling sessions, followed by a later video-stimulated recall interview based on incidents selected by the researchers after viewing lesson material. Lesson observation data included video-recordings of teachers, small groups of target students, and student written work. Teacher data were gathered via pre- and post-lesson interviews and collaborative discussion that took place during researcher/teacher whole-day meetings. These augmented data were generated, as noted above, by the teachers' own prior anticipations in respect to the design and adaptation of tasks and planning for their subsequent implementation. As this article is concerned with the development of the DIFMT in collaboration with teachers, we draw on data that is reflective of their input into its design.

4.3 Instrumentation

Data collection procedures, focused on the researcher/teacher collaboration that generated the DIFMT, are presented in the following section.

Modelling problems: Sample modelling problems included the following: a bushwalking problem which involved the faster of two walkers estimating how much further to travel after the other had started to return, so that they arrived home at the same time (Galbraith, 2015); a waste problem which involved predicting the amount of rubbish generated by Australians over coming decades; a petrol pricing problem (motivated by Blum & Leiß, 2005) which involved selecting the best option for filling a tank from outlets offering different prices per litre; a sausage sizzle task which involved pricing sausages at a school fete to maximize profit; and a problem involving calculation of costs associated with the conversion of a bullring in Barcelona (Las Arenas) to a shopping centre. Relevant assumptions and data for all problems could be generated either on the basis of observation, experience, or through internet searches. An exemplar item is presented in Fig. 4.

Interview schedules: Pre- and post-lesson teacher interviews probed what teachers expected to unfold in the lesson, and reflection on what had actually happened. These included both cognitive and affective components. Examples of these questions are presented below:

- Where do you anticipate students will need the most support, for example, specific aspects of modelling, the modelling process, content knowledge, etc.? How might you deal with this?
- What do you think your students learned in terms of the task? Is this different from what you anticipated? In what ways?

THE BEST PRICE FOR PETROL

The rapid change in the price of petrol at the bowser has become common place in recent years. Prices also vary significantly between suburbs towns and states. Some have developed the habit of scouring apps such as *Fair Fuel Prices* hosted by the RACQ (https://www.racq.com.au/cars-and-driving/driving/fair-fuel-prices) to determine the best location at which to fill up. But is simply finding the cheapest price and driving to the relevant location effective in terms of minimizing your costs?

Think about your own circumstances, where you live, the type of car your family owns, and any other relevant factors before developing a plan to fill up the tank of your car. This plan should include travel to and from the petrol station you select. Write up your plan as a report that includes all relevant factors, how these are accommodated and justifications for any decisions you make.

To assist in your report, consider the following situation to start. Sam has just finished his shopping at the 'The Gap Village Shopping Center' and realises that they are almost out of fuel with only 4 litres left! Although Sam lives just across the road from the Shopping Center it is important the car returns home with a full tank (or near to). Sam checks a phone fuel app and is presented with the following data for surrounding petrol stations:

Name	Petrol Cost (Cents/Litre).
7 Eleven Albany Creek	120.7
BP Noonans Garage	135.7
BP Stafford	130.6
Puma Everton Park	125.7

Sam's car has the following attributes
Car: Toyota Yaris Ascent Hatch Manual
Fuel Tank: 42 L
Fuel Consumption: 7.1L / 100km or 14.08km/L
Current Fuel Tank Level: about 4 L

What Petrol Station should Sam choose to fill the tank and spend the least amount of money? Is this the best choice for Sam?

Fig. 4 Petrol pricing problem (motivated by Blum & Leiß, 2005)

Research/teacher whole-day meetings: Researcher/teacher meetings were recorded and transcribed. These focused on both retrospective reflection on the most recent task implementation, forward reflection focusing on planning for the next implementation in the sequence, and the collaborative development of the DIFMT.

5 Developing the DIFMT

In this section, we describe and illustrate the development of the DIFMT by connecting literature relevant to the components/dimensions of the DIFMT to teachers' perspectives on the connection of the framework to changes in their instructional modelling competency.

5.1 The design and implementation framework for mathematical modelling tasks

The DIFMT consists of two major components—principles for modelling task design and pedagogical architecture—each of which is structured around a number of dimensions. The first component is primarily concerned with the development of a modelling task itself, while the second relates to its classroom implementation—consistent with the dual notions of *design as intention* and *design as implementation* (Czocher, 2017). The dimensions of the DIFMT draw on research related to teacher instructional competency in modelling or aligned fields such as numeracy; principles for modelling task design (e.g., Galbraith, 2006); pre-engagement (e.g., Goos et al., 2019); initial problem presentation (Borromeo Ferri, 2018); body of lesson (e.g., Maaß, 2010; Schukajlow et al., 2015); and conclusion and reporting (e.g., Haines et al., 2001; Stillman et al., 2007). The complete DIFMT is depicted in Table 2.

Table 2 The Design and Implementation Framework for Mathematical Modelling Tasks (DIFMT)

Principles for modelling task design		
Nature of problem	Problems must be open-ended and involve both intra- and extra- mathematical information. The degree of open-endedness is dependent on students' previous experience with modelling. Less experienced students may need additional scaffolding questions or information. More experienced students should be expected to engage with less defined problems.	
Relevance and motivation	There is some genuine link with the real-world of the students. This will depend on factors including students' age, year level, personal circumstances, etc. Problems may need to be contextualised for specific student groups.	
Accessibility	It is possible to identify and specify mathematically tractable questions from a general problem statement. Is there a mathematical approach accessible to students? Problems must be tractable from the perspective of the student group.	
Feasibility of approach	Formulation of a solution process is feasible, involving (a) the use of mathematics available to students, (b) the making of necessary assumptions, and (c) the assembly of necessary data. Teachers must work through the problem.	
Feasibility of outcome	Solution of the mathematics for a basic problem is possible for the students, together with interpretation. Expectations in relation to the type of response, for example, arithmetical versus generalised solutions, are dependent on the characteristics and year level of the specific student group being engaged.	
Didactical flexibility	The problem may be structured into sequential questions that retain the integrity of the real situation. (Having worked through the problem, how can it be implemented?) For example, can prompts/assistance to students be structured into sequential questions (identify sub-sections of the problem)?	
Pedagogical architecture		
Pre-engagement: Understanding of the modelling process and its application including support materials (learn/illustrate what the modelling process is)	Students need to be initially familiarised with the modelling process. This can be supported via materials including: - A copy of the of the modelling process (diagram/graphic) [modelling infrastructure] [also for students to map their way around the graphic during implementation]. - An example of a simple modelling problem matched to the phases of the cycle: problem statement; formulation; solution; interpretation; and evaluation - A copy of report structure. Students should have a clear idea what their report should look like at the end.	
Modelling process review	Reviewing pre-engagement as required. - The length of the discussion is dependent on students' prior experience with the modelling process. - Each student is provided with a copy of the modelling task, and a representation of the modelling process (e.g., a diagram) that is a depiction of the logical process that will guide their efforts.	Points that may be considered by teachers and students: - It is necessary to leave the realm of pure mathematics to build a model, e.g. by procuring extra-mathematical information and data. - Several different models may be reasonable. There is rarely a unique, or a best, answer to a modelling problem.

Table 2 (continued)

		- That modelling is not a five-minutes-to-get-an answer activity. - Simplifications are likely to be needed and assumptions may be necessary to reduce the complexity of the extra-mathematical domain being modelled or to make the mathematics tractable. - Assumptions can be made at any point in the cycle. - Students should be encouraged to ask clarifying questions.
Initial problem presentation	- Teacher provides brief general description of the task scenario [2-3 minutes]. - Students should be organised into small groups. They are provided with time to read the task description and identify a mathematical question(s) [5 minutes] - Teacher calls the class back together to discuss their initial understanding of the task and possible mathematical questions. Each group contributes via a representative. - Students in groups then consider assumptions and variables relevant to the mathematical question as well as other observations such as trends in data, etc. [5 minutes] - Teacher once more calls the class back together. Each group reports back to whole class by group representative. Teacher synthesizes/prioritizes students' initial assumptions and variables sufficient to begin modelling process for an initial model.	Points that may be considered by teachers and students: - Teachers use facilitating questions that emerge from students' engagement in the task rather than clarify problem contexts or ask questions up front. Responses should align with the question "What should a modeller be asking himself/herself at this point in the modelling process?" (metacognitive connection) - There should be a focus on student decision making – with students required to initiate suggestions regarding relevant mathematical content, assumptions, variables; and for the more experienced, possible alternative questions. - Students should be encouraged to pose explorative questions as to the nature of the endeavour as well
Body of lesson	Students: - Proceed in their groups to create model, solve, interpret, etc., in terms of the question they are addressing. - Engage in productive student-student collaboration - Identify and make productive use of technology where applicable, for example, to source relevant information, check calculations and/or generate solutions. - Develop a report of their progress in terms of the stages of the	Points that may be considered by students: - Documenting progress against a visual representation of the modelling process. Problem statement → Formulate → Mathematical solutions → Interpreting outcomes → Evaluation. - Forms of collaboration: Working separately and then coming together; Working together from the beginning; Negotiating/confirming

Table 2 (continued)

	modelling process (e.g., formulate, solve, interpret, evaluate) Teachers: - Bring to consciousness those things that are implicit …actions are then deliberate. - Activate teacher meta-meta cognition: (a) How will the students be interpreting what I as a teacher am doing/saying at this point? (b) What should the students be asking themselves at this point in the modelling process? - Support students with making progress through the modelling process. - Anticipate where students might have problems, e.g., interpreting the problem, generalizing the solution. - Employ measured responsiveness – rather than providing specific advice about the problem, teachers should prompt students to think about where they are in the modelling process. Structure mathematical questions that promote a viable solution pathway. - Encourage the use of digital or other tools as appropriate. - Support student development of a modelling report.	consensus; Explaining external to the group [Teacher/Researcher – Student). Students also encouraged to identify groups working on a similar problem/issue and extend collaborations. - Points that may be considered by teachers: - Checking if documenting progress against the modelling process is taking place (both in the doing and in the recording). - Focusing students' attention on phases of the modelling cycle (there should be no specific direction towards a solution). - Support student decision making – multiple solution pathways should be encouraged. - Responses to students' questions or requests for assistance could include: What are you doing? What are you trying to do? Where are you in the modelling process? How have you checked your answer? (both mathematically and in terms of context); Can your solution be generalized? - Take account of student capability (catering for diversity)
Conclusion: Presentations of findings and teacher summary	- Students share what they have found with justification (representative from each group as spokesperson). Findings should be reported in a succinct fashion (e.g., via 3-4-minute video) - Teachers/students ask questions of clarification as required or to test arguments.	Points to be considered by teachers and students: - Students in the audience should provide commentary that includes questions, elaborations, clarifications (e.g., each student to write down one question or comment about the presented model). - Comments could also be directed towards criteria related to making judgements about the quality of the presentation of findings (e.g., Problem statement → Formulate → Mathematical solutions → Interpreting outcomes → Evaluation). All students should have access to these criteria. - Teacher clarification questions can include: How does that work with your model? (e.g., teacher has identified an error); Will your solution work for other situations?

Table 2 (continued)

		(e.g., teacher encouraging students to generalize). What did you do to evaluate the model? (e.g., teacher encouraging students to validate and verify am proposed solution). - The focus should be on what was learnt about the modelling process
Report (if required)	- Students should communicate their findings via a succinct, coherent, systematic report. The report must make use of appropriate mathematical language. - Teacher checks for the validity of the solution and supporting justification.	- Students report findings should address Formulating, Solving, Interpreting, Evaluating (FSIE).

Within the cycle of researcher/teacher meetings and classroom visits, the following process was employed in the development of the DIFMT: presentation of tasks designed according to preliminary DIFMT principles at researcher/teacher meetings; individual completion of the task by teachers; teacher accepting or adapting task for own classroom; teacher implementing task; teachers/researchers meeting to analyse and reflect on implementation; refinement of the DIFMT after researcher/teacher reflections upon data collected during implementation. While each DIFMT dimension and elaboration is an outcome of this cycle, space allows for only illustrative examples of teacher input into the DIFMT, as set out in the following text.

5.2 Teachers' response to initial theoretical perspectives

We initiated discussion about the design of tasks by presenting the original draft of the DIFMT during the first whole-day teachers' meeting. A question asked consistently by all teachers was "What would this principle look like in a classroom?"

Teacher F: From the framework, I can tell at each implementation step what to do… [but] I need to go and look at how do I actually deal with the formulating bit of it. The implementation framework has some ideas for me there…on how I can bring it into the class.

This was a typical comment as teachers interrogated the developing framework through each cycle of refinement. Thus, when providing clarification, we emphasised that task design principles be thought of as *necessary* criteria for a task to be considered suitable for possible classroom implementation. The design principles were not sufficient in themselves, as teachers had to then consider whether and how a task may require adaptation for a particular class.

5.3 Principles of task design

Typical examples of collective discussions about the principles of task design are illustrated below.

Teachers asked for clarification of an original aspect of the DIFMT, Galbraith's (2006) *Principle 1*: *There is some genuine link with the real-world of the students.*

This resulted in a revision—an expansion of this aspect of the DIFMT into two new components—the *nature of the problem* and *relevance and motivation* (see Table 2). The former related to authenticity as a real-world problem and the later to students' perceived readiness. This included the need for task contexts to be relevant to factors such as year level and personal circumstances, key when adapting existing tasks. The significance of this change was noted by Teacher A in a response to the revised framework.

Teacher A: It [the DIFMT] certainly helped in terms of me designing tasks that I think are accessible for students... I think it's really good if you can find something that the students have got some sort of interest in…if it's something that is relevant to their lives…they're more likely to persevere through that.

In a similar vein, Galbraith's *Principle 2* was refined under the dimension of *accessibility* by adding that problems must be tractable from the perspective of the specific student group.

Teacher B, while agreeing with the need for tasks to be accessible, also argued that tasks remain challenging in order to develop students' reasoning capability.

Teacher B: The tasks need to be accessible to the kids, but it can't be trivial. They need to be able to reason it and work it through... Also, teachers should make the task clear to the students.

Similar data-informed discussion between researchers and teachers contributed to the refinement of the elaborations of each principle presented in Table 2. It should be noted, however, that these focused centrally around changes deemed necessary when an authentic task, written in general terms, required adaptation for particular classroom settings. When considering adaptations of existing tasks, it must be recognized that teachers' anticipation of student responses plays a central role.

5.4 Pedagogical architecture

The pedagogical architecture component of the DIFMT was also developed from a synthesis of relevant research literature and refined through ongoing discussion with teachers. As the project progressed, teachers placed an increasing importance on the role of anticipation in planning for the implementation of a task—exemplified by the following comment:

Teacher E: In terms of the Design and Implementation Framework, the part I think is most important is the questioning and the anticipation parts…to guide my thinking. I recognize that I need to be able to do that.

In response to classroom observations in which half the teachers presented students with tasks following limited preparation, we included a dimension dedicated to *Pre-engagement*, in which initial (at least) familiarity with the modelling process is promoted. This familiarization includes discussion of a graphical representation of the modelling process, a straightforward modelling task including a solution pathway aligned with the phases of modelling process, and expectations for reporting back on a solution. This phase is consistent with our addition to Niss' (2010) enablers that students be specifically acquainted with the modelling process. The latter requirement needs to be made explicit for novice modellers. For those with significant prior or recent experience, it can receive less teaching emphasis but needs to be consciously reviewed.

Later in the project, Teacher D stressed the importance of illustrating the process with a less sophisticated modelling problem in this introductory phase.

Teacher D: …I think that whole lesson we spent on Friday where we had the students focus on learning how to model was critical… It's about taking that real-world problem and simplifying it.

Also, after initial classroom observations, we noted a tendency for teachers to take one of two directions when beginning a modelling session:

(1) Ask students to work on a task with very little introduction.

(2) Provide too much preliminary information and direction, making the pathway to solution obvious.

To mitigate both approaches, we introduced a *modelling process review* to the DIFMT. The time that should be dedicated to this review is dependent on the experience and confidence of the students in relation to modelling. This dimension includes the distribution of a graphical representation of the modelling process.

Other aspects of implementation were made explicit in this dimension because they interact with notions of socio-mathematical norms and didactic contracts students have previously acquired. These included explicit advice that different pathways to a solution were acceptable and that a range of valid but different assumptions and associated simplifications were possible. After incorporating these elaborations into the DIFMT and subsequently classroom practice, teachers noted that students' appropriation of the modelling process proceeded at different rates—with some needing to be convinced that the process of simplification, for example, was an essential element of modelling and that solving such problems took time and persistence:

Researcher: Have the students taken that on board much so far? [the modelling process]

Teacher F: A number of them have really jumped on board with that modelling cycle. Some of them are still kind of in the throes of trying to make sense of the modelling cycle…other things that I anticipate is trying to convince students to start with something simple and then come back and refine it later…they think they need to have automatically got the answer from just reading the question.

We suggested that teachers, after revising the modelling process with their students, structure the remainder of the *review* according to whether their students were novice or experienced modellers—a further clarification made within the DIMFT after classroom observations.

A further refinement included the addition of an *initial problem presentation* phase to provide structure when introducing a task to a class. This phase included advice on classroom organisation and a focus on the identification of key mathematical questions. Student decision-making was also emphasised, with no specific mathematical suggestions from teachers regarding relevant content, assumptions, or choice of variables. This advice represented a significant change to previous practice for all participant teachers. A typical reflection on the value of this phase of the lesson is captured in the following comments:

Researcher: Two things that happened at the beginning of the lesson were your introduction and then you came back again later on and hauled things together…a bit different to what you've done before.

Teacher A: It certainly helped the kids relax a little bit. That we were just going to take an initial look and then we were going to all come back together. So, having told them "Here's your task. Have a read. Have a think about what you might do in terms of the modelling cycle but that's all I want you to do for now." They were all happy to read about it and to start a conversation about what they might need to do…They were then going to have a little safety net: "let's just make sure we're all on the same page here".

I think that bringing them back together really did enable the students to get further into the task quickly and just minimize the number of tangents that could be gone on. Obviously, some students were still going off in different directions…but they were different directions still associated with the task…Because they had that key mathematical question that needed to be answered.

At this stage, Teacher A can see advantage in a new approach but does not (yet) seem to have a clear idea how it can be structured for maximum benefit. This is a reminder that the didactical contract and socio-mathematical norms apply to teachers as well as to students and that such change takes time.

Within the *body of the lesson*, students were expected to apply the modelling process. We noted that in at least half the classrooms we visited, in the early stages of the project, teachers were inclined to intervene often and sometimes provide too much input. In response, we emphasised the notion of *measured responsiveness*, where teachers should provide prompts that focused students on the modelling process or ask questions structured to promote exploration of a range of possible solution pathways without directing them to a single predetermined response. While this had been discussed early in the project, it was apparent further discussion of the issue was needed in researcher/teacher meetings with subsequent and more detailed elaborations added to the DIMFT. This change eventually made impact on Teacher F's practice, who had previously tended to provide too much support.

Teacher F: I was just trying to make sure that I didn't give too much away. And ask the right questions, not explain things too much...

Researcher: So, you're looking at that *measured responsiveness* part of it. Anything else sort of ring a bell with you as you were going through?

Teacher F: Not to explain things too much or panic because they're running out of time or something.

Teacher C reported a similar change to their approach.

Teacher C: At the start I think I was trying to lead them too much and guide them…I think leaving them alone is a lot better because they discover it for themselves rather than, "Oh, yeah, I would have used that." Thanks for telling me it doesn't work!

In concluding the lesson or lesson sequence, we have recommended a *presentation of findings and teacher summary* in which students outline the outcomes of their work to their peers—perhaps via a representative from each group. The teacher's role during this activity is to facilitate discussion by asking clarifying questions and encouraging students to make comments about the quality of the findings and their justification. This dimension should also incorporate a conversation about what was learned about the modelling process itself. To date, we have observed limited occasions in which this dimension was enacted because of the constraints of time.

In cases where a formal *report* is required, teachers were advised to request that students provide a succinct, coherent, systematic report of their findings—an area in which students often had trouble knowing how to proceed. In one instance, a teacher provided scaffolding via a report template in which students were expected to record the outcomes of their thinking and reflections against the phases of the modelling process. Another teacher asked students to document the results of their work phase by phase on an A3-sized graphic of the modelling cycle. He also advised students that they should continually revise and update these records. In both cases, these devices were used to scaffold the writing of the final report.

Teacher D: I hadn't anticipated that they would work so effectively through the booklet… They very much clarified the question quite quickly and were able to move, almost through the flowchart, as you would like them to move through it.

Teacher A: So, I've given [the students] an A3 page with the cycle on it. I found it was useful last time. And I think that being able to refer the students to what is important. Then, when they say "Oh, we've got our answer…I can go, "This iteration. You've done here, here and here. The first three steps and you've stopped. We've got a lot more we can do. Let's go and think about those things."

As part of the process of consolidating teachers' views of the holistic effectiveness of the DIFMT, we asked if it was helpful in their efforts to design and implement modelling tasks.

Teacher B: It's very good. No, really! I can't say I've internalized it, but I certainly think about it a lot. And it fits nicely with that thing that QCAA [local curriculum authority] put together in terms of a problem solving and modelling approach.

6 Discussion and conclusion

In this section, we address the two research questions which provided the foci of this article.

6.1 What are the principles for the design and implementation of mathematical modelling tasks that support instructional competency in mathematical modelling of practicing secondary teachers?

The DIFMT, an outcome of a research/teacher collaboration, embodies characteristic principles for the design and implementation of modelling tasks and provides guidance for teachers' instructional competency in modelling. In developing the DIFMT, we adopted Niss et al.'s (2007) perspective on holistic modelling competency as the focus of the project was on the implementation of the complete modelling process. This by no means suggests that individual sub-competencies are unimportant, but we take the position that a holistic perspective is necessary for their appropriate selection and coordination across the modelling process. While the capabilities identified in the DIFMT are related to the sub-competencies identified in previous research (e.g., Borromeo Ferri & Blum, 2010; Maaß, 2006), they represent a departure from earlier work in that they were developed in collaboration with teachers as an outcome of a three-year long longitudinal study. This long-term engagement enabled teacher input from the perspective of classroom practice as the DIFMT was developed. As far as we can determine, this approach is unique in its approach to developing guidelines to instructional competency in mathematical modelling.

While there has been previous research into task design (e.g., Czocher, 2017; Galbraith, 2006; Maaß, 2010) and task implementation (e.g., Geiger, 2019; Goos et al., 2019; Hernandez-Martinez & Vos, 2018; Schukajlow et al., 2015; Tan & Ang, 2016) in mathematical modelling, the DIFMT is the first attempt to bring these components together into a single cohesive framework—a contribution to new knowledge in the field. Furthermore, in developing the DIFMT, we have extended what has been previously known about both task design and implementation separately. In the case of task design, our principles are seen as necessary but not sufficient criteria for classroom effectiveness, as it is teachers who must also determine how a task must be shaped (or not) to meet the needs of their own students and to align with the classroom/curriculum/system contexts. This represents a departure from previous research (e.g., Maaß, 2010) which has tended to generate modelling tasks that teachers themselves are expected to adapt. This finding provides a new focus for Niss' (2010) construct of implemented anticipation, conceptualized as an enabler for modellers themselves, in that it also has relevance to those who provide instruction in modelling.

The task implementation component of the DIFMT extends Geiger's (2019) notion of pedagogical architecture, developed within the context of numeracy education, into the realm of mathematical modelling. This component of the DIFMT also takes into account the specific circumstances in which modelling is being taught and learned. This is distinct from previous research (e.g., Borromeo Ferri, 2018) which has tended to generate schemes of modelling sub-competencies to which teachers and classrooms need to adapt. A specific case in point is the degree of attention devoted to the modelling process review conducted before the implementation of a new task. The degree of attention to this phase is dependent on a teacher's capacity to anticipate how a task will be received and acted upon by their students, and what support will be required. This is an anticipatory capability not previously discussed in other approaches that emphasise the need to provide instruction related to the modelling process.

While teachers' responses to the usefulness of the DIFMT have been unanimously positive, we have noted that the changes to their instructional practices in modelling depend upon existing didactical contracts (Brousseau, 1984) and socio-mathematical norms (Yackel & Cobb, 1996). The DIMFT is unique, in terms of a framework related to the developing of modelling competency, in that it accommodates aspects of theses constructs rather than taking a purely cognitive approach (e.g., Borromeo Ferri, 2018).

6.2 What role can teachers play in developing principles for the design and implementation of mathematical modelling tasks that support their own instructional competency?

Teachers provided vital input when co-developing tasks with researchers. In the initial stages, teachers provided advice about adaptions to researcher-developed tasks to ensure relevance to specific classroom contexts. Over time, teachers accepted increasing responsibility for the development of tasks, incorporating their detailed knowledge of essential contextual features, such as local curriculum requirements, and students' previous learning experiences in mathematics. The development of this capability is consistent with approaches advocated within the field of task design (e.g., Jones & Pepin, 2016).

In a similar way, teachers provided input into the clarifications and extensions needed to enhance the DIMFT. This insight grounded the dimensions and elaborations of the DIFMT in the practical reality of classroom practice and indicated where refinements to the DIMFT could make it more accessible to other teachers. As a consequence of their active involvement in the research-development process, teachers developed an intimate understanding of both the principles underpinning tasks and the dimensions of the pedagogical architecture. These outcomes are consistent with approaches to research conducted with teachers rather than to them, such as that advocated by Jung and Brady (2016).

6.3 Future research directions and limitations

This study opens up research around teachers' instructional competency in mathematical modelling that is distinct from a focus on their proficiency with sub-competencies. We argue that this can best be achieved by considering task design and task implementation as different sides of the same coin, as embodied in the DIFMT. The feature that moderates the successful deployment of these components of the DIFMT, in an interactive way, is teachers' capacity to anticipate, as design and implementation must each inform the enaction of the other. While this co-dependency has been noted in our study as essential for effective instructional practice in mathematical modelling, further research is needed to identify what enables teachers' anticipation in designing tasks and planning for their implementation.

We also note that existing didactical contracts (Brousseau, 1984) and socio-mathematical norms (Yackel & Cobb, 1996) are strongly influential when attempting to change classroom practice. These constructs have had limited attention in mathematical modelling research to date, but our experience in the current study suggests these ideas represent potentially rich avenues for further research into how to manage effective change in mathematical modelling practice. Our current study indicates that both teachers and students need to be convinced of the benefits of new ways of working, a process that may involve multiple trials and some persistence before success is forthcoming.

The strengths of this study include its longitudinal nature, which means that claims are not based on brief engagement with participants, but rather on involvement over an extended period of time—allowing teachers' perspectives and understandings to stabilise. This study provides for a strong alignment between the type of educator involved in developing the DIMFT and those to whom it is applicable—practising teachers to practising teachers—an alignment that allows for fewer qualifications when applying research findings to practice. At the same time, it must be acknowledged that the research has been conducted with a relatively small number of participants, drawn from a limited pool of school contexts and circumstances. For this reason, we are aware that general advice about the effectiveness of the DIMFT can only be extended beyond the cohort of teachers involved in the study in a tentative manner. This means that further research into the effectiveness of the DIMFT needs to be conducted at greater scale before we can lay claim to robust and generalisable findings.

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