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On Stalnaker's Simple Theory of Propositions*

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Final draft

Abstract

Robert Stalnaker recently proposed a simple theory of propositions using the notion of a set of propositions being consistent, and conjectured that this theory is equivalent to the claim that propositions form a complete atomic Boolean algebra. This paper clarifies and confirms this conjecture. Stalnaker also noted that some of the principles of his theory may be given up, depending on the intended notion of proposition. This paper therefore also investigates weakened constraints on consistency and the corresponding classes of Boolean algebras.

1 Introduction

Robert Stalnaker (2012, pp. 23–27) puts forward a simple theory of propositions. The theory operates with two primitive notions: truth of propositions, and consistency of sets of propositions. It consists of a short list of principles, using a few further notions defined from consistency. The theory has its roots in a discussion of possible worlds and propositions in (Stalnaker, 1976). An intermediary version of the theory can be found in a revised version of this article (Stalnaker, 1984), also reprinted as (Stalnaker, 2003). Stalnaker (2018) conjectures that this simple theory is equivalent to the claim that propositions form a complete atomic Boolean algebra, and that a weaker version corresponds to a suitably more inclusive class of Boolean algebras. The purpose of this paper is to clarify and confirm these conjectures. The main focus here is on purely formal questions, with occasional motivating philosophical discussion.

The remainder of this section introduces a basic form of Stalnaker's simple theory invoking only the notion of consistency. It is shown how to define the Boolean connectives in this theory, and the definition of a Boolean algebra is stated. Section 2 shows how the structures satisfying this theory relate to Boolean algebras. Subsequent sections consider various additional principles, and their relationship to suitable classes of Boolean algebras. Step-by-step, these sections develop a hierarchy of variants of Stalnaker's theory and their corresponding classes of Boolean algebras. The main result (Proposition 61) confirms one of the conjectures of Stalnaker (2018): Stalnaker's full theory is equivalent to the claim that propositions form a complete atomic Boolean algebra. The matter turns out to be more complicated for weaker versions of the theory. Section 7 summarizes the findings in a diagram (figure 1), which the reader may find useful to consult as they go through the different variants in previous sections.

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1.1 Consistency Structures

The simple theory formulated by Stalnaker (2012, 2018) concerns both consistency and truth. Boolean algebras provide notions like entailment which are closely related to consistency, but nothing corresponding to the notion of truth. However, the only way in which truth figures in Stalnaker's theory is in a principle stating that the set of true propositions is maximal consistent. To allow a direct comparison with Boolean algebras, we therefore set aside this principle and the notion of truth more generally.

Similar to the theory of Boolean algebras, Stalnaker's simple theory can be treated as defining a class of algebraic structures. Such structures provide a set A of propositions and a set C of consistent sets of propositions.

Definition 1. *A* consistency structure *is a pair $\langle A, C\rangle$ where A is a non-empty set (of* propositions*) and $C \subseteq \mathcal{P}(A)$ (the set of* consistent sets *of propositions).*

Stalnaker defines three notions from consistency: equivalence, entailment and contradictoriness. Sets of propositions X and Y are equivalent if they make the same demands on the world, which amounts to being consistent with the same hypotheses. X entails Y just in case Y makes no demands of the world in addition to what is demanded by X, which amounts to X and Y together being equivalent to X. Finally, propositions x and y are contradictories if they make opposing demands of the world, which amounts to being jointly inconsistent but such that any consistent hypothesis is consistent with at least one of them.

Definition 2. *For any $x, y \in A$ and $X, Y \subseteq A$ in a consistency structure $\langle A, C\rangle$, define:*

$X \sim Y$ *(X and Y are* equivalent*) iff*

for all $Z \subseteq A$, $X \cup Z \in C$ iff $Y \cup Z \in C$.

$X \leq Y$ *(X* entails *Y) iff*

$X \sim X \cup Y$.

$x \times y$ *(x and y are* contradictories*) iff*

$\{x, y\} \notin C$ and for all $X \in C$, C contains $X \cup \{x\}$ or $X \cup \{y\}$ (or both).

It will be useful to extend talk of consistency, entailment, and equivalence from sets of propositions to single propositions. In these cases, any proposition is treated as its singleton set. For example, x is said to be consistent just in case $\{x\}$ is consistent. Lowercase letters will generally be used for propositions, and uppercase letters for sets of propositions.

An immediate consequence of the definitions of equivalence and entailment is the following lemma. It notes that equivalence is mutual implication, which partitions the propositions, refining the partition dividing propositions into consistent and inconsistent propositions.

Lemma 3. *In any consistency structure:*

(i) $X \sim Y$ iff $X \leq Y$ and $Y \leq X$.

(ii) $\sim$ is an equivalence relation.

(iii) If $X \sim Y$, then $X \in C$ iff $Y \in C$.

Proof. Immediate. □

With these defined notions, we can state a basic version of Stalnaker's simple theory of propositions.

Definition 4. *Let $\langle A, C\rangle$ be a consistency structure. $\langle A, C\rangle$ is* simple *if it satisfies the following constraints:*

(1) Every subset of a consistent set of propositions is consistent:
For all $X \in C$ and $Y \subseteq X$, $Y \in C$.

(2) Every propositions has a contradictory:
For all $x \in A$, there is a $y \in A$ such that $x \times y$.

(3) Any two propositions are equivalent to some proposition:
For all $x, y \in A$, there is a $z \in A$ such that $\{x, y\} \sim \{z\}$.

(4) Equivalent propositions are identical:
For all $x, y \in A$, $\{x\} \sim \{y\}$ only if $x = y$.

This is weaker than the theory of Stalnaker (2012, 2018), in three ways, which facilitate the comparison with Boolean algebras. The first was noted already above: we have dispensed with the notion of truth, since this has no correspondent in Boolean algebras. Second, Stalnaker includes, instead of (3), a principle according to which *any* set of propositions is equivalent to a single proposition. We consider such a strengthening of (3) in section 4.1. Third, Stalnaker includes a further principle stating that every consistent set of propositions can be extended to a maximal consistent set. We consider adding such a strengthening in section 4.2.

Of the four principles of simplicity, the first three seem highly plausible. (1) is clearly motivated given our conception of consistency. (2) is motivated by the thought that every proposition has a negation. (3) is motivated by the thought that any two propositions have a conjunction. (4) is likely to be considered more controversial. It can be read as saying that propositions which make the same demands on the worlds are identical. Thus, it enforces a rather coarse-grained individuation of propositions, which some may want to reject. We will therefore consider the theory which omits this principle, i.e., (1–3), in section 6.

1.2 Negation, Conjunction and Disjunction

(4) entails that the propositions witnessing (2) and (3) are unique, as we now show. Thus, in any simple consistency structure, we can formally define the negation $-x$ of any given x, and the conjunction $x \sqcap y$ of any given x, y.

Lemma 5. *In any simple consistency structure, every proposition has a unique contradictory.*

Proof. By (2), every proposition has some contradictory. Suppose, for reductio, that x has contradictories $y \neq z$. Then:

(i) $\{x, y\}$ and $\{x, z\}$ are inconsistent.

(ii) For every $X \in C$, if $X \cup \{x\}$ is inconsistent then both $X \cup \{y\}$ and $X \cup \{z\}$ are consistent.

(iii) By (4), there is – without loss of generality – some Y such that $Y \cup \{y\} \in C$ and $Y \cup \{z\} \notin C$.

Let Y be a witness to (iii). By (i), $\{x, y\}$ is inconsistent, so $Y \cup \{y, x\}$ is inconsistent. By (iii), $Y \cup \{y\}$ is consistent, so with (ii), it follows that $Y \cup \{y, z\}$ is consistent. But this contradicts (iii). □

Lemma 6. *In any simple consistency structure, for any x and y, there is a unique z such that $\{x, y\} \sim \{z\}$.*

Proof. By (3), some such z exists. If z_1 and z_2 are each equivalent to $\{x, y\}$, then as $\sim$ is an equivalence relation, $z_1 \sim z_2$, whence $z_1 = z_2$ by (4). □

These lemmas establish that the following is well-defined.

Definition 7. *In any simple consistency structure:*

$-x$, the negation *of x, is the unique contradictory of x.*

$x \sqcap y$, the conjunction *of x and y, is the unique z such that $\{x, y\} \sim \{z\}$.*

With negation and conjunction defined, we can also define disjunction using the de Morgan laws. Furthermore, we can show that a unique contradiction can be identified as the conjunction of any given element with its negation, and a unique tautology as its negation.

Lemma 8. *In any simple consistency structure, $x \sqcap -x = y \sqcap -y$.*

Proof. $\{x, -x\}$ and $\{y, -y\}$ are both inconsistent, and so equivalent. Thus $x \sqcap -x$ and $y \sqcap -y$ are equivalent and so identical. □

Definition 9. *In any simple consistency structure:*

$x \sqcup y$, the disjunction *of x and y, is $-(-x \sqcap -y)$.*

$\bot$, the contradiction, *is $x \sqcap -x$, for any element x.*

$\top$, the tautology, *is $-\bot$.*

1.3 Entailment Orders

Simple consistency structures will be related to Boolean algebras. Boolean algebras can be defined in several different ways. For present purposes, it will be convenient to understand Boolean algebras as ordered sets, with the elements taken to be propositions and the order to be entailment. That is, Boolean algebras can be understood as certain entailment orders. Since we have already defined entailment in terms of consistency, there is a straightforward way of mapping consistency structures to entailment orders.

Definition 10. *For any consistency structure $\mathcal{S} = \langle A, C \rangle$, define $\mathcal{S}^*$ to be $\langle A, \leq \rangle$.*

To investigate $\mathcal{S}^*$, for various consistency structures $\mathcal{S}$, it is useful to note that given (1), the definitional characterization of entailment can be simplified slightly.

Lemma 11. *In a consistency structure satisfying (1):*

$$X \leq Y \text{ iff for all } Z,\ X \cup Z \in C \text{ only if } X \cup Y \cup Z \in C.$$

Proof. Immediate. □

$\cdot^*$ provides a mapping from consistency structures to entailment orders. We can thus investigate the relationship between simple consistency structures and Boolean algebras by relating the entailment orders which are images of simple consistency structures under $\cdot^*$ to Boolean algebras. We will do so in the next section; the remainder of this section recalls the central notions of Boolean algebras which will be needed. Very basic notions and observations concerning partial orders will be assumed to be familiar, such as the notion of a lower bound, or the observation that greatest lower bounds of partial orders are unique. All of the necessary mathematical background can be found in standard textbooks such as Davey and Priestley (2002) and Givant and Halmos (2009).

Boolean algebras can be defined as a special kind of ordered set using the following chain of definitions:

A *partial order* is a reflexive, transitive and antisymmetric relation.

A *lattice* is a partial order in which for any elements x and y, the set $\{x, y\}$ has a greatest lower bound $x \wedge y$ and a least upper bound $x \vee y$.

A lattice is *bounded* if there is a least element 0 and a greatest element 1.

In a bounded lattice, x is a *complement of* y if $x \wedge y = 0$ and $x \vee y = 1$.

A *complemented* lattice is a bounded lattice in which every element has a complement.

A lattice is *distributive* if $x \wedge (y \vee z)$ is always $(x \wedge y) \vee (x \wedge z)$.

A *Boolean algebra* is a complemented distributive lattice.

The simplest examples of Boolean algebras are *powerset algebras*: For any set W, the set $\mathcal{P}(W)$ of subsets of W, ordered by the subset relation, is a Boolean algebra. We notate it $\mathfrak{P}(W) = \langle \mathcal{P}(W), \subseteq \rangle$, leaving the restriction of $\subseteq$ to $\mathcal{P}(W)$ implicit.

It turns out to be difficult to establish directly that $\mathcal{S}^*$ is distributive, for a suitable consistency structure $\mathcal{S}$. We therefore make use of a variant characterization of Boolean algebras due to Birkhoff (1940, p. 171, Theorem X.17), as uniquely complemented orthocomplemented lattices. To define orthocomplementation, we call a function f on a set X an *involution* if $x = f(f(x))$, for all $x \in X$. Relative to an order $\leq$ on X, we call f *order-reversing* if $x \leq y$ only if $f(y) \leq f(x)$, for all $x, y \in X$.

A bounded lattice is *orthocomplemented* if there is an order-reversing involution f which maps every element x to a complement $f(x)$.

Note that orthocomplements are complements, and being orthocomplemented entails being complemented. An alternative proof of Birkhoff's result can be found in (Saliĭ, 1988, p. 48).

2 From Simplicity to Boolean Algebras

We are now ready to start investigating the claim that Stalnaker's theory of propositions is equivalent to the claim that propositions form a suitable Boolean algebra. As we have seen, any simple consistency structure determines an entailment order, so the first natural question is whether any such entailment order is a Boolean algebra. We will first show that this is indeed so.

2.1 Simple Consistency Structures are Mapped to Boolean Algebras

For this section, let $\mathcal{S} = \langle A, C \rangle$ be a simple consistency structure. Let $\mathcal{S}^* = \langle A, \leq \rangle$. We start by showing that $\mathcal{S}^*$ is a partial order, which is just a matter of checking the constraints. We will generally make use of Lemmas 3 and 11 without referring to them explicitly.

Lemma 12. $\leq$ *is reflexive.*

Proof. $x \leq x$ iff $x \sim x$, which is immediate. □

Lemma 13. $\leq$ *is transitive.*

Proof. Assume $x \leq y$ and $y \leq z$. Consider any X such that $X \cup \{x\} \in C$. Since $x \leq y$, $X \cup \{x, y\} \in C$, and so with (1), $X \cup \{y\} \in C$. Since $y \leq z$, $X \cup \{x, z\} \in C$, as required. □

Lemma 14. $\leq$ *is antisymmetric.*

Proof. If $x \leq y$ and $y \leq x$, then $x \sim y$, and so with (4), $x = y$. □

The next step is to show that $\mathcal{S}^*$ is a lattice. For this, we first prove a lemma on negation. This will help us establish orthocomplementation later on, but will be useful in the meantime as well.

Lemma 15. *Negation is an order-reversing involution.*

Proof. Since $\times$ is a symmetric relation, it is immediate from the uniqueness of contradictories that any x is identical to $--x$; so negation is an involution. Assume for contradiction that there is a case in which $x \leq y$ but not $-y \leq -x$. Then there is some X such that $X \cup \{-y\} \in C$ and $X \cup \{-x, -y\} \notin C$. Then $X \cup \{x, -y\} \in C$. As $x \leq y$, $X \cup \{x, y, -y\} \in C$, so $\{y, -y\} \in C$, which is false. □

$\mathcal{S}^*$ can be shown to be a lattice by showing that conjunction and disjunction are greatest lower bounds and least upper bounds.

Lemma 16. $x \sqcap y$ *is a lower bound of* $\{x, y\}$ *under* $\leq$.

Proof. Consider any X such that $\{x \sqcap y\} \cup X$ is consistent. Since $\{x \sqcap y\} \sim \{x, y\}$, $\{x \sqcap y, x, y\} \cup X$ is consistent, whence $\{x \sqcap y, x\} \cup X$ and $\{x \sqcap y, y\} \cup X$ are consistent. Thus $x \sqcap y \leq x$ and $x \sqcap y \leq y$, as required. □

Lemma 17. $x \sqcap y$ *is the greatest lower bound of* $\{x, y\}$ *under* $\leq$.

Proof. As shown in Lemma 16, $x \sqcap y$ is a lower bound of $\{x, y\}$. So assume for contradiction that $x \sqcap y$ is not the *greatest* lower bound of $\{x, y\}$. Then there is a lower bound z of $\{x, y\}$ such that $z \not\leq x \sqcap y$. So there is a set X such that $X \cup \{z\} \in C$ and $X \cup \{z, x \sqcap y\} \notin C$. Since $\{x \sqcap y\} \sim \{x, y\}$, $X \cup \{x, y, z\} \notin C$. But as $z \leq x$, it follows from $X \cup \{z\} \in C$ that $X \cup \{x, z\} \in C$, with which it follows from $z \leq y$ that $X \cup \{x, y, z\} \in C$, which was shown to be false. □

Lemma 18. *$x \sqcup y$ is the least upper bound of $\{x, y\}$ under $\leq$.*

Proof. By Lemma 17, $-x \sqcap -y \leq -x$ and $-x \sqcap -y \leq -y$, so with Lemma 15, $x \leq x \sqcup y$ and $y \leq x \sqcup y$; thus $x \sqcup y$ is an upper bound of $\{x, y\}$. Consider any upper bound z of $\{x, y\}$. By Lemma 15, $-z \leq -x$ and $-z \leq -y$, so with Lemma 17, $-z \leq -x \sqcap -y$, and thus $x \sqcup y \leq z$, appealing to Lemma 15 again. Hence $x \sqcup y$ is the least upper bound of $\{x, y\}$. □

Corollary 19. *$\mathcal{S}^*$ is a lattice.*

In the following, we will make use of Birkhoff's characterization of Boolean algebras as uniquely complemented and orthocomplemented lattices. Since complementation requires being bounded, we first show that $\mathcal{S}^*$ is bounded, using $\bot$ and $\top$ as witnesses. Next, we show that it is orthocomplemented. Since orthocomplements are complements, it then only remains to show that these complements are unique.

Lemma 20. *x is inconsistent iff $x = \bot$.*

Proof. Since $\bot \sim \{x, -x\}$, $\bot$ is inconsistent. If x is inconsistent, then $x \sim \bot$, and so $x = \bot$. □

Lemma 21. *$\bot$ and $\top$ are least and greatest elements, respectively.*

Proof. By Lemma 20, $\bot \leq x$ for all x. So for all y, $\bot \leq -y$, whence with Lemma 15, $y \leq \top$. □

Since we have already shown negation to be an order-reversing involution, all that is needed to show orthocomplementation is to show that negations, i.e., contradictories, are complements.

Lemma 22. *Contradictories are orthocomplements.*

Proof. If $x \times y$, then $\{x, y\} \notin C$, so $\{x \sqcap y\} \notin C$. Thus with Lemma 20, $x \sqcap y = \bot$. Since contradictories are unique and conjunction by construction symmetric, $-x \sqcap -y = y \sqcap x = x \sqcap y = \bot$. Hence $x \sqcup y = -(-x \sqcap -y) = -\bot = \top$. So contradictories are complements. Orthocomplementation follows with Lemma 15. □

Finally, we show that complements are unique. To do so, we first show that any complements are jointly inconsistent.

Lemma 23. *If x and y are complements, $\{x, y\}$ and $\{-x, -y\}$ are inconsistent.*

Proof. If x and y are complements, then (i) $x \sqcap y = \bot$ and (ii) $x \sqcup y = \top$. (ii) states that $-(-x \sqcap -y) = \top$, so with Lemma 15, (iii) $-x \sqcap -y = \bot$. By (i), $\{x, y\} \sim x \sqcap y = \bot$, and so $\{x, y\} \notin C$. Similarly, $\{-x, -y\} \notin C$ by (iii). □

Lemma 24. *$\mathcal{S}^*$ is uniquely complemented.*

Proof. Let y, z be complements of x. By Lemma 23:

($*$) $\{x, y\}$, $\{-x, -y\}$, $\{x, z\}$ and $\{-x, -z\}$ are all inconsistent.

Suppose, for reductio, that $y \neq z$. Then without loss of generality, there is a set X such that $X \cup \{y\}$ is consistent and $X \cup \{z\}$ is inconsistent. Then $X \cup \{y, z\}$ is inconsistent as well, so by definition of contradictories, $X \cup \{y, -z\}$ must be consistent. Then $\{y, -z\}$ is consistent as well, which means that one of $\{x, y, -z\}$ and $\{-x, y, -z\}$ is consistent, but this contradicts ($*$). □

Proposition 25. *If $\mathcal{S}$ is a simple consistency structure, then $\mathcal{S}^*$ is a Boolean algebra.*

Proof. Corollary 19 and Lemmas 22 and 24 establish that $\mathcal{S}^*$ is a uniquely complemented orthocomplemented lattice. As shown by Birkhoff (1940, p. 171, Theorem X.17), such a structure is a Boolean algebra. □

2.2 Surjectively

We have shown that simple consistency structures determine Boolean entailment orders. Thus, assuming that entailment can be characterized in terms of consistency as captured in Definition 2, we can conclude from principles (1–4) that propositions form a Boolean algebra under entailment.

This does not yet show that principles (1–4) are equivalent to the claim that propositions form a Boolean algebra. One reason for this is that we have not ruled out the possibility of consistency structures imposing constraints on propositions which are not imposed by Boolean algebras. For example, it may be that some Boolean algebras are not the entailment order of any simple consistency structure. We now show that this does not happen: Every Boolean agebra is the entailment order of some simple consistency structure. That is, the mapping $\cdot^*$ from simple consistency structures to Boolean algebras is surjective (onto).

To show that every Boolean algebra is determined by some simple consistency structure, we will find a way of mapping any Boolean algebra $\mathfrak{A}$ to a simple consistency structure whose entailment order is $\mathfrak{A}$. One natural idea is to let a set of propositions be consistent just in case it has a lower bound other than 0. It turns out that variations on this idea are possible. For instance, we can impose a cardinality constraint on consistent sets, counting any set above a certain infinite cardinality threshold as inconsistent. Or we could assess consistency of a set not by considering whether it has a lower bound other than 0 itself, but by considering whether all of its subsets below a certain infinite cardinality threshold have a lower bound other than 0. Indeed, these two variations can be combined. For later purposes, it will be useful to make use of these variations, so we introduce a mapping from Boolean algebras to consistency structures which is parametric to two cardinal parameters. To state the definition formally, let, for any infinite cardinal κ, $X \subseteq_\kappa Y$ just in case $X \subseteq Y$ and $|X| < \kappa$. In a bounded lattice, we call an element *non-zero* if it is distinct from 0.

Definition 26. *For any infinite cardinals κ, λ and Boolean algebra $\mathfrak{A} = \langle A, \leq \rangle$, let $\mathfrak{A}^\kappa_\lambda$ be the consistency structure $\langle A, C \rangle$ such that $X \in C$ just in case:*

(i) $X \subseteq_\kappa A$, and

(ii) every $Y \subseteq_\lambda X$ has a non-zero lower bound.

The definition of consistency structures requires the set of propositions to be non-empty. But $\mathfrak{A}^\kappa_\lambda$ is guaranteed to satisfy this constraint: Boolean algebras are complemented distributive lattices, and complemented lattices are non-empty as they are required to include least and greatest elements.

In this definition of $\mathfrak{A}^\kappa_\lambda$, κ and λ serve only as bounds to the cardinality of subsets of $|A|$. Thus, any differences between cardinals larger than $|A|$ are immaterial for such parameters. To underscore this fact, we will use '∞' instead of '$|A|^+$' (denoting the successor cardinal of the cardinality of A) or any larger cardinal in this context. Thus, e.g., $\mathfrak{A}^\infty_\lambda = \mathfrak{A}^{|A|^+}_\lambda = \mathfrak{A}^{|A|^{++}}_\lambda \ldots$.

We can now show that this mapping does the intended job: it maps any Boolean algebra $\mathfrak{A}$ to a simple consistency structure which itself is mapped back to $\mathfrak{A}$ by $\cdot^*$:

Lemma 27. *For any infinite cardinals κ, λ and Boolean algebra $\mathfrak{A}$:*

(i) $\mathfrak{A} = (\mathfrak{A}^\kappa_\lambda)^*$.

(ii) $\mathfrak{A}^\kappa_\lambda$ *is simple.*

Proof. Let $\mathfrak{A} = \langle A, \leq \rangle$ be a Boolean algebra, $\mathfrak{A}^\kappa_\lambda = \langle A, C \rangle$ and $(\mathfrak{A}^\kappa_\lambda)^* = \langle A, \leq' \rangle$. We will use $\leq, \wedge, -, \ldots$ for the relevant algebraic notions in $\mathfrak{A}$, and $\leq', \sqcap', \wedge', -', \ldots$ for the relevant notions in $\mathfrak{A}^\kappa_\lambda / (\mathfrak{A}^\kappa_\lambda)^*$. Recall that Lemmas 17 and 18 show that $\sqcap' = \wedge'$ and $\sqcup' = \vee'$.

Proof of claim (i): By construction of C, $x \leq' y$ iff the following condition holds:

($*$) For all $X \subseteq_\kappa A$:
every $Y \subseteq_\lambda X \cup \{x\}$, there is a $z \neq 0$ such that $z \leq w$ for all $w \in Y$
only if
for all $Y \subseteq_\lambda X \cup \{x, y\}$, there is a $z \neq 0$ such that $z \leq w$ for all $w \in Y$.

Assume first $x \leq y$. To establish ($*$), let $X \subseteq_\kappa A$ such that for all $Y \subseteq_\lambda X \cup \{x\}$, there is a $z \neq 0$ such that $z \leq w$ for all $w \in Y$. For any such Y and z, since $z \leq x$ and $x \leq y$, $z \leq y$ as well, whence $z \leq w$ for all $w \in Y \cup \{y\}$. So for all $Y \subseteq_\lambda X \cup \{x, y\}$, there is a $z \neq 0$ such that $z \leq w$ for all $w \in Y$, as required.

Assume for contradiction that $x \nleq y$ but $x \leq' y$. Since $x \nleq y$, $x \wedge -y \neq 0$. So for all $Y \subseteq_\lambda \{x, -y\}$, there is a $z \neq 0$, namely $x \wedge -y$, such that $z \leq w$ for all $w \in Y$. By ($*$), it follows from $x \leq' y$ that for all $Y \subseteq_\lambda \{x, y, -y\}$, there is a $z \neq 0$ such that $z \leq w$ for all $w \in Y$. In particular, there is then a $z \neq 0$ such that $z \leq w$ for all $w \in \{y, -y\}$, which is impossible. So $x \nleq y$ only if $x \nleq' y$.

Proof of claim (ii): We verify conditions (1–4):

(1) Immediate.

(2) We show that any x has $-x$ as a contradictory. Since $x \wedge -x = 0$, $\{x, -x\} \notin C$. Assume for contradiction that there is some consistent set X such that neither $X \cup \{x\}$ nor $X \cup \{-x\}$ is consistent. Since $X \subseteq_\kappa A$, there are then $Y_1, Y_2 \subseteq_\lambda X$ such that there is (i) no $z_1 \neq 0$ such that $z_1 \leq y$ for all $y \in Y_1 \cup \{x\}$ and (ii) no $z_2 \neq 0$ such that $z_2 \leq y$ for all $y \in Y_2 \cup \{-x\}$. Since X is consistent, there is a $z_0 \neq 0$ such that $z_0 \leq y$ for all $y \in Y_1 \cup Y_2$. Since $z_0 \neq 0$, $z_0 \wedge x \neq 0$ or $z_0 \wedge -x \neq 0$. In the first case, $z_0 \wedge x \leq y$ for all $y \in Y_1 \cup \{x\}$, which contradicts (i), and in the second case, $z_0 \wedge -x \leq y$ for all $y \in Y_2 \cup \{-x\}$, which contradicts (ii).

(3) We show that $\{x, y\} \sim \{x \wedge y\}$. Consider any X. If $|X| \geq \kappa$, then neither $\{x \wedge y\} \cup X$ nor $\{x, y\} \cup X$ is consistent. So assume that $|X| < \kappa$, and consider any $Y \subseteq_\lambda X$. It suffices to show that for any z, $z \leq y$ for all $Y \cup \{x \wedge y\}$ iff $z \leq y$ for all $Y \cup \{x, y\}$. This follows from the fact that $x \wedge y$ is the greatest lower bound of x and y in $\leq$.

(4) Assume $x \neq y$, so without loss of generality $x \not\leq y$. As shown in (i), this entails $x \not\leq' y$, so there is an X such that $X \cup \{x\} \in C$ and $X \cup \{x, y\} \notin C$. Thus $X \cup \{x\}$ witnesses $\{x\} \not\sim \{y\}$.

□

Corollary 28. $\cdot^*$ *maps simple consistency structures surjectively to Boolean algebras.*

2.3 But not Injectively

So far, we have seen that the mapping from simple consistency structures to entailment orders is a surjective mapping to Boolean algebras. This means that what principles (1–4) say about the order of entailment is precisely that it is a Boolean algebra. Does this mean that principles (1–4) just amount to the claim that propositions form a Boolean algebra?

It turns out that this is not so: there are distinct simple consistency structures which give rise to the same entailment order. That is, the mapping from simple consistency structures to Boolean algebras is not injective (one-to-one). This is where the parameters of the mapping from Boolean algebras to consistency structures come in useful: By choosing different parameters, different sets come out as consistent, but without affecting the entailment order.

Proposition 29. $\cdot^*$ *is not injective on simple consistency structures.*

Proof. Consider the powerset algebra $\mathfrak{P}(W)$ for any infinite set W. By Lemma 27, $\mathfrak{P}(W)^{\aleph_0}_{\aleph_0}$ and $\mathfrak{P}(W)^{\infty}_{\infty}$ are simple, and mapped to $\mathfrak{P}(W)$ by $\cdot^*$. Let $w \in W$; then $X = \{x \in \mathcal{P}(W) : w \in x\}$ is infinite. (In the terminology introduced in the next section, this is the principal ultrafilter generated by the atom $\{w\}$.) Since $\{w\}$ is a non-zero lower bound of X, X is consistent in $\mathfrak{P}(W)^{\infty}_{\infty}$. But X is inconsistent in $\mathfrak{P}(W)^{\aleph_0}_{\aleph_0}$ due to being infinite. Hence $\mathfrak{P}(W)^{\aleph_0}_{\aleph_0} \neq \mathfrak{P}(W)^{\infty}_{\infty}$. □

This shows that in some sense, consistency structures record more information than Boolean algebras: from the consistency facts, we can recover the entailment facts, but we cannot necessarily recover the consistency facts from the entailment facts. It is important to note that this asymmetry arises from an asymmetry in assumptions we have made (and Stalnaker makes) about the relationship between consistency and entailment: we have assumed that entailment can be characterized in terms of consistency (x entails y just in case x is equivalent to $\{x, y\}$), but we have not assumed any way of characterizing consistency in terms of entailment.

This, one might say, is easily fixed: Why don't we assume that a set is consistent just in case it has a non-zero lower bound in the entailment order? On this assumption, the way to recover consistency facts from entailment facts is by the mapping which takes any Boolean algebra $\mathfrak{A}$ to the consistency structure $\mathfrak{A}^{\infty}_{\infty}$. One may of course endorse this view, but it is not the only philosophically viable candidate. Consider, by way of example, a view on which propositions are a kind of idealized sentences – sentences up to logical equivalence. A natural model for this view is the Lindenbaum-Tarski algebra of classical propositional logic, which is obtained by taking sentences ordered by derivability and then quotienting this order under provable equivalence. (If the language is countable, this amounts to requiring propositions to form a countable atomless Boolean algebra, which pins down the entailment order up to isomorphism.) On this view, the natural notion of consistency is the deductive consistency of the corresponding set of sentences, and that – due to the finitary nature of deduction – means that consistency does not amount to having a non-zero lower bound, but to having no finite subset with a non-zero lower bound. On this particular conception of propositions, the consistency structure determined by the entailment order $\mathfrak{A}$ is $\mathfrak{A}^{\infty}_{\aleph_0}$.

Any characterization of consistency in terms of entailment is therefore going to pronounce on philosophically contentious question. This raises the question whether the characterization of entailment in terms of consistency which was assumed above is similarly contentious. It is of course possible to question this characterization, but at least on the assumption that consistency conforms to the constraints of simple consistency structures, it seems hard to deny: If x is consistent with $-y$, then x should not entail y. And if x does not entail y, then x should be consistent with $-y$. So it seems hard to deny that x entails y just in case x is inconsistent with $-y$, and this is equivalent to the definition of entailment given above.

Proposition 30. *In any simple consistency structure, $x \leq y$ just in case $\{x, -y\} \notin C$.*

Proof. If $x \nleq y$, then there is an X such that $X \cup \{x\} \in C$ but $X \cup \{x, y\} \notin C$. So $X \cup \{x, -y\} \in C$, whence $\{x, -y\} \in C$. If $\{x, -y\} \in C$, then as $\{x, y, -y\} \notin C$, $\{-y\}$ witnesses that $x \nleq y$. □

So, while consistency facts seem to determine entailment facts without any further substantial commitment (assuming the principles of simplicity), entailment facts do not seem to determine all consistency facts without substantial further assumptions.

3 Closure

The preceding discussion establishes the relationship between simple consistency structures and Boolean algebras. Stalnaker's own theory imposes further constraints on consistency. In the rest of the paper, we will explore such strengthenings of simplicity, alongside one weakening. The final section 7 sums up the resulting landscape of classes of structures in the form of a diagram (figure 1). Stalnaker's additional principles will be discussed in section 4. In this section, we prepare the ground by discussing two relatively weak further principles which are of independent interest, and which will be useful later on.

The first of these principles can be motivated by considering again the proof of Proposition 29. This proof brings out an odd situation allowed by simple consistency structures: $\{w\}$ is consistent in $\mathfrak{P}(W)^{\aleph_0}_{\aleph_0}$, and a lower bound of the set X of propositions it entails. But since X is infinite, it is inconsistent. Yet, if some propositions are all entailed by the same consistent proposition, then the set of these propositions should plausibly be consistent as well. This motivates the following additional principle.

Definition 31. *A consistency structure is* upward closed *if it satisfies:*

($\uparrow$) *A set is consistent if it has a consistent lower bound in the entailment order:*

If there is some x such that $\{x\} \in C$ and $x \leq y$ for all $y \in Y$, then $Y \in C$.

It turns out that the converse of upward closure is an interesting principle as well. According to this converse principle, if a set of propositions is consistent, then some consistent proposition entails all of its members. Without the *prima facie* stronger claim that every set of propositions can be conjoined to a single proposition (to which we return below), it is hard to see a philosophical motivation for this. But in mathematically investigating different classes of consistency structures, it is helpful to consider this converse alongside upward closure.

Definition 32. *A consistency structure is* downward closed *if it satisfies:*

($\downarrow$) *A set is consistent only if it has a consistent lower bound in the entailment order:*

If $Y \in C$, then there is some x such that $\{x\} \in C$ and $x \leq y$ for all $y \in Y$.

A consistency structure is closed *if it is upward closed and downward closed.*

While the philosophical motivations of (downward) closure may not be immediately transparent, the notion is mathematically very natural. Indeed, once closure is added to simplicity, consistency can be defined in terms of entailment, since in this theory, a set is consistent just in case it has a non-zero lower bound in the entailment order. Thus the mapping from closed simple consistency structures to Boolean algebras is injective, with its inverse given by $\cdot^{\infty}_{\infty}$. And $\mathfrak{A}^{\infty}_{\infty}$ is closed for any Boolean algebra $\mathfrak{A}$, so closed consistency structures and Boolean algebras are one-to-one correlated, as the next results show.

Lemma 33. *For any infinite cardinal κ and Boolean algebra $\mathfrak{A} = \langle A, \leq \rangle$:*

(i) $\mathfrak{A}^{\infty}_{\kappa}$ is upward closed.

(ii) $\mathfrak{A}^{\kappa}_{\kappa}$ is downward closed.

Proof. (i): Let x be a consistent lower bound of X. Trivially, $X \subseteq_{\infty} A$, and for any $Y \subseteq_{\kappa} X$, $x \leq y$ for all $y \in Y$. Hence $X \in C$.

(ii): Let $X \in C$. Then $X \subseteq_{\kappa} A$. So there is a $x \neq 0$ such that $x \leq y$ for all $y \in X$, as required. □

Lemma 34. *For any closed simple consistency structure $\mathcal{S}$, $(\mathcal{S}^*)^{\infty}_{\infty} = \mathcal{S}$.*

Proof. Let $\mathcal{S} = \langle A, C \rangle$ and $(\mathcal{S}^*)^{\infty}_{\infty} = \langle A, C' \rangle$. $X \in C'$ iff X has a consistent (i.e., non-zero) lower bound. That this is the case iff $X \in C$ is guaranteed by upward and downward closure. □

Proposition 35. *$\cdot^*$ bijectively maps closed simple consistency structures to Boolean algebras, with $\cdot^{\infty}_{\infty}$ its inverse.*

Proof. By Proposition 25 and Lemmas 27, 33 and 34. □

Since $\cdot^*$ is not injective on all simple consistency structures, it follows that simplicity does not entail closure – not all simple consistency structures are closed. In fact, upward closure and downward closure are logically independent relative to simplicity. To prove this, and for further applications below, it will be useful to recall some further standard order-theoretic notions.

A lattice (such as a Boolean algebra) is *complete* if every set of elements X has a greatest lower bound $\bigwedge X$, which is the case just in case every set of elements has a least upper bound $\bigvee X$. An *atom* of a bounded lattice is a non-zero element which is not greater than any other non-zero element. Correspondingly, a *co-atom* is a non-top ($\neq 1$) element which is not below any other non-top element. A partial order is *atomless* if it has no atoms, and *atomic* if for every non-zero element x, there is an atom $a \leq x$. Powerset algebras are complete and atomic; in fact, a Boolean algebra is complete and atomic just in case it is isomorphic to a powerset algebra. In powerset algebras, greatest lower bounds, least upper bounds and complements are intersection, union and relative complement, respectively.

A set X of elements of a partial order is an *upset* if $x \in X$ and $x \leq y$ only if $y \in X$. The *upward closure* of an element x is the upset $\{y : x \leq y\}$. A *filter* is a non-empty upset X such that $x, y \in X$ only if there is a $z \in X$ such that $z \leq x$ and $z \leq y$. A filter X is *proper* if there is some element $x \notin X$. An *ultrafilter* is a proper filter X of which no proper filter Y is a proper superset. Using the axiom of choice, it can be shown that every proper filter can be extended to an ultrafilter. A filter is *principal* if it is the upward closure of an element. In lattices, a filter can also be characterized as a non-empty upset X such that $x, y \in X$ only if $x \wedge y \in X$. In Boolean algebras, an ultrafilter can also be characterized as a filter X such that $-x \in X$ iff $x \notin X$, and a principal ultrafilter as a filter which is the upward closure of an atom.

Proposition 36. *All four logical combinations of upward and downward closure are satisfiable in simple consistency structures. Indeed, let W be an infinite set, and $\mathfrak{A}$ be $\mathfrak{P}(W)$, the powerset algebra on W; then:*

(i) $\mathfrak{A}^{\infty}_{\infty}$ *is both upward and downward closed.*

(ii) $\mathfrak{A}^{\infty}_{\aleph_0}$ *is upward closed but not downward closed.*

(iii) $\mathfrak{A}^{\aleph_0}_{\aleph_0}$ *is downward closed, but not upward closed.*

(iv) $\mathfrak{A}^{\aleph_1}_{\aleph_0}$ *is neither upward nor downward closed.*

Proof. (i) $\mathfrak{A}^{\infty}_{\infty}$ is both upward and downward closed:

This follows by Lemma 33.

(ii) $\mathfrak{A}^{\infty}_{\aleph_0}$ is upward closed but not downward closed:

The former follows by Lemma 33 (i). For the latter, consider the set of co-atoms: since every finite subset has a non-zero greatest lower bound (i.e., non-empty intersection), this is consistent, but since only the empty set is a subset of all of them, no consistent element entails all its members.

(iii) $\mathfrak{A}^{\aleph_0}_{\aleph_0}$ is downward closed, but not upward closed:

The former follows by Lemma 33 (ii). For the latter, consider any principal ultrafilter: all of its element are entailed by the atom in the ultrafilter, but it is inconsistent as it is infinite.

(iv) $\mathfrak{A}^{\aleph_1}_{\aleph_0}$ is neither upward nor downward closed:

As in (iii), the failure of upward closure is witnessed by any principal ultrafilter U, as $|U| = 2^{|W|} \geq \aleph_1$. For the failure of downward closure, consider a countably infinite set $V \subseteq W$, and let $X = \{V \backslash \{v\} : v \in V\}$. X is countable and every finite subset of X has a non-zero greatest lower bound, hence X is consistent. But since $\bigcap X = \emptyset$, no consistent element entails every member of X.

□

4 Completeness and Atomicity

We now return to the principles of Stalnaker's theory omitted in the definition of simplicity: the strengthening of the conjunctive principle (3) to arbitrary sets, and the extendibility of consistent sets to maximal consistent sets. We will see that each one of these two principles entails one of the directions of closure, and once the other direction is added, one principle enforces the completeness of the entailment order, and the other the atomicity of the order. This section explores these two principles.

4.1 Completeness

Principle (3) of simplicity tells us that any two propositions x, y may be conjoined to a proposition $x \sqcap y$ which is equivalent to $\{x, y\}$. A simple induction shows that this extends to arbitrary finite sets. The argument uses the fact that in a lattice, every finite set X has a greatest lower bound $\bigwedge X$.

Lemma 37. *In any simple consistency structure, if X is finite, then $\{\bigwedge X\} \sim X$.*

Proof. By induction on the cardinality of X.

$n = 0$: We show that $\bigwedge \emptyset \sim \emptyset$, i.e., $1 = \top \sim \emptyset$. So consider any set X. If $X \cup \emptyset \in C$, then $X \cup \{\bot\} \in C$ or $X \cup \{\top\} \in C$. The former is not the case, so the latter is. The converse entailment is immediate.

$n + 1$: Assuming $|X| = n$, we show that $\bigwedge(X \cup \{x\}) \sim (X \cup \{x\})$. First, $\bigwedge(X \cup \{x\}) = \bigwedge X \wedge x$, which by Lemma 17 is $\bigwedge X \sqcap x$, and $\bigwedge X \sqcap x \sim \{\bigwedge X, x\}$; therefore, $\bigwedge(X \cup \{x\}) \sim \{\bigwedge X, x\}$. Second, by induction hypothesis, $\bigwedge X \sim X$, so $\{\bigwedge X, x\} \sim X \cup \{x\}$. Thus $\bigwedge(X \cup \{x\}) \sim X \cup \{x\}$. □

One of Stalnaker's principles extends this feature from finite sets to arbitrary sets of propositions: every set of propositions may be conjoined. We call this completeness.

Definition 38. *A consistency structure is* complete *if it satisfies:*

(C) Every set of propositions is equivalent to some proposition:

For all $X \subseteq A$, there is an $x \in A$ such that $\{x\} \sim X$.

One way of motivating this is as follows: for any set of propositions X, there is the proposition that they are all true, and this proposition ought to be equivalent to X. But not every conception of propositions supports this. Recall the conception of propositions as idealized sentences. Since sentences (on a suitable conception) are finite, we may not be able to conjoin infinite sets of sentences in general.

As in the binary case, we can use the coarse-grained individuation of propositions captured by (4) to show that the propositions witnessing (C) are uniquely determined, which allows us to formally introduce the notion of a conjunction of an arbitrary set of propositions.

Lemma 39. *In any complete simple consistency structure, every set of propositions is equivalent to a unique proposition.*

Proof. By (C), every set of propositions is equivalent to some proposition. If x and y are both equivalent to X, then as $\sim$ is an equivalence relation, $x \sim y$, whence $x = y$ by (4). □

This shows that the following is well-defined:

Definition 40. *In any complete simple consistency structure:*

$\bigsqcap X$*, the* conjunction *of* X*, is the unique* x *such that* $\{x\} \sim X$*.*

$\bigsqcup X$*, the* disjunction *of* X*, is* $-\bigsqcap\{-x : x \in X\}$*.*

Corresponding to the binary case (see Lemma 17), one might expect $\bigsqcap X$ always to be the greatest lower bound of X, so that complete consistency structures determine complete Boolean algebras. But while $\bigsqcap X$ is always a lower bound, it is not guaranteed to be the greatest one. In fact, $\bigsqcap X$ is guaranteed to be the greatest lower bound just in case the consistency structure is closed. And it turns out that closure does not follow from completeness: while completeness entails downward closure, it does not entail upward closure.

In order to investigate these claims, we need a way of constructing complete consistency structures from Boolean algebras. As one might suspect, any complete Boolean algebra can be turned into a complete consistency structure: for any X, the greatest lower bound $\bigwedge X$ will serve to ensure that (C) is satisfied. But we don't always need full completeness: if we map a Boolean algebra $\mathfrak{A}$ to $\mathfrak{A}^\kappa_\kappa$, then all sets of size κ and larger will be counted as inconsistent, so these will trivially have a conjunction, namely $\bot$. So to ensure that $\mathfrak{A}^\kappa_\kappa$ is a complete consistency structure, it suffices to ensure that every set of size under κ has a greatest lower bound. We call this κ-completeness, a notion which is naturally extended to cover filters on such Boolean algebras as well.

Definition 41. *Let* κ *be an infinite cardinal. A Boolean algebra* $\mathfrak{A} = \langle A, \leq \rangle$ *is* κ-complete *if every* $X \subseteq_\kappa A$ *has a greatest lower bound* $\bigwedge X$*. In this case, a filter* F *of* $\mathfrak{A}$ *is* κ-complete *if for every* $X \subseteq_\kappa F$*,* $\bigwedge X \in F$*.*

Note that every Boolean algebra and filter is $\aleph_0$-complete, as this only requires finite sets to have greatest lower bounds.

Lemma 42. *For any infinite cardinal* κ *and* κ*-complete Boolean algebra* $\mathfrak{A}$*,* $\mathfrak{A}^\kappa_\kappa$ *is a complete consistency structure.*

Proof. We distinguish two cases. If $|X| \geq \kappa$, then $X \notin C$, so $X \sim \bot$. If $|X| < \kappa$, then we show that $X \sim \bigwedge X$. Consider any Y. If $|Y| \geq \kappa$, then neither $\{\bigwedge X\} \cup Y$ nor $X \cup Y$ is consistent. If $|Y| < \kappa$, then $\bigwedge(X \cup Y) = \bigwedge(\{\bigwedge X\} \cup Y)$, so $X \cup Y \in C$ just in case $\{\bigwedge X\} \cup Y \in C$. □

In cases where $\kappa < \infty$, the witnesses of completeness in $\mathfrak{A}^\kappa_\kappa$ may be unexpected: for example, if $\mathfrak{A}$ is an infinite atomic algebra and U is a principal ultrafilter generated by an atom a, $\bigsqcap U$ is not $\bigwedge U = a$, but $\bot$.

We can now show that completeness does not entail upward closure, as $\mathfrak{A}^\kappa_\kappa$ need not be upward closed. To show that completeness does entail downward closure, we note that $\bigsqcap X$ is a lower bound of X:

Lemma 43. *In any complete simple consistency structure, $\bigsqcap X$ is a lower bound of X under $\leq$.*

Proof. Consider any $x \in X$ and Y such that $\{\bigsqcap X\} \cup Y \in C$. Since $\bigsqcap X \sim X$, $\{\bigsqcap X\} \cup X \cup Y \in C$. Therefore $\{\bigsqcap X, x\} \cup Y \in C$, and so $\bigsqcap X \leq x$. □

Lemma 44. *Among simple consistency structures, completeness entails downward closure, but not upward closure.*

Proof. For downward closure, let X be a consistent set in a complete consistency structure. Then by Lemma 43, $\bigsqcap X$ is a consistent lower bound of X, as required.

For upward closure, by Proposition 36 (iii), there is a Boolean algebra $\mathfrak{A}$ such that $\mathfrak{A}^{\aleph_0}_{\aleph_0}$ is not upward closed. By Lemma 42, $\mathfrak{A}^{\aleph_0}_{\aleph_0}$ is complete. □

We now return to the issue of $\bigsqcap X$ being the greatest lower bound of X: this is guaranteed only in complete consistency structures which are closed, which, as we just saw, is a non-trivial restriction.

Proposition 45. *A complete simple consistency structure is closed iff for all X, $\bigsqcap X$ is the greatest lower bound of X.*

Proof. Assume first that for all X, $\bigsqcap X$ is the greatest lower bound of X. By Lemma 44, it suffices to establish upward closure. So consider any set X with a consistent lower bound x. Then $x \leq \bigwedge X = \bigsqcap X$, which means that $\bigsqcap X$ is consistent. As $\bigsqcap X \sim X$, X is consistent as well.

For the converse, assume for contradiction that there is a complete closed simple consistency structure in which $\bigsqcap X$ is no greatest lower bound of some set X. By Lemma 43, $\bigsqcap X$ is a lower bound, so there is some lower bound y of X such that $y \not\leq \bigsqcap X$. Let $z = y \wedge -\bigsqcap X$. Then (i) $z \leq y$, (ii) $\bigsqcap X \leq -z$ (with Lemma 15, as $z \leq -\bigsqcap X$), and (iii) $z \neq \bot$. By (ii), for all Y, $Y \cup \{\bigsqcap X\} \in C$ only if $Y \cup \{\bigsqcap X, -z\} \in C$. Since $\{z, \bigsqcap X, -z\} \notin C$, it follows that $\{z, \bigsqcap X\} \notin C$, and so $\{z\} \cup X \notin C$. But by (i) and the fact that y is a lower bound of X, z is a lower bound of $\{z\} \cup X$. By (iii), z is consistent, so with closure, $\{z\} \cup X \in C$, which was shown to be false. □

With these results at hand, it is easy to show how complete consistency structures relate to Boolean algebras: First, even if the requirement of completeness is added to simplicity, we still get all Boolean algebras, and there are still cases in which distinct consistency structures give rise to the same Boolean algebra. This changes once (upward) closure is added: then, failures of injectivity disappear, but moreover, entailment orders are guaranteed to be complete.

Proposition 46. *$\cdot^*$ maps complete simple consistency structures surjectively, but not injectively, to Boolean algebras.*

Proof. For surjectivity, consider any Boolean algebra $\mathfrak{A}$. $\mathfrak{A}^{\aleph_0}_{\aleph_0}$ is complete by Lemma 42, and mapped to $\mathfrak{A}$ by $\cdot^*$ with Lemma 27 (i).

For the failure of injectivity, recall from Proposition 36 (i & iii) that for some complete Boolean algebra $\mathfrak{A}$, $\mathfrak{A}^{\infty}_{\infty}$ and $\mathfrak{A}^{\aleph_0}_{\aleph_0}$ are distinct. By Lemma 27 (i), both are mapped to $\mathfrak{A}$ by $\cdot^*$, and by Lemma 42, both are complete. □

Proposition 47. *$\cdot^*$ bijectively maps complete closed simple consistency structures to complete Boolean algebras.*

Proof. For any complete closed simple consistency structure $\mathcal{S}$, $\mathcal{S}^*$ is a complete Boolean algebra by Proposition 25 and Lemma 45. For any complete Boolean alegbra $\mathfrak{A}$, $\mathfrak{A}^{\infty}_{\infty}$ is complete by Lemma 42 and closed by Lemma 33. The claim follows with Proposition 35. □

Since $\cdot^*$ surjectively maps closed simple consistency structures to Boolean algebras, it immediately follows that closure does not entail completeness among simple consistency structures.

4.2 Atomicity

We turn to the second way in which Stalnaker's theory goes beyond simplicity: According to it, every consistent set of propositions can be extended to a maximal consistent one. Here, maximality can be defined in the straightforward way, according to which a consistent set is maximal if no proper superset is consistent. Given (1), this may be formulated as follows.

Definition 48. *Let $\langle A, C\rangle$ be a consistency structure. For any $X \subseteq A$, define:*

X *is an* MCS *(a* maximal consistent set *of propositions) iff*

$X \in C$ *and for all* $x \in A$, $X \cup \{x\} \in C$ *only if* $x \in X$.

Definition 49. *A consistency structure is* atomic *if it satisfies:*

(A) Every consistent set is included in a maximal consistent set:

For all $Y \subseteq A$, $Y \in C$ only if there is an MCS X such that $Y \subseteq X$.

The constraints of atomicity and completeness on consistency structures display a certain symmetry, with upward and downward closure playing corresponding roles. First, as in the case of complete simple consistency structures, atomic simple consistency structures give rise to all Boolean algebras, and distinct atomic simple consistency structures may determine the same Boolean algebra. The next few results establish this. Again, we can make use of the parametrized mapping from Boolean algebras to consistency structures by choosing parameters suitably (depending on the features of the algebra). It helps to start by showing certain tight connections between MCSs and ultrafilters.

Lemma 50. *In any simple consistency structure, every MCS is an ultrafilter.*

Proof. Let X be an MCS. Since X is consistent, $\bot \notin X$. To show that X is an ultrafilter, it thus suffices to show that for all x, y, (i) $x \sqcap y \in X$ if $x, y \in X$, and (ii) X contains one of x and $-x$. (i): If $x, y \in X$, then $X \sim X \cup \{x \sqcap y\}$, so $x \sqcap y \in X$. (ii): As one of $X \cup \{x\}$ and $X \cup \{-x\}$ is consistent, one of x and $-x$ is a member of X. □

It follows that for every infinite Boolean algebra $\mathfrak{A}$ and $\kappa < \infty$, there is no MCS in $\mathfrak{A}^{\kappa}_{\lambda}$, since every ultrafilter of an infinite Boolean algebra has the size of the algebra. In cases where $\kappa = \infty$ and $\mathfrak{A}$ is λ-complete, we can characterize the MCSs of $\mathfrak{A}^{\infty}_{\lambda}$ exhaustively as follows.

Lemma 51. *For any infinite cardinal λ and λ-complete Boolean algebra $\mathfrak{A}$, a set is an MCS in $\mathfrak{A}^{\infty}_{\lambda}$ iff it is a λ-complete ultrafilter in $\mathfrak{A}$.*

Proof. If X is an MCS in $\mathfrak{A}^{\infty}_{\lambda}$, then by Lemma 50, it is an ultrafilter. Consider any $Y \subseteq_{\lambda} X$, and assume for contradiction that $\bigwedge Y \notin X$. Then as X is an MCS, $-\bigwedge Y \in X$. Since X is consistent, $Y \cup \{-\bigwedge Y\}$ must have a non-zero lower bound, which is false. Thus X is λ-complete.

If X is a λ-complete ultrafilter, then for every $Y \subseteq_{\lambda} X$, $\bigwedge Y$ is non-zero as it is contained in X; so X is consistent. Clearly, X cannot be consistently extended, so X is an MCS. □

The special case of $\lambda = \aleph_0$ states that the MCSs of $\mathfrak{A}^{\infty}_{\aleph_0}$ are the ultrafilters of $\mathfrak{A}$. The special case of $\lambda = \infty$ states that if $\mathfrak{A}$ is complete, then the MCSs of $\mathfrak{A}^{\infty}_{\infty}$ are the principal ultrafilters of $\mathfrak{A}$.

Lemma 52. *For any Boolean algebra $\mathfrak{A}$, $\mathfrak{A}^{\infty}_{\aleph_0}$ is atomic.*

Proof. Consider any consistent set X. Let Y be the smallest filter including X. Since every finite subset of X has a non-zero lower bound, Y is consistent as well, and so a proper filter. Since Y is a proper filter, there is an ultrafilter $U \supseteq Y$, which by Lemma 51 is an MCS. □

Lemma 53. *For any atomic Boolean algebra $\mathfrak{A}$, $\mathfrak{A}^{\infty}_{\infty}$ is atomic.*

Proof. Consider any consistent set X. Then X has a non-zero lower bound, and so has an atom a as a lower bound. Let U be the principal ultrafilter generated by a; this is an MCS including X. □

Proposition 54. *$\cdot^*$ maps atomic simple consistency structures surjectively, but not injectively, to Boolean algebras.*

Proof. For surjectivity, consider any Boolean algebra $\mathfrak{A}$. Lemma 52 states that $\mathfrak{A}^{\infty}_{\aleph_0}$ is atomic, and by Lemma 27 (i), $(\mathfrak{A}^{\infty}_{\aleph_0})^* = \mathfrak{A}$.

For the failure of injectivity, recall from Proposition 36 (i & ii) that there is a (complete) atomic Boolean algebra $\mathfrak{A}$ such that $\mathfrak{A}^{\infty}_{\aleph_0}$ and $\mathfrak{A}^{\infty}_{\infty}$ are distinct. By Lemmas 52 and 53, both are atomic consistency structures. And by Lemma 27 (i), $\cdot^*$ maps both to $\mathfrak{A}$. □

As in the case of completeness, things change when we consider only atomic simple consistency structures which are closed. The failures of injectivity disappear, while only atomic Boolean algebras arise as entailment orders.

Proposition 55. *$\cdot^*$ bijectively maps atomic closed simple consistency structures to atomic Boolean algebras.*

Proof. We first show that if $\mathcal{S}$ is an atomic closed simple consistency structure, then $\mathcal{S}^*$ is an atomic Boolean algebra. Consider any non-zero element x of $\mathcal{S}^*$. Let X be an MCS containing x, which by Lemma 50 is an ultrafilter. By closure, X has a non-zero lower bound, which must be an atom.

Now, let $\mathfrak{A}$ be an atomic Boolean algebra. Then $\mathfrak{A}^{\infty}_{\infty}$ is atomic by Lemma 53 and closed by Lemma 33. The claim follows with Propositions 35. □

With these results, it follows that closure and atomicity are independent principles. In fact, as indicated, the relationship between closure and atomicity mirrors that of closure and completeness: atomicity entails upward closure but not downward closure. We could prove this directly, but it turns out that there is an interesting variant of atomicity, stated in (Stalnaker, 2018), which can be shown to be in strength properly between upward closure and atomicity. According to this weak version of atomicity, every *single* consistent proposition can be extended to a maximal consistent set.

Definition 56. *A consistency structure is* weakly atomic *if it satisfies:*

(wA) Every consistent proposition is contained in a maximal consistent set:
For all $x \in A$, if $\{x\} \in C$ then there is an MCS X such that $x \in X$.

We first establish the entailment relations between atomicity, weak atomicity and upward closure. That atomicity entails weak atomicity is immediate. That weak atomicity entails upward closure can be shown as follows.

Lemma 57. *Among simple consistency structures, weak atomicity entails upward closure.*

Proof. Consider any x and Y such that x is a consistent lower bound of Y. By weak atomicity, there is an MCS X containing x. Assume for contradiction that Y is inconsistent; then $Y \nsubseteq X$, so there is a $y \in Y$ such that $y \notin X$. Since X is an MCS, it follows with Lemma 50 that X is an ultrafilter. Since $y \notin X$, $-y \in X$, and since $x \leq y$, $y \in X$. This contradicts the consistency of X. □

We turn to showing that these entailments are proper. First, that (upward) closure does not entail weak atomicity follows from the following lemma.

Lemma 58. *Among simple consistency structures, weak atomicity and downward closure entail atomicity.*

Proof. Consider any consistent set Y. By downward closure, Y has a non-zero lower bound x. Given weak atomicity, $\{x\}$ can be extended to an MCS X, which must include Y: if $y \notin X$, then $-y \in X$, whence $x \nleq y$, from which we can conclude that $y \notin Y$. □

Proposition 59. *Among simple consistency structures, upward closure does not entail weak atomicity.*

Proof. If upward closure were to entail weak atomicity, then by Lemma 58, being closed would entail being atomic, contradicting Propositions 35 and 55. □

It remains to show that weak atomicity does not entail atomicity.

Proposition 60. *Among simple consistency structures, weak atomicity does not entail atomicity.*

Proof. Let κ be any uncountable but not measurable cardinal, such as $\aleph_1$. Let $\mathfrak{A}$ be $\mathfrak{P}(\kappa)$, the powerset algebra on κ. We show that $\mathfrak{A}^{\infty}_{\kappa}$ is weakly atomic but not atomic. For weak atomicity, note that every consistent element is above an atom, and so a member of a principal ultrafilter, which is an MCS by Lemma 51. For the failure of atomicity, let X be the set of co-atoms. This is consistent, since there are κ co-atoms, whence every $Y \subseteq_\kappa X$ has a non-empty intersection. But X cannot be expanded to an MCS, since by Lemma 51, this would have to be a κ-complete ultrafilter. And there are no such ultrafilters, since κ is not measurable; see Jech (2002, p. 127). □

We have shown that among closed simple consistency structures, the complete ones correspond bijectively to complete Boolean algebras, and the atomic ones correspond bijectively to atomic Boolean algebras. Hence even among closed consistency structures, completeness and atomicity are independent: neither principle entails the other.

Finally, we arrive at the full theory of Stalnaker (2012) (minus the notion and principle governing truths), which combines simplicity with completeness and atomicity. Since completeness and atomicity each entail one direction of closure, this theory implies closure, and so all of the principles considered so far. With the results established above, it follows that the consistency structures satisfying this theory correspond bijectively to complete atomic Boolean algebras.

Proposition 61. $\cdot^*$ *bijectively maps complete atomic simple consistency structures to complete atomic Boolean algebras.*

Proof. Let $\mathfrak{A}$ be a complete atomic Boolean algebra. Then $\mathfrak{A}^\infty_\infty$ is complete by Lemma 42 and atomic by Lemma 53. The claim follows with Propositions 35, 47 and 55. □

Stalnaker's full theory of consistency is thus equivalent to the claim that propositions form a complete atomic Boolean algebra. We can think of complete atomic simple consistency structures as a variant presentation of complete atomic Boolean algebras.

5 Compactness and Finitarity

Recall that proposition 35 shows that closed simple consistency structures correspond bijectively to Boolean algebras, with $\cdot^\infty_\infty$ the inverse correspondence. Thus, on the assumption that to be consistent is to have a non-zero lower bound in the entailment order, claiming that propositions form a closed simple consistency structure amounts just to claiming that propositions form a Boolean algebra. As we have just seen, this extends to the addition of completeness and atomicity principles on both sides. But it is important to keep in mind that this assumption about the relationship between consistency and entailment is not trivial.

In order to illustrate this, it is helpful to show that there are natural constraints other than closure which lead to classes of consistency structures which stand in a bijective correspondence to Boolean algebras. We consider two such classes here: one which requires inconsistent sets to have finite inconsistent subsets, and one which requires consistent sets to be finite. The constraints could naturally be generalized, by allowing the relevant cardinality bound to be given by any infinite cardinal, and the two constraints so generalized could be combined. But such generality is not needed here.

5.1 Compactness

The first principle to be investigated can be motivated by the conception of propositions as idealized sentences, and consistency as derivability, mentioned above. Since proofs are finite, any inconsistent set should have an inconsistent finite subset, or conversely, a set should be consistent if all of its finite subsets are consistent. Call this compactness.

Definition 62. *A consistency structure is* compact *if it satisfies:*

(Ct) Every inconsistent set has a finite inconsistent subset:

If $X \in C$ for all $X \subseteq_{\aleph_0} Y$, then $Y \in C$.

Assuming (1), a set with a finite inconsistent subset is inconsistent itself, so compactness entails that a set is consistent just in case all of its finite subsets are consistent. And it follows from simplicity that a finite set X is consistent just in case $\bigwedge X$ is consistent, which is the case iff $\bigwedge X$ is non-zero. So given compactness, entailment facts determine consistency facts: a set is consistent just in case every one of its finite subsets has a non-zero (greatest) lower bound. This definition is captured by $\cdot^{\infty}_{\aleph_0}$. Indeed, using the fact that $\mathfrak{A}^{\infty}_{\aleph_0}$ is compact for every Boolean algebra $\mathfrak{A}$, we can show that compact simple consistency structures and Boolean algebras stand in bijective correspondence.

Lemma 63. *For any Boolean algebra $\mathfrak{A}$, $\mathfrak{A}^{\infty}_{\aleph_0}$ is compact.*

Proof. Consider any inconsistent set X. By construction, there is a finite subset with greatest lower bound $\bot$; this is itself inconsistent. □

Proposition 64. *$\cdot^*$ bijectively maps compact simple consistency structures to Boolean algebras, with $\cdot^{\infty}_{\aleph_0}$ its inverse.*

Proof. For any Boolean algebra $\mathfrak{A}$, $(\mathfrak{A}^{\infty}_{\aleph_0})^* = \mathfrak{A}$ by Lemma 27 (i). Since $\mathfrak{A}^{\infty}_{\aleph_0}$ is compact, surjectivity follows. For injectivity, it suffices to show that $\cdot^{\infty}_{\aleph_0}$ is the inverse of $\cdot^*$ on compact simple consistency structures. So let $\mathcal{S}$ be a compact simple consistency structure. Consider any set X. X is consistent iff every $Y \subseteq_{\aleph_0} X$ is consistent, which by Lemma 37 is the case iff $\bigwedge Y \neq \bot$ for all $Y \subseteq_{\aleph_0} X$, and this is equivalent to X being consistent in $(\mathcal{S}^*)^{\infty}_{\aleph_0}$. Thus $(\mathcal{S}^*)^{\infty}_{\aleph_0} = \mathcal{S}$. □

In terms of the principles considered above, compactness is a strengthening of atomicity. Essentially, this is because compactness allows us to prove atomicity along the lines of the standard proof of Lindenbaum's Lemma: we well-order the propositions, and extend any consistent set to an MCS by adding, for each proposition in the order, either it or its negation, depending on which one can be added consistently. At limit stages, we take unions, and it is there that compactness guarantees that we don't move from an infinite sequence of consistent sets to a single inconsistent set.

Lemma 65. *Among simple consistency structures, compactness entails atomicity.*

Proof. Assume compactness and simplicity, and let $\langle x_\beta : \beta < \alpha \rangle$ be a well-order of all elements. Consider any consistent set X. Define:

$X_0 = X$

$$X_{\beta+1} = \begin{cases} X_\beta \cup \{x_\beta\} & \text{if } X_\beta \cup \{x_\beta\} \text{ is consistent} \\ X_\beta \cup \{-x_\beta\} & \text{otherwise} \end{cases}$$

$$X_\lambda = \bigcup_{\beta<\lambda} X_\beta$$

We argue inductively that X_α is consistent: X_0 is consistent by assumption. If X_β is consistent, then $X_\beta \cup \{x_\beta\}$ or $X_\beta \cup \{-x_\beta\}$ is consistent, hence $X_{\beta+1}$ is consistent. If X_β is consistent for all $\beta < \lambda$, then X_λ has no finite inconsistent subset, whence X_λ itself is consistent by compactness.

By construction, if $x \notin X_\alpha$, then $-x \in X_\alpha$, so $X_\alpha \cup \{x\}$ is inconsistent. So X_α is the desired MCS including X. □

Since $\cdot^*$ is not injective on atomic simple consistency structures, it is clear that compactness is stronger than atomicity. But in terms of the principles considered above, this is all it entails: it does not entail any principle not entailed by atomicity, not even the weak constraint of downward closure. In fact, adding downward closure to compactness forces a consistency structure to be finite (i.e., to be based on a finite set A, which we label (F)).

Lemma 66. *A compact downward closed simple consistency structure is finite.*

Proof. Assume for contradiction that there is an infinite consistency structure $\mathcal{S}$ which is both compact and downward closed. Since $\mathcal{S}$ is infinite, there is a non-principal ultrafilter U of $\mathcal{S}^*$. By compactness, U is consistent, but by downward closure, U is inconsistent. □

By Proposition 36, there are infinite downward closed simple consistency structures, so it follows that among simple consistency structures, downward closure does not entail compactness.

5.2 Finitarity

For symmetry's sake, let us consider a principle which strengthens completeness corresponding to how compactness strengthens atomicity. This is the principle which requires consistent sets to be finite, which, to be clear, seems lacking in philosophical motivation.

Definition 67. *A consistency structure is* finitary *if it satisfies:*

(Fy) Every consistent set is finite:
If $X \in C$ then $|X| < \aleph_0$.

Note that a finitary (Fy) structure need not be finite (F). As in the case of compactness, finitarity ensures that consistency can be defined in terms of entailment: a set is consistent just in case it is finite and has a non-zero (greatest) lower bound in the entailment order. This definition is codified in the mapping $\cdot^{\aleph_0}_{\aleph_0}$. Thus, finitary consistency structures also correspond bijectively to Boolean algebras.

Proposition 68. $\cdot^*$ *bijectively maps finitary simple consistency structures to Boolean algebras, with* $\cdot^{\aleph_0}_{\aleph_0}$ *its inverse.*

Proof. For any Boolean algebra $\mathfrak{A}$, $(\mathfrak{A}^{\aleph_0}_{\aleph_0})^* = \mathfrak{A}$ by Lemma 27 (i). Since $\mathfrak{A}^{\aleph_0}_{\aleph_0}$ is finitary, surjectivity follows. It suffices to show that $\cdot^{\aleph_0}_{\aleph_0}$ is the inverse of $\cdot^*$ on finitary simple consistency structures. So let $\mathcal{S}$ be a finitary simple consistency structure. Then X is consistent only if X is finite. Furthermore, by Lemma 37, a finite set X is consistent iff $\bigwedge X$ is non-zero. Thus, X is consistent iff (i) X is finite and (ii) $\bigwedge X$ is non-zero. And (i) and (ii) are the case iff X is consistent in $(\mathcal{S}^*)^{\aleph_0}_{\aleph_0}$. Thus $(\mathcal{S}^*)^{\aleph_0}_{\aleph_0} = \mathcal{S}$. □

Corresponding to the fact that among simple consistency structures, compactness entails atomicity, finitarity entails completeness.

Lemma 69. *Among simple consistency structures, finitarity entails completeness.*

Proof. Consider any consistent set X in a finitary consistency structure. Since X is finite, by Lemma 37, $\bigwedge X \sim X$. □

Since $\cdot^*$ is not injective on complete consistency structures, it is clear that finitarity is stronger than completeness. And finitarity relates to upward closure as compactness relates to downward closure.

Lemma 70. *A finitary upward closed simple consistency structure is finite.*

Proof. Assume for contradiction that there is an infinite consistency structure $\mathcal{S}$ which is both finitary and upward closed. We first show that there is an element $x \neq \bot$ such that its upset $\{y : x \leq y\}$ is infinite by distinguishing two cases: If $\mathcal{S}^*$ is atomless, any $x \notin \{\top, \bot\}$ has an infinite upset. Otherwise, there is an atom x, which must have an infinite upset. By upward closure, X is consistent, but by finitarity, X is inconsistent. □

By Proposition 36, there are infinite upward closed simple consistency structures, so it follows that among simple consistency structures, upward closure does not entail finitarity.

With the results established here, we can immediately infer that a simple consistency structure is finitary and compact just in case it is finite. Finite simple consistency structures of course correspond bijectively to finite Boolean algebras. This means that there is a unique (up to isomorphism) consistency structure of size 2^n, for each natural number n, and that these are all the finite consistency structures. The case of $n = 1$ is of particular interest: this is the Fregean view according to which there are two propositions: the True and the False. Note that the case of $n = 0$ is included as well. This is the view on which there is only one proposition x. On this view, no set is consistent, not even the empty set $\emptyset$: if $\emptyset$ was consistent, then one of $\{x\}$ and $\{-x\}$ would have to be consistent as well. But since $x = -x$, it would then follow that $\{x, -x\}$ was consistent as well, contradicting the definition of contradictories. This is the only case in which $\emptyset$ is inconsistent: if $\emptyset$ is inconsistent, then every set is inconsistent, so all propositions are identical.

6 Pre-Simplicity

Finally, we return to the controversial fourth principle of simplicity, according to which equivalent propositions are identical. Many who question this principle will consider principles (1–3) plausible. It is therefore of interest to investigate the weakening of simplicity obtained by omitting (4), and the entailment relations to which consistency structures satisfying it give rise.

Definition 71. *A consistency structure is* pre-simple *if it satisfies constraints (1), (2) and (3).*

The corresponding entailment relations turn out to be what are sometimes called pre-Boolean algebras, or Boolean pre-algebras, e.g., by Fox and Lappin (2005) and Pollard (2008). Corresponding to different ways of defining Boolean algebras, there are different ways of defining pre-Boolean algebras. The present treatment of Boolean algebras as partial orders allows a satisfyingly simple definitions: a pre-Boolean algebra is a pre-order which, when quotiented under equivalence (mutual entailment), constitutes a Boolean algebra.

Here, the notion of quotienting is the following: Let a *pre-order* be an ordered set $\mathfrak{A} = \langle A, \leq \rangle$ which is reflexive and transitive. Thus, a partial order is an anti-symmetric pre-order. Let *equivalence*, written $\sim$, be mutual entailment: $x \sim y$ iff $x \leq y$ and $y \leq x$. Then, $\mathfrak{A}$ *quotiented by* $\sim$, written $\mathfrak{A}/{\sim}$, is the order $\langle A/{\sim}, \leq_\sim \rangle$, where $A/{\sim}$ is the set $\{[x]_\sim : x \in A\}$ of equivalence classes of $\sim$, and $[x]_\sim \leq_\sim [y]_\sim$ iff $x \leq y$. For this to be well-defined, $\sim$ has to be a congruence with respect to $\leq$, i.e., $\sim$ has to be an equivalence relation on A such that $x \sim x'$ and $y \sim y'$ only if $x \leq y$ iff $x' \leq y'$. But this is guaranteed by the assumption of $\leq$ being a pre-order. Likewise, by construction, $\mathfrak{A}/{\sim}$ is a partial order. Thus, any class of partial orders can naturally be generalized to a corresponding class of pre-orders: anti-symmetry is factored out by taking the pre-orders which, when quotiented by equivalence, produce partial orders in the relevant class. The special case of generalizing Boolean algebras in this way gives us the definition of a pre-Boolean algebra.

Definition 72. *Let a* pre-Boolean algebra *be a pre-order* $\mathfrak{A} = \langle A, \leq \rangle$ *such that* $\mathfrak{A}/{\sim}$ *is a Boolean algebra.*

Note that when discussing pre-simple consistency structures, two readings for the symbol $\sim$ are available: it can be read to indicate equivalence as mutual entailment, or to indicate equivalence as defined initially in terms of being consistent with the same sets. But there is no danger of equivocation, since given constraint (1), the two notions coincide, as shown in Lemma 3 (i).

We will show that pre-simple consistency structures relate to pre-Boolean algebras as simple consistency structures relate to Boolean algebras. Therefore, we first show that each pre-simple consistency structure $\mathcal{S}$ gives rise to a pre-Boolean entailment order $\mathcal{S}^*$; i.e., an entailment order which, when quotiented under mutual entailment, constitutes a Boolean algebra $\mathcal{S}^*/{\sim}$. To show this, we define a way of quotienting the pre-simple consistency structure $\mathcal{S}$ itself under mutual entailment, producing a simple consistency structure $\mathcal{S}_\sim$. It then suffices to show that the entailment order of $\mathcal{S}_\sim$ coincides with $\mathcal{S}^*/{\sim}$. In effect, we show that the operations of quotienting under mutual entailment and taking the entailment order commute. Once we have shown that $\mathcal{S}_\sim$ is simple and gives rise to the entailment order $\mathcal{S}^*/{\sim}$, previous results tell us that $\mathcal{S}^*/{\sim}$ is a Boolean algebra, and thus that $\mathcal{S}^*$ is a pre-Boolean algebra, as intended.

There are several possible ways of quotienting a pre-simple consistency structure $\mathcal{S} = \langle A, C \rangle$, not all of which work for the present proof. It is clear enough that the quotiented structure needs to be based on the quotient set $A/{\sim}$, but there are several options concerning what it should take for a set X of equivalence classes to count as consistent. E.g., one could require $\bigcup X$ to be consistent, but this does not lead to the desired results. Instead, we fix what we will call a contraction function γ, which determines, for every equivalence class, a designated member. Consistency of a set of equivalence classes is then defined as consistency of the set of designated members of these classes. Which sets are consistent in the resulting consistency structure may in fact depend on the choice of contraction function. But for present purposes, the choice does not matter, since any such differences are not reflected in the entailment order.

To define this construction formally, we extend the notation for quotient sets to subsets of the given domain: for any equivalence relation $\sim$ on a set A and set $X \subseteq A$, $X/{\sim} = \{[x]_\sim : x \in X\}$.

Definition 73. *Let $\mathcal{S} = \langle A, C\rangle$ be a pre-simple consistency structure. A* contraction *of $\mathcal{S}$ is a function $\gamma : A/{\sim} \to A$ such that $\gamma(x) \in x$ for all $x \in A/{\sim}$. For any such function γ and $X \subseteq A/{\sim}$, let $\gamma(X) = \{\gamma(x) : x \in X\}$. Let $\mathcal{S}_\gamma = \langle A/{\sim}, C_\gamma\rangle$ such that for all $X \subseteq A/{\sim}$:*

$$X \in C_\gamma \textit{ iff } \gamma(X) \in C$$

We can now carry out the proof strategy outlined above, starting with the simplicity of any quotiented pre-simple consistency structure.

Lemma 74. *If $\mathcal{S}$ is a pre-simple consistency structure and γ a contraction of $\mathcal{S}$, then $\mathcal{S}_\gamma$ is a simple consistency structure.*

Proof. We distinguish defined notions in the two structures by indexing the notion in $\mathcal{S}_\gamma$ with γ. E.g., for contradictoriness, we use $\times$ in $\mathcal{S}$ and $\times_\gamma$ in $\mathcal{S}_\gamma$.

(1): If $Y \subseteq X \in C_\gamma$, then $\gamma(Y) \subseteq \gamma(X) \in C$. Since $\mathcal{S}$ satisfies (1), $\gamma(Y) \in C$, and so $Y \in C_\gamma$.

(2): Consider any $x \in A/{\sim}$. Since $\mathcal{S}$ satisfies (2), there is a $y \in A$ such that $\gamma(x) \times y$. Let $\bar{x} = [y]_\sim$. We show that $x \times_\gamma \bar{x}$. First, $\{\gamma(x), y\} \notin C$, whence $\{\gamma(x), \gamma(\bar{x})\} \notin C$; thus $\{x, \bar{x}\} \notin C_\gamma$. Second, consider any $X \in C_\gamma$ such that $X \cup \{x\} \notin C_\gamma$. Then $\gamma(X) \cup \{\gamma(x)\} \notin C$. Since $X \in C_\gamma$, $\gamma(X) \in C$, so it follows with $\gamma(x) \times y$ that $\gamma(X) \cup \{y\} \in C$. Therefore $\gamma(X) \cup \{\gamma(\bar{x})\} \in C$, and so $X \cup \{\bar{x}\} \in C_\gamma$.

(3): Consider any $x, y \in A/{\sim}$. Since $\mathcal{S}$ satisfies (3), there is a $w \in A$ such that $\{\gamma(x), \gamma(y)\} \sim \{w\}$. Let $z = [w]_\sim$. We show that $\{x, y\} \sim_\gamma \{z\}$. So consider any $X \subseteq A/{\sim}$. $X \cup \{x, y\} \in C_\gamma$ iff $\gamma(X) \cup \{\gamma(x), \gamma(y)\} \in C$, which is the case iff $\gamma(X) \cup \{w\} \in C$. This is the case iff $\gamma(X) \cup \{\gamma(z)\} \in C$, which in turn is the case iff $X \cup \{z\} \in C_\gamma$.

(4): Consider any $x, y \in A/{\sim}$ such that $x \neq y$. Then $\gamma(x) \not\sim \gamma(y)$. Without loss of generality, we may assume that there is an $X \subseteq A$ such that $X \cup \{\gamma(x)\} \in C$ but $X \cup \{\gamma(y)\} \notin C$. Since $\mathcal{S}$ satisfies (2), there is a $z \in A$ such that $\gamma(y) \times z$. Let $\bar{y} = [z]_\sim$. We show that $x \not\sim_\gamma y$ by showing that $\{x, \bar{y}\} \in C_\gamma$ and $\{y, \bar{y}\} \notin C_\gamma$. First, as $X \cup \{\gamma(x), \gamma(y)\} \notin C$, it follows that $X \cup \{\gamma(x), z\} \in C$. Therefore $\{\gamma(x), z\} \in C$, and so $\{\gamma(x), \gamma(\bar{y})\} \in C$, whence $\{x, \bar{y}\} \in C_\gamma$. Second, $\{\gamma(y), z\} \notin C$, and so $\{\gamma(y), \gamma(\bar{y})\} \notin C$, whence $\{y, \bar{y}\} \notin C_\gamma$. □

Next, note that the proofs of Lemmas 12 and 13 rely only on (1), so that for every pre-simple consistency structure $\mathcal{S}$, $\mathcal{S}^*$ is a preorder. Thus, $\mathcal{S}^*/{\sim}$ is well-defined, and we can show the following.

Lemma 75. *Let $\mathcal{S}$ be a pre-simple consistency structure and γ a contraction of $\mathcal{S}$. Then:*

$$\mathcal{S}_\gamma{}^* = \mathcal{S}^*/\sim$$

Proof. We write $\leq_\gamma$ for the order of $\mathcal{S}_\gamma{}^*$, and $\leq_\sim$ for the order of $\mathcal{S}^*/\sim$. We show that $\leq_\gamma = \leq_\sim$. Consider any $x, y \in A/\sim$.

Assume first $x \leq_\sim y$, whence $\gamma(x) \leq \gamma(y)$. To show $x \leq_\gamma y$, consider any $X \subseteq A/\sim$ such that $X \cup \{x\} \in C_\gamma$. Then $\gamma(X) \cup \{\gamma(x)\} \in C$, from which $\gamma(X) \cup \{\gamma(x), \gamma(y)\} \in C$ follows. Thus $X \cup \{x, y\} \in C_\gamma$, as required.

Assume now $x \leq_\gamma y$. To show that $x \leq_\sim y$, we show that $\gamma(x) \leq \gamma(y)$. So consider any $X \subseteq A$ such that $X \cup \{\gamma(x)\} \in C$. Since $\mathcal{S}$ satisfies (2), there is a $z \in A$ such that $z \times \gamma(y)$. Let $\bar{y} = [z]_\sim$. Thus $X \cup \{\gamma(x), z\} \in C$ or $X \cup \{\gamma(x), \gamma(y)\} \in C$. We establish the latter by ruling out the former. For if the former is the case, $\{\gamma(x), \gamma(\bar{y})\} \in C$, whence $\{x, \bar{y}\} \in C_\gamma$. Since we assumed $x \leq_\gamma y$, it follows that $\{x, \bar{y}, y\} \in C_\gamma$, and so that $\{\bar{y}, y\} \in C_\gamma$. Thus $\{\gamma(\bar{y}), \gamma(y)\} \in C$, and so $\{z, \gamma(y)\} \in C$, contradicting $z \times \gamma(y)$. □

Proposition 76. *If $\mathcal{S}$ is a pre-simple consistency structure, then $\mathcal{S}^*$ is a pre-Boolean algebra.*

Proof. Let $\mathcal{S}$ be a pre-simple consistency structure. By the axiom of choice, $\mathcal{S}$ has a contraction γ. By Lemma 74, $\mathcal{S}_\gamma$ is a simple consistency structure, so with Proposition 25, $\mathcal{S}_\gamma{}^*$ is a Boolean algebra. By Lemma 75, it follows that $\mathcal{S}^*/\sim$ is a Boolean algebra. So $\mathcal{S}^*$ is a pre-Boolean algebra. □

It remains to show that the mapping from pre-simple consistency structures to pre-Boolean algebras is surjective but not injective. The failure of injectivity is immediate from the case of simple consistency structures. For surjectivity, we can also make use of previous constructions, in this case the mappings from Boolean algebras to simple consistency structures: For any pre-Boolean algebra $\mathfrak{A}$, define a consistency structure by counting a set as consistent if the set of equivalence classes of its members is consistent in $(\mathfrak{A}/\sim)^\infty_\infty$ (one of the simple consistency structures associated with the quotient algebra of $\mathfrak{A}$).

Lemma 77. *Consider any pre-Boolean algebra $\mathfrak{A} = \langle A, \leq \rangle$. Let $(\mathfrak{A}/\sim)^\infty_\infty = \langle A/\sim, C \rangle$, and let $\mathcal{S}$ be the consistency structure $\langle A, C' \rangle$ such that for all $X \subseteq A$:*

$X \in C'$ *iff* $X/\sim \in C$.

Then:

(i) $\mathcal{S}^* = \mathfrak{A}$

(ii) $\mathcal{S}$ *is pre-simple.*

Proof. (i): Let $\leq_\sim$ be the order of $\mathfrak{A}/\sim$, and $\leq'$ the order of $\mathcal{S}^*$. We show that $\leq = \leq'$. Consider any $x, y \in A$.

Assume first $x \leq y$. Then $[x]_\sim \leq_\sim [y]_\sim$. To show $x \leq' y$, consider any $X \subseteq A$ such that $X \cup \{x\} \in C'$. Then $X/\sim \cup \{[x]_\sim\} \in C$, whence there is a non-zero lower bound of $X/\sim \cup \{[x]_\sim\}$ under $\leq_\sim$. The same thus applies to $X/\sim \cup \{[x]_\sim, [y]_\sim\}$, from which it follows that $X \cup \{x, y\} \in C'$.

Assume now $x \not\leq y$. Then $[x]_\sim \not\leq_\sim [y]_\sim$. Let z be the complement of $[y]_\sim$ in $\mathfrak{A}/\sim$. Then $\{[x]_\sim, z\}$ has a non-zero lower bound under $\leq_\sim$, but $\{[y]_\sim, z\}$ does not. So $\{[x]_\sim, z\} \in C$ while $\{[y]_\sim, z\} \notin C$. By construction, there is some $\bar{y} \in z$. Thus $\{x, \bar{y}\} \in C'$ and $\{y, \bar{y}\} \notin C'$. So $x \not\leq' y$.

(ii): For the following arguments, note that by Lemma 27 (ii), $(\mathfrak{A}/\sim)^\infty_\infty$ is a simple consistency structure.

(1): If $Y \subseteq X \in C'$, then $Y/\sim \subseteq X/\sim \in C$. Since $(\mathfrak{A}/\sim)^\infty_\infty$ satisfies (1), $Y/\sim \in C$, whence $Y \in C'$.

(2): Consider any $x \in A$. Since $(\mathfrak{A}/\sim)^\infty_\infty$ satisfies (2), there is a $\bar{x} \in A$ such that $[x]_\sim$ and $[\bar{x}]_\sim$ are contradictories in $(\mathfrak{A}/\sim)^\infty_\infty$. So $\{[x]_\sim, [\bar{x}]_\sim\} \in C$, whence $\{x, \bar{x}\} \notin C'$. Consider any $X \in C'$. Then $X/\sim \in C$, so $X/\sim \cup \{[x]_\sim\} \in C$ or $X/\sim \cup \{[\bar{x}]_\sim\} \in C$. Thus $X \cup \{x\} \in C'$ or $X \cup \{\bar{x}\} \in C'$.

(3): Consider any $x, y \in A$. Since $(\mathfrak{A}/\sim)^\infty_\infty$ satisfies (3), there is a $z \in A$ such that $\{[x]_\sim, [y]_\sim\}$ is equivalent to $\{[z]_\sim\}$ in $(\mathfrak{A}/\sim)^\infty_\infty$. Consider any $X \subseteq A$. Then $X \cup \{x, y\} \in C'$ iff $X/\sim \cup \{[x]_\sim, [y]_\sim\} \in C$. This is the case iff $X/\sim \cup \{[z]_\sim\} \in C$, which in turn is the case iff $X \cup \{z\} \in C'$. □

Proposition 78. $\cdot^*$ *maps pre-simple consistency structures surjectively but not injectively to pre-Boolean algebras.*

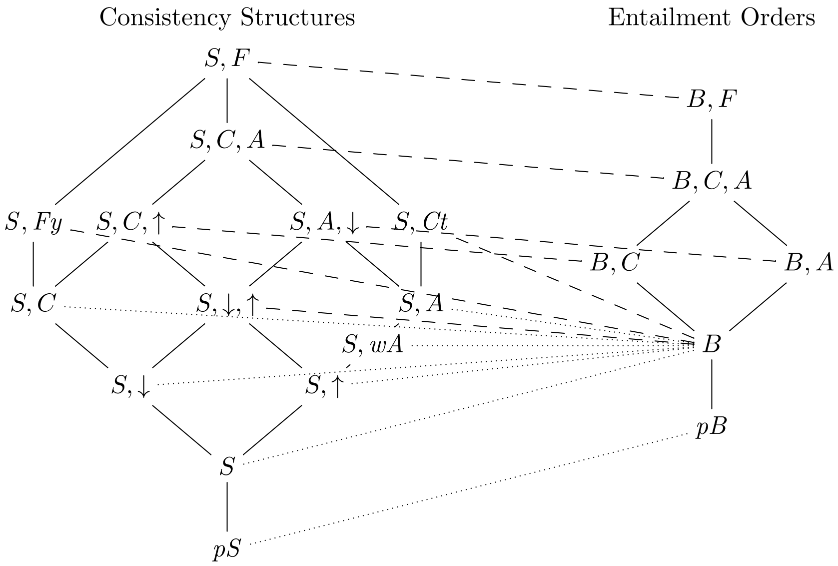
Proof. By Proposition 76, $\cdot^*$ maps every pre-simple consistency structure to a pre-Boolean algebra. Surjectivity follows from Lemma 77. The failure of injectivity follows from Proposition 29. □

The results established here show that endorsing pre-simplicity imposes on entailment exactly the requirements of pre-Boolean algebras. As in the case of simple consistency structures, consistency facts cannot be recovered from entailment facts without further assumptions, and this leads to cases in which distinct pre-simple consistency structures give rise to the same entailment order. We have also seen that any pre-simple consistency structure, quotiented by equivalence, is a simple consistency structure. This supports a suggestion by Stalnaker (2012, p. 26), which in the present setting amounts to the idea that those disagreeing on whether equivalent propositions are identical may still agree that *truth-conditions*, construed as propositions quotiented by equivalence, satisfy this principle. In these terms, we have shown that those agreeing on propositions satisfying pre-simplicity thereby agree on truth-conditions satisfying simplicity.

7 Summary

We have considered a number of properties of consistency structures, which have been shown to correspond – in various ways – to properties of entailment orders. Figure 1 gives an overview of the classes of structures which arise from combinations of these principles, and their relations established here. Combinations of principles are written as comma-separated lists; e.g., closed simple consistency structures are written $S, \downarrow, \uparrow$.

The theory of consistency presented by Stalnaker (2012) appears here as S, C, A: the class of complete atomic simple consistency structures, corresponding bijectively to the class of complete atomic Boolean algebras. It is worth noting that Stalnaker (2012, pp. 25–26) only defends atomicity (A) and the identity of equivalents (4) in the sense of claiming that there is *some* notion



pS:	pre-simple	S:	simple	pB:	pre-Boolean
↓:	downward closed	↑:	upward closed	B:	Boolean
C:	complete	A:	atomic	C:	complete
wA:	weakly atomic	Ct:	compact	A:	atomic
Fy:	finitary	F:	finite	F:	finite

——— : proper inclusion (going upward)
– – – : bijective mapping by $\cdot^*$ (going right)
········ : surjective but not injective mapping by $\cdot^*$ (going right)

Figure 1: Overview

or domain of propositions satisfying these axioms. Other notions or domains of propositions may only satisfy some of the weaker theories.

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References

Garrett Birkhoff. *Lattice Theory.* New York: American Mathematical Society, 1940.

B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order.* Cambridge: Cambridge University Press, second edition, 2002.

Chris Fox and Shalom Lappin. *Foundations of Intensional Semantics.* Malden, MA: Blackwell Publishing, 2005.

Steven Givant and Paul Halmos. *Introduction to Boolean Algebras.* New York: Springer, 2009.

Thomas Jech. *Set Theory.* Berlin: Springer, third edition, 2002.

Carl Pollard. Hyperintensions. *Journal of Logic and Computation*, 18:257–282, 2008.

V. N. Saliĭ. *Lattices with Unique Complements.* Providence: American Mathematical Society, 1988.

Robert Stalnaker. Possible worlds. *Noûs*, 10:65–75, 1976.

Robert Stalnaker. Possible worlds. In *Inquiry*, chapter 3, pages 43–58. Cambridge, MA: MIT Press, 1984.

Robert Stalnaker. Possible worlds. In *Ways a World Might Be*, chapter 1, pages 25–39. Oxford: Clarendon Press, 2003.

Robert Stalnaker. *Mere Possibilities.* Princeton: Princeton University Press, 2012.

Robert Stalnaker. *A Quinean Foundation for Modal Logic.* 2018. Handouts for *The Rutgers Lectures in Philosophy* 2018. A book based on these lectures is to be published by Oxford University Press.