STUDENTS' PROPORTIONAL REASONING IN MATHEMATICS AND SCIENCE

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Proportional reasoning is increasingly being recognised as fundamental for successful operation in many topics within both the mathematics and science curriculum. However, research has consistently highlighted students' difficulties with proportion and proportion-related tasks and applications, suggesting the difficulty for many students in these core school subjects. As a first step in a major research project to support the design of integrated curriculum across these two disciplines, this paper reports on students' results on a proportional reasoning pretest of mathematics and science items. Administered to approximately 700 students across grades 4 to 9, results anticipated increased gradual progression in results, but surprising similarities in performance on particular items for student groups at each year level.

PROPORTIONAL REASONING IN MATHEMATICS AND SCIENCE

Many topics within the school mathematics and science curriculum require knowledge and understanding of ratio and proportion and being able to reason proportionally. In mathematics, for example, problem solving and calculation activities in domains involving scale, probability, percent, rate, trigonometry, equivalence, measurement, algebra, the geometry of plane shapes, are assisted through ratio and proportion knowledge. In science, calculations for density, molarity, speed and acceleration, force, require competence in ratio and proportion. Proportional reasoning, according to Lamon (2006) is fundamental to both mathematics and science.

Proportional reasoning means being able to understand the multiplicative relationship inherent in situations of comparison (Behr, et al., 1992). The study of ratio is the foundation upon which situations of comparison can be formalised, as a ratio, in its barest form describes a situation in comparative terms. For example, if a container of juice is made up of 2 cups of concentrated juice and 5 cups of water, then a container triple the size of the original container will require triple the amounts of concentrate and water (that is, 6 cups of concentrated juice and 15 cups of water) to ensure the same taste is attained. Proportional thinking and reasoning is knowing the multiplicative relationship between the base ratio and the proportional situation to which it is applied. Further, proportional reasoning is also dependent upon sound foundations of associated topics, particularly multiplication and division (Vergnaud, 1983), fractions (English & Halford, 1995) and fractional concepts of order and equivalence (Behr, et al. 1992). Although understanding of ratio and proportion is intertwined with many mathematical topics, the essence of proportional reasoning is the understanding of the multiplicative structure of proportional situations (Behr, et al., 1992).

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In the middle years of schooling, ratio and proportion are typically studied in mathematics classes. In fact, ratio and proportion have been described as the cornerstone of middle years mathematics curriculum (Lesh, Post & Behr, 1988). However, research has consistently highlighted students' difficulties with proportion and proportion-related tasks and applications (e.g, Behr, Harel, Post & Lesh, 1992; Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller, 1998; Lo & Watanabe, 1997), which means that many students will struggle with topics within both the middle years mathematics and science curriculum due to their lack of understanding of ratio and proportion. Understanding ratio and proportion is more than merely being able to perform appropriate calculations and being able to apply rules and formulae, and manipulating numbers and symbols in proportion equations. It is well-accepted that students' computational performances are not a true indicator of the degree to which they understand the concepts underlying the calculations.

THE STUDY

The research reported in this paper is part of a larger study entitled the MC SAM project, the acronym for 'Making Connections in Science and Mathematics'. The project aims to take a "conscious, systematic and explicit…. structured and goal-oriented" learning by design approach (Kalantzis & Cope, 2004, p. 39) to support the careful design of an integrated curriculum to promote students' connected knowledge development across these two disciplines. In this project, researchers and teachers are collaboratively developing, implementing and documenting innovative, relevant and connected learning in mathematics and science, and hence redefining classroom culture as well as redefining curriculum. This paper presents results of a proportional reasoning pretest, the results of which highlight great variance of proportional reasoning in students across Years 4 to 9, and simultaneously underscores the importance of a more systematic and structured approach to promoting proportional reasoning across mathematics and science.

The pretest was to designed to provide a snapshot of a large group of students' proportional reasoning on tasks that relate to mathematics and science curriculum in the middle years of schooling. This aspect of the research was concerned with the development of an instrument that would provide a 'broad brush' measure of students' proportional reasoning and their thinking strategies, and that would have some degree of diagnostic power. This challenge was undertaken with full awareness of both the pervasiveness and the elusiveness of proportional reasoning throughout the curriculum and that its development is dependent upon many other knowledge foundations in mathematics and science.

Instrument design

A large corpus of existing research has provided analysis of strategies applied by students to various proportional reasoning tasks (e.g., Misailidou & Williams, 2003; Hart, 1981). Such research has highlighted issues associated with the impact of 'awkward' numbers (that is, common fractions and decimals as opposed to whole numbers), the common application of an incorrect additive strategy, and the blind application of rules and formulae to proportion problems.

To identify more specific links across both mathematics and science, we consulted the Atlas of Scientific Literacy (American Association for the Advancement of Science (AAAS), 2001). The AAAS has identified two key components of proportional reasoning: Ratios and Proportion (parts and wholes, descriptions and comparisons and computation) and Describing Change (related changes, kinds of change, and invariance). Using this as a frame, we devised the test to incorporate items on direct proportion (whole number and fractional ratios), rate, and inverse proportion as well as items relating to fractions, probability, speed and density. Guided by the words of Lamon (2006) who suggested that students must be provided with many different contexts, 'to analyse quantitative relationships in context, and to represent those relationships in symbols, tables, and graphs' (p. 4), the items included contexts of shopping, cooking, mixing cordial, painting fences, graphing stories, saving money, school excursions, dual measurement scales. For each item on the test, students were required to provide the answer and explain the thinking they applied to solve the problem.

The pretest consisted of 16 items, split into two sections of 8 items each. Bearing in mind that the test would be administered to 4th Grade students, we wanted to avoid test fatigue and provided students with 30 minutes to complete each section of the test on two different days. Most students completed each section of the test within 15 minutes. Table 1 provides the title of each test item and a brief description of its focus.

A1	*Butterflies*. 5 drops of nectar for 2 butterflies; *x* drops of nectar for 12 butterflies? Missing value – simple numbers.
A2	*Chance Encounters*. Which of 4 bags of counters (B/W) has best chance of selecting black: 4B 4W; 1B 1W; 2B 1W; 4B 3W. Probability
A3	*Shopping Trip*. \$6 remaining after spending $^1/_3$ of money. How much at the start? Part/part/whole – complex ratio.
A4	*Three Cups*. Full cup, ½ cup, $^1/_3$ cup water; 3, 2, 1 lumps sugar respectively. Which is sweetest? Intuitive proportion, small numbers.
A5	*Sticky Mess*. Recipe: 4 cups of sugar, 10 cups of flour; 6 sugar for *x* flour? Missing value – complex ratio.
A6	*Fence Painting*. 6 people take 3 days; how many in 2 days? Inverse.
A7	*End of Term*. Comparison of preferences for an end of term activity in two classes of students (different totals). Absolute vs relative thinking.
A8	*Number Line*. Reading dual-scale representation using two measurement scales. Scaling.
B1	*Speedy Geoff*. Distance covered when speed halved. Speed.

B2	*Balancing*. Identifying impact of weights on Balance Scale.
B3	*Washing Days*. Powder A: 1kg, 20 loads, $4; Powder B: 1.5 kg, 30 loads, $6.50. Which is better buy?
B4	*Funky Music*. Mum pays $5 for every $2 saved to buy item for $210. How much did each person paying? Part/part/whole
B5	*Cycling Home*. Matching graph to speed of bicycle.
B6	*Sinking and Floating*. Density of object in liquid.
B7	*Juicy Drink*. Mixing cordial; two-step ratio problem.
B8	*Tree Growth*. Non-proportional situation, trees grow at same rate.

Table 1: Proportional Reasoning Pretest Item Overview.

RESULTS

Approximately 700 students across Grades 4-9 completed the test. Students' results on this assessment are presented in Figure 1.

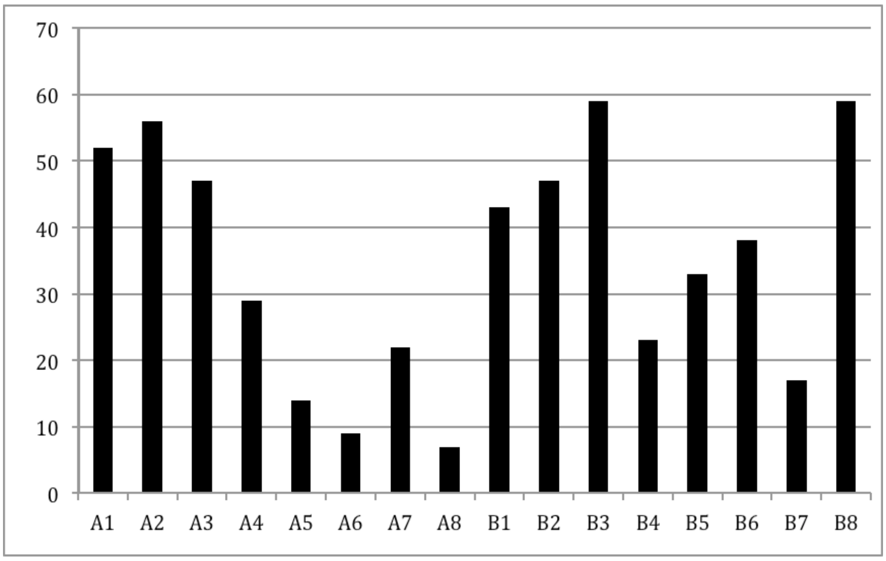

Figure 1: **Percentage correct for each test item**

Students' responses for each test item were coded. Coding occurred at two levels, and hence a two-digit code was assigned to each response. The first digit in the code identified whether the item was correct (code 1), incorrect (code 2), or not attempted (code 0). The second digit in the code identified the thinking strategy utilised by the student in solving the problem, as gleaned from the explanation of how he/she solved each problem. In particular, a solution strategy that showed application of elegant ratio thinking (that is, direct use of multiplication and division strategies) was assigned a code of 1, with a solution strategy that showed application of a repeated addition strategy (use of tables of values) assigned a code of 2. These two codes were considered indicative of appropriate proportional reasoning. A code of 3 was given to thinking that suggested (incorrect) additive thinking had been applied, and a code of 4 was given to thinking that suggested that the student's strategy would never lead him/her to the correct solution. A code of 0 was given when the student left this section blank. Scores of 11 or 12 thus indicated a correct solution and application of proportional reasoning. A score of 23 indicated an incorrect solution with inappropriate additive thinking. Table 2 shows the percentage of responses for each particular code.

	Response Code										
Item	**11**	**12**	**13**	**14**	**10**	**21**	**22**	**23**	**24**	**20**	**00**
A1	31	19	0	0	2	4	11	2	28	1	2
A2	16	2	36	2	0	2	1	32	8	1	0
A3	31	9	4	3	2	2	4	5	32	7	3
A4	12	0	8	6	3	0	0	27	38	4	2
A5	9	4	0	1	0	1	0	66	7	8	4
A6	1	0	4	2	2	1	0	33	42	11	4
A7	7	0	6	7	2	1	0	57	15	3	2
A8	2	1	2	2	0	2	0	13	53	13	12
B1	35	5	0	1	2	0	0	4	40	10	3
B2	10	10	20	5	2	0	1	6	40	3	3
B3	22	3	20	10	4	1	0	25	9	3	3
B4	11	9	0	1	2	1	10	1	32	12	21
B5	11	16	2	2	2	0	15	27	15	6	4
B6	1	6	27	1	3	0	8	33	9	3	9
B7	15	1	0	0	1	7	0	15	34	11	16
B8	0	0	56	0	3	3	1	14	13	3	7

Table 2: Percentage of strategy use for correct and incorrect responses.

DISCUSSION

Lamon (2006) described proportional reasoning as a web of interrelated "concepts, operations, contexts, representations, and ways of thinking" (p. 9) to highlight the complexity of proportional reasoning and hence advocating a rich, recursive curriculum across rational number domains for promoting proportional reasoning. Central core ideas for proportional reasoning, as identified by Lamon, include rational number interpretation, measurement, quantities and covariation, relative thinking, unitizing, sharing and comparing, reasoning up and down. And all these are "recurrent, recursive and of increasing complexity across mathematical and scientific domains" (p. 9). Inherent in these words is a call for change of focus to mathematics instruction in ratio and proportion topics, and a new look at the traditional separatist demarcation of mathematics and science curricular.

The pretest designed in the MC SAM project is only a tentative first step for emphasising the centrality of proportional reasoning across mathematics and science topics. In this test, the items were designed to capture students' proportional reasoning in its broadest sense. Some items were very typical ratio tasks (items A1, A5, B7), but some were specifically linked more directly to science. Item B1 was a simple speed situation: *Geoff runs 100 metres in 12 seconds. If he runs the same distance at half the speed, how long will it take him?* This item was correctly answered by less than 50% of the students, but was comparatively well-answered by the fourth graders (just above 30%), and was one of the best-answered items on the test for these students (see figure 2). This suggests that intuitively, fourth graders can understand simple speed situations. Interestingly, the ninth graders' mean score for this item was only approximately 63%, and was not the best-answered item for this cohort. Item B2 was a classic balance beam problem, frequently cited in science research as a science reasoning task (see for example, Shayer & Adey, 1981). The mean score for this item was 47%, and was also well-answered by the fourth graders (35%). Item B5 required students to select (from 6) the appropriate graph for the following situation: *Anne was cycling home from school. She rode for a short time at a steady speed then stopped for a rest. When se started again, she rode twice as fast to get home quickly*. This item was devised to link to the AAAS's 'Describing Change' component of scientific reasoning, but clearly graph interpretation is a key component of rational number understanding (Lamon, 2006). The mean score for this item was 31%, with the fourth graders responding relatively well at 15%, which is higher than for many other items. This suggests that fourth graders can interpret situations graphically, and has implications for instruction at much earlier junctions than typically occurs in primary school. Compared to the seventh to ninth graders, the fourth and fifth graders's results were impressive. However, not as impressive as for item B6, which was also a specific science item relating to density. This item had several parts, providing students with a data table that displayed the mass and volume of a collection of cubes and information about one cube in the collection that is known to sink. The students had to determine which other cubes would sink. The fourth graders scored higher than the ninth graders

(38% and 22% respectively), and the fifth graders scored highest of all cohorts (48%). The reasons for these curious differences can only be speculated, but the impact of instruction upon students' intuitive knowledge of density warrants scrutiny in relation to performance.

Item B3 and B8 were the best answered of all items on this test (both 59%). B8 was a non-proportional situation (two trees of different height grow at the same rate; find the height of the second tree after a period of time given the height of the first tree). Students' capacity to distinguish proportional from non-proportional situations is a key for proportional reasoning that indicates reasoning capacity as compared to blind application of formulae (Lamon, 2006). B3 was a 'better buy' situation (briefly described in table 1), and results of this task may not be as exciting as they appear, as this item was essentially a two-choice item (A or B). This is where the second level of coding gives further insight into students' reasoning. From Table 2, it can be seen that students who selected the correct washing powder equally used multiplicative and additive reasoning (22% responses coded 11 and 20% coded 13). Ten percent of students selected the correct answer (code 10) without stating how they achieved this answer. Approximately 25% of students selected the wrong powder (code 23) and used additive thinking in their response. Hence, for this particular item, students may have selected the correct powder but used inappropriate faulty additive reasoning.

The coding of responses and the use of additive and multiplicative thinking is most starkly revealed in items A1 and A5 (see table 1 for an overview of these items). Approximately 50% of students used appropriate multiplicative thinking for item A1, but for A5, 66% of students used inappropriate additive thinking on a standard ratio task that involved a fractional ratio. Item A1 was one of the better-answered of all items by the ninth graders, where the mean score for this cohort was approximately 73%. But for item A5 involving a fractional ratio, performance overall is merely 15% overall, and 20% for ninth graders. Students clearly recognised the multiplicative relationship of the butterflies to drops of nectar in item A1, but alarmingly abandoned this thinking and used an additive strategy for item A5 in the recipe question. The power and stability of additive thinking is clearly an issue for successful operation in domains that require proportional reasoning. Although this finding is not new, the overwhelming incorrect use of additive thinking for this item further highlights the instability of relational thinking of students in the middle years of schooling.

Conclusion

The results reported in this paper are the first steps towards taking a more structured approach to a connected curriculum across the domains of mathematics and science. The proportional reasoning test devised for this project makes no claims of comprehensively assessing students' proportional reasoning for mathematics and science. However, its purpose was more fundamentally to raise awareness of the pervasiveness of proportional reasoning across the domains of mathematics and science and to assist teachers to target instruction more specifically to promote students' proportional reasoning.

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