The Impact of Computer Algebra Systems in Assessment in Undergraduate Calculus : A Commognitive Approach

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THE IMPACT OF COMPUTER ALGEBRA SYSTEMS IN ASSESSMENT IN UNDERGRADUATE CALCULUS: A COMMOGNITIVE APPROACH

Submitted by

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Statement of Authorship and Sources

This thesis contains no material that has been extracted in whole or in part from a thesis that I have submitted towards the award of any other degree or diploma in any other tertiary institution.

No other person's work has been used without due acknowledgment in the main text of the thesis.

All research procedures reported in the thesis received the approval of the relevant Ethics/Safety Committees (where required).

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Abstract

Calculus examination questions can sometimes be judged as having similar mathematical content when, due to textual aspects and visual mediators, they are quite different. Computer Algebra Systems (CAS) technology can have a further impact on their nature and on students' expected answer approaches. Little research has been done to assess the impact of CAS technology on these characteristics from a *discursive* perspective.

This thesis investigates the impact of CAS technology on mathematical discourse. The first research question asks how a commognitive analysis framework (Morgan & Sfard, 2016) can be applied to effectively capture the complexity and difficulty level of written answer and multiple-choice examination questions in undergraduate Calculus units where use of CAS technology is available. This includes considering how such questions position students as knowers and users of CAS technology. The second research question asks (a) how effectively undergraduate students use their CAS calculator and use and interpret CAS output, especially when it is in a different format to what they are familiar with from the methods of working by-hand that they have been taught in class; as well as (b) what we can learn from a commognitive analysis of task-based interviews of students in relation to how they reflect on their answers in this situation. Finally, the third research question asks to what extent undergraduate students use CAS and when they believe its use to be most beneficial.

Commognitive analysis was used for qualitative analysis of nine questions purposively selected from 7 of 11 undergraduate Calculus assessments (examinations and tests) collected. To capture how students approach similar questions using CAS, task-based interviews were conducted with four students taking a first- or second-year Calculus unit. Their written and spoken discourse was analysed qualitatively, using commognitive analysis, to determine the extent to which they used ritual or explorative routines, aspects of how they used CAS in their problem-solving approaches, how they navigated the classroom discourse of Calculus and that of CAS, and whether any indication of the *discursive footprint* of high school mathematics on their Calculus discourse was present. Evidence was sought of any intrapersonal commognitive conflicts arising from their interactions with a CAS calculator, due to its different discursive requirements. Their preferences and reasons for using CAS or working by-hand were also examined. To contextualise the findings of the task-based interviews, basic descriptive statistics were

produced, showing the percentage frequency distribution of responses for questionnaire data on undergraduate students' use of, and attitudes towards, CAS technology at the time of the study.

Analysis of the test and examination scripts indicated that, while reducing the number of procedural steps involving working by-hand, questions more logically and grammatically complex in other ways were asked when CAS technology use was expected, including some with increased symbolic and graphical CAS outputs. A finding was that complexity of questions with CAS outputs also depended on how individual students were *positioned* to answer them, as, in positioning students as knowers of technology, such questions can involve greater logical complexity if a student is not familiar with CAS outputs.

From the task-based interviews analysis, a finding was identifying unresolved intrapersonal commognitive conflicts encountered by students, between the discourse of CAS technology itself and the more familiar classroom discourse of Algebra and Calculus. The participants mostly used ritual routines, which could have been an artefact of tasks chosen. They typically used CAS to employ a *direct procedure* in a single step, but also in an *inter-representational fashion* and as *a procedure within a more complex process.* The *discursive footprint* of high school mathematics was evident in their written and spoken discourse, with few instances of transitioning to more scholarly, university level Calculus discourse. Where they preferred CAS, reasons included saving time, visualising graphs and not being confident in solving a problem by-hand.

Applying commognitive analysis to test and examination scripts where CAS technology is available or present as outputs, contributes to research on assessment by identifying more aspects to consider in evaluating complexity of questions and how they are positioning students to respond to them. The identification of commognitive conflicts due to the different discourse of CAS can inform teaching practice about using and interpreting results with CAS technology, and effectively integrating CAS into learning materials.

Glossary of Key Terms

Alienated Discourse: Discourse where phenomena are presented impersonally as if they occur without the involvement of humans. [See section 2.5.3]

Artefact: An object made by a human so *symbolic artefacts* in discourse are written words, algebraic ideographs, diagrams, graphs, iconic drawings. An artefact can also be a physical object, such as a CAS calculator, that is used as a tool. [See section 2.5.3]

Colloquial discourse: discourse involving words or expressions that are neither formal nor literary. [See section 2.5.3]

Commognitive approach: A discursive approach to doing mathematics which links thinking and communication, with an emphasis on language and communication, where learning is viewed as communicating in the discourse of a specific community. [see Section 2.1]

Commognitive conflict: A situation that arises when "communication occurs across incommensurable discourses" (Sfard, 2008, p. 296), that make different use of words, visual mediators or routines. This can cause seemingly contradictory narratives to be endorsed. [see Section 2.7]

Discourse: in the commognitive approach is a form of communication that can be distinguished by four characteristics— its objects (vocabulary and syntax), the kinds of visual mediators used, routines and endorsed narratives (rules followed by participants in the discourse). [see Section 2.3]

Discursive objects: Mathematical objects (e.g., numbers, variables and functions) are not accessible to our senses. Learners of mathematics therefore construct abstract objects through their discourse, which are discursive objects. [see Section 2.3.3]

Discursive footprint: The discursive characteristics of how particular mathematical objects such as tangents are presented in different areas of mathematics during a student's studies. [see Section 3.2.1]

Doing mathematics in the **commognitive approach** is the active engagement with mathematical discourse. [see Section 2.5.3]

Dynamic interactive mediators (DIMs) are mediators which both change over time (i.e., they are dynamic) and respond to a person's manipulations (i.e., they are interactive). [see Section 2.6]

Endorsed narratives: These are those stories about mathematical objects and the relationships between them that the mathematical community accepts as valid (e.g., definitions, proofs, theorems). We can, for example, look at whether learners use keywords and visual mediators to construct endorsable narratives pertaining to various types of problem solving in mathematics (e.g., solving linear equations). [see Section 2.5.3]

Explorative discourse is discourse which contains endorsable narratives about mathematical objects. [see Section 2.5.3]

Grammatical complexity is the use of "grammatical devices such as complex nominal groups and repetitive or recursive use of subordinate clauses" (Morgan, 2016, p. 127) that need to be unpacked by the reader of a mathematical problem/question to identify the mathematical information involved. [see section 5.2]

Instrument: An artefact which has a special relationship with its user, for carrying out particular tasks. [see Section 3.3.2]

Instrumental thinking: Recalling and executing ritualised procedures, without understanding the reasoning behind them. Rote learning is an example of this. [see Section 2.2.1]

Keywords: Words of great significance in a discourse. [see section 2.5.3]

Logical Complexity measures the complexity of logical relationships that are present both within and between statements. [see section 5.2.1]

Meta-rules: Rules that govern other rules or describe how they should be used. [see Section 3.2.1]

Misconception: The systematic use of a concept across different contexts in a way that is different from the way in which it is used by experts (for example, the 'change sign change side' technique for solving equations which involve addition and subtraction). [see Section 3.2.2]

Narrative: Any set of utterances, “spoken or written, that is framed as a description of objects and relations between objects or about processes with, or by, objects” (Sfard, 2008, p. 300). [see Section 2.3.1]

Nominalisation is a grammatical metaphor which is used when a mathematical process is converted into a noun associated with a mathematical object. [see Section 5.2.1]

Objectification is the process of naming objects to allow a focus on their properties and the relationships between them (rather than having a focus on processes). [see Section 2.3.3]

Objectified discourse: The learner is acting with mathematical objects. [see Section 2.3.3]

Precedent events: Events associated with past situations that are interpretated by a learner as being sufficiently similar to the current setting to be appropriate to replicate in responding to the current situation, in which they consider that they are required to act [see section 3.2.1].

Precedent search space: The set of all precedent events that are accessible and relevant to a learner when entering a particular environment, in deciding how to respond to a current situation in which they consider that they are required to act. A default precedent search space is based on “the unarticulated assumption that precedents for whatever happens in this setting should come from the same discursive, material, institutional and historical context” (Lavie et al., p. 160) [see section 3.2.1].

Realisation: Response produced by a signifier (where a signifier can produce different responses in different people). [see Section 2.3.3]

Reification: This occurs when moving from talking about actions or processes to talking about associated objects. [see Section 2.3.3]

Rituals: These occur when a learner follows or mimics rules that have been have been set by an authority figure such as an instructor. [see Section 2.5.3]

Routines are the types of repetitive discursive actions, governed by rules, which are demanded of particular actors in a mathematical community. [see Section 2.5.3]

Signifier: A primary object which mediates meaning between two entities. Each signifier has particular significance for a learner. [see Section 2.7]

Subordinate clauses: A clause that is part of, and depends on, a main clause. [see Section 5.2.1]

Visual mediators: Physical objects which assist participants in mathematical discourse in communicating about mathematical topics, with communication being the reason for the creation of a visual mediator. They include “different types of symbolic artefacts”, such as “written words, algebraic ideographs, diagrams, graphs, and various iconic drawings” (Morgan & Sfard, 2016, p. 101). [see Section 2.5.3]

Publications during Candidature

Tobin, P., & Weiss, V. (2011). Teaching differential equations in undergraduate mathematics: Technology issues for service courses. In J. Hannah, & M. Thomas (Eds.), *Te Ara Mokaroa: The long abiding path of knowledge*, *Proceedings of volcanic Delta 2011* (pp. 375–385). Rotarua NZ.

Tobin, P., & Weiss, V. (2015). Teaching undergraduate mathematics using CAS technology: Issues and prospects. *International Journal for Technology in Mathematics Education, 23*(1), 35–41. DOI: 10.1564/TME_V23.1.04

Weiss, V., & Tobin, P. (2016). Use of calculators with computer algebra systems in test assessment in engineering mathematics. *ANZIAM Journal, 57(EMAC2015*), C51–C65. https://journal.austms.org.au/ojs/index.php/ANZIAMJ/issue/view/80

CHAPTER 1

THE NATURE AND PURPOSE OF THESIS

1.1 Rationale

Technology has frequently had a role to play in curriculum change in mathematics units, although such curriculum change frequently also generates debate and controversy (see, e.g., Mao et al., 2017; Risser, 2011; Sevimli, 2016; Weigand, 2017; Weiss & Tobin, 2016). Technology affects how mathematics problems can be solved (Borba & Villareal, 2005; Jacinto & Carreira, 2017). Hence, both the teaching and learning environment and the assessment environment can be impacted (Buteau et al., 2014; Stacey & Wiliam, 2013). The combination of graphing capabilities and algebraic functions present in Computer Algebra Systems (CAS) in calculators and computer packages, for example, can have an especially large effect, particularly in some Calculus topics (Hong & Thomas, 2015; Sevimli, 2016; Shahriari, 2019; Weigand, 2017). CAS can greatly reduce the number of steps required to solve some types of mathematics problems (Varbanova, 2017), and its multiple representation algebraic and graphing capabilities can help in determining and interpreting solutions to a wide range of problems (Leigh-Lancaster & Stacey, 2022).

Written assessments are impacted by CAS (Leigh-Lancaster & Stacey, 2022) both in preparing students to take them (Weigand, 2017) and in the actual texts and visual imagery used in them. The presence of CAS could also change the nature of online assessments, but I am not targeting those in this thesis. Use of CAS software on computers commonly occurs on student assignments and class exercises in units where use of CAS is encouraged. In recent years, CAS software has also become available on other types of portable devices such as Smart phones (Roanes-Lozano, 2017), often at a more affordable price. However, handheld CAS calculators are still useful due to their portability and their practical convenience for use in examinations, in part because they do not have the remote access capabilities of these other types of devices which can lead to “concerns about inequities associated with students in examinations accessing extraneous information on a hard drive, or via a piece of software, or by personal communication of some kind” (Kissane, 2020).

In this thesis, in agreement with Morgan (2016), mathematics is “understood as a discursive practice, [which] includes not only the product of mathematical activity but also the processes that give rise to it and the values of the practice” (p. 122). The language of written and spoken communication, combined with the manipulation of physical objects and artefacts, are features which determine the nature of the discourse of tertiary level teaching and learning, where this study is set. Assessment of this learning is often by static written and graphical forms such as diagrams, graphs, and screenshots of mathematically-enabled calculators or computer software outputs that appear on hardcopy tests and examination papers. My focus is on the impact of CAS technology on mathematical problem solving in undergraduate Calculus units as evidenced by changes in assessment from the perspectives of what is required of students, how a test or examination paper speaks to students and positions them as knowers and users of CAS technology, and how they are likely to interact with such questions. Undergraduate Arts students, including some who want to prepare to be secondary school mathematics teachers enrol in mathematics units where, in most cases, use of CAS is embedded in the learning materials, including the unit notes. In this thesis, commognitive analysis (Alshwaikh, 2016; Morgan & Sfard, 2016; Thoma, 2018) has been used to analyse the discourse of examination questions and to capture the nature of the potential approaches taken by students to typical types of undergraduate Calculus questions when they have access to CAS technology, especially in types of questions where intrapersonal commognitive conflicts between the technology and the student can occur, or where the resulting CAS output is different from the results they would have obtained if solving the same mathematics problems by-hand.

1.2 Background to the Study

CAS, an abbreviation for Computer Algebra System, is software on a computer, calculator or other handheld device that is distinguished from other types of software by its ability to solve and represent solutions to mathematics problems algebraically. CAS calculators and computer software were introduced in the late 1980s. They have been mandated in final-year school mathematics examinations in Victorian schools since 2010 (Leigh-Lancaster & Stacey, 2022) and in Western Australia since 2008 (Kissane et al., 2015), while by contrast in New South Wales, only scientific calculators are permitted in final year mathematics examinations at the end of secondary schooling. However, their use at university level, especially in examination assessments, has often generated debate and controversy, not just in Australia but also internationally (Bressoud et al., 2016; Risser, 2011). In some countries, part of this concern has in the past related to practical issues around how CAS can be incorporated into examination assessments fairly in cases where there might be some students who do not have access to the CAS technology including, for example, in Sweden (Pantzare, 2012) and New Zealand (Oates, 2011). More recently, given ongoing upgrades to CAS technology in Australia and elsewhere, it is important to assess students "equitably considering the several brands and models of devices that students can use" (Leigh-Lancaster & Stacey, 2022, p. 9). Concerns have also been raised around how to write CAS-active examinations in a way that still "tests the student's understanding of the underlying mathematical concepts and problem solving strategies" (Pountney et al., 2002, p. 15), so as to still have a comparable level of difficulty to previous CAS-free questions and to also still "test mathematically important content [and] to encourage students to learn mathematics deeply" (Leigh-Lancaster & Stacey, 2022, p. 9). However, CAS also presents opportunities to test conceptual knowledge in different ways, through its multiple representation capabilities, its capacity to solve problems (e.g., some equations) that cannot be solved analytically by-hand, and its ability to save time to allow more focus on testing conceptual knowledge by removing the need for some procedural steps that are necessary when working by-hand (Hong & Thomas, 2015; Jankvist et al., 2021).

Following on from this, while CAS technology has been available for some decades, the extent to which it actually has been adopted both in schools and universities varies widely, both in Australia and overseas (Billman et al., 2018; Buteau, et al., 2014; Lavicza, 2010). As discussed in Heid et al. (2013), in most countries "the impact of technology on school mathematics has to date been marginal and the incorporation of CAS in classrooms even slower" (p. 599). The use of CAS at university level has also been less common in summative assessments such as examinations (Buteau et al., 2014) than in "normal teaching" (Jankvist et al., 2021, p. 116). In contrast, Mao et al. (2017), surveying a nationally representative sample of 7087 students enrolled in first-year Calculus at 314 colleges and universities in USA, found 94% of students reported using calculators, describing calculator use in high school mathematics classes as ubiquitous, with the vast majority having graphing facilities, and 42% of students reporting using them in every lesson. Even in settings where use of CAS technology is approved for use in high stakes pre-tertiary entry examinations, such as schools in Victoria and Western Australia, the actual extent of CAS calculator competency that can be assumed at tertiary level varies. This is in part due to the variation in proficiency with, and attitudes towards, the calculators held by individual school teachers and students themselves. Pierce and Bardini (2015), for example, report a survey of first-year Victorian university students' perception of the use of technology for teaching and learning mathematics they previously received in their final years at school. They found that "at one extreme a few teachers were perceived to use CAS often in most topics … however, this pattern of use is evident for only a small percentage of the students' teachers … most teachers apparently made very little use of CAS" (p. 39).

At university level, many mathematics and engineering departments ban use of CAS in examinations with the survey of Canadian mathematicians by Buteau et al. (2014), for example, showing that only 22% of all CAS-user respondents integrated CAS, at least occasionally, in final examinations, and only 27% in classroom tests (which was 19% of all respondents). Similarly, even though CAS calculators are required in year 12 mathematics examinations in Western Australia, a survey of CAS usage in universities in Western Australia (Kissane et al., 2015) found that handheld CAS calculators were generally banned in university mathematics examinations. Kissane et al. (2015) concluded that CAS calculators are "regarded as computational devices and not regarded as tools that might be used for either teaching or learning" (p. 49). The use of CAS (and other types of educational technology) for university-level mathematics teaching and learning varies from one unit to the next, depending largely on the attitude to, and proficiency with, CAS calculators of individual instructors (Buteau et al. 2014; Jarvis et al., 2014). CAS integration in tertiary mathematics classes has in the past been found to occur more frequently in mathematical majors rather than in service units in other courses (Buteau et al., 2010). In a study of two mathematics departments in Canada and the United Kingdom, Jarvis et al. (2014) concluded that implementing and sustaining use of CAS at departmental level required, among other things, support from a group of academics who are committed to curriculum change using CAS, led by an advocate with power or influence within the department and supported by an administration who are in favour of creative reform of teaching, including "well-considered risk taking" (p. 117). Buteau et al. (2014) noted that "[t]wo main factors impeding CAS integration are the departmental culture and the time required for designing CAS-based resources" (p. 35). Jankvist et al. (2021) also observe that at universities in the United Kingdom the use of CAS "relies heavily upon the teaching staff of the university in question" (p. 106) and is still not common.

In cases where CAS is permitted in university level examinations, some existing research studies (e.g., Jankvist et al., 2021) include examples of questions from actual examinations that have been specifically designed with CAS in mind, often with the objective of testing students' level of problem-solving ability and/or conceptual understanding. There are also research studies which include examples and recommendations on how past examination questions written for an environment in which CAS was not permitted can be modified and/or extended in the presence of CAS to better assess students' conceptual understanding (e.g., Brown, 2001). In looking at considerations in writing appropriate examination questions when CAS is permitted for upper secondary external examinations, Flynn and McCrae (2001) emphasised that appropriately wording and structuring such questions is very important, as well as ensuring that the capabilities of CAS are taken into account. Clearly, similar considerations apply in such examinations at university level (Thomas et al., 2017).

In determining the impact of CAS on examination questions and on assessment more generally, one consideration is the extent to which students are being tested on their knowledge of the required mathematics and the extent to which they are being tested on their technical ability in using the CAS (Jankvist et al., 2021) as "[t]here is value in both, but there is a distinction" (Meagher, 2001, p. 2). Concerns have also been raised about questions which could be considered to be "CAS trivial" (MacAogáin, 2000) and "no longer suitable" to ask in examinations in which use of CAS is permitted, for instance as the result of "reduc[ing] down to two or three steps, such as enter the expression and differentiate" (Flynn & McCrae, 2001, p. 224). This suggests that altering examination processes is necessary when CAS is permitted (Stacey, 2005), whether this be at upper secondary or tertiary level. However, there are also circumstances where an argument could be made for including a few questions of this type in examinations: for example, if instructors wanted to include questions that focus solely on testing students' technical capability with the CAS device.

However, the number of 'steps' required for the student to obtain the required answer is not the only consideration when analysing the nature of examination questions. Previous research on discourse has also shown that "non-identical statements regarded by a mathematically versed person as having 'the same mathematical content' may be seen by the student as anything but equivalent" (Morgan & Sfard, 2016, p. 98). The nature of the discourse present in a given question, including both textual indicators and the presence or absence of visual mediators, contribute to the “dissimilarities between examination questions that can make a difference to the student’s vision of mathematics and to their performance” (Morgan & Sfard, 2016, p. 98). These aspects of the discourse are therefore central to evaluating the level of difficulty and complexity of examination questions. Little research has been done to assess and compare the impact of CAS technology on these characteristics.

This demonstrates the need for a framework for further examining the likely impact and level of difficulty of CAS-active examination questions, by analysing the discourse of examination questions in terms of how they speak to students and the level of autonomy they provide to students, regarding whether or not the students have a choice to use CAS technology to successfully answer a given question.

1.3 Different Types of Learning

Given the capabilities of handheld Computer Algebra Systems calculators, they offer many opportunities to enhance students’ learning experience and, as argued in Stacey and Wiliam (2013), “technology has the potential to alter all aspects of the assessment process” (p. 722). Depending on how much freedom staff are given in implementation of CAS (Buteau et al., 2014) and on their previous experience in integrating technology into the classroom, in cases where it is part of a unit, the way and extent to which it is incorporated can vary. A major source of debate in relation to use of CAS in teaching is in the effect it has on students’ understanding of concepts (Buteau et al., 2014). However, there are also different definitions given of what is meant by “understanding” or different types of “learning”, with a common distinction being that between “procedural learning” and “conceptual learning”. For example, Arslan (2010, p. 94) defines procedural learning as that which “involves only memorising operations with no understanding of underlying meanings” while “conceptual learning involves understanding and interpreting concepts and the relations between concepts”.

Because of its multiple-representation capabilities, which in many cases include both algebraic and graphical options for representing functions and solutions of equations, some argue that CAS can enhance students’ level of conceptual understanding. For example, Beaudin and Picard (2010) describe the experience of using CAS calculators in teaching mathematics units over a decade and note that these can be used to enhance students’ conceptual understanding, although this is often not done if calculators are merely used to generate answers replicating traditional procedural learning. Similarly, Nieto and Ramos (2021) observe that CAS can be “a useful support in the understanding of mathematical concepts and not only a tool for making tedious calculations” (p. 157). When discussing the impact of CAS on assessing students, Jankvist et al. (2021) also argue that technology can be used in “shifting the focus from any technical aspects to the learning of mathematics, problem solving and reasoning” (p. 101). However, in most mathematics units, much of the time teaching focuses on technique and procedure, as discussed in Tall et al. (2010). Many mathematics examination questions correspondingly test procedural understanding through expecting students to apply ritual routines involving procedural steps, rather than testing conceptual understanding through requiring students to think more creatively (Bergqvist, 2012; Matic, 2014) in interpreting a question asked, linking multiple representations of a function or solutions and/or using problem-solving and reasoning skills.

1.4 CAS-enabled Technologies and Calculus

Although CAS software is available on handheld calculators, desktop computers and increasingly on a wide range of other portable devices such as iPads and iPhones, handheld CAS calculators are the main focus of this research project as they are still the most commonly permitted form of CAS in mathematics examinations. However, use of CAS on handheld devices is much lower than use of computer-based CAS (20% to 80%) (Buteau et al., 2014). CAS computer outputs produced by programs such as Wolfram Alpha will thus also be considered when they appear in examination questions.

Calculus topics form a large component of typical first-year undergraduate mathematics units (Bressoud et al., 2016) and Calculus is also a topic area where CAS can have an especially large impact, as it is able to “compute virtually all limits, derivatives and integrals posed in most Calculus texts... and can handle the problem either symbolically, numerically, or graphically” (Palmiter, 1991, p. 151) as well as also having a big impact in other areas of Calculus including differential equations (Beaudin & Picard, 2010). This is, in part, because some types of Calculus problems (e.g., certain integrals, solving differential equations) can be time consuming to solve by-hand but can be done in one step using CAS. It is also because the multiple representation capabilities of CAS allow for linking of different types of representations of solutions to many Calculus problems (e.g., observing slope fields as a representation of the solution of a first order Ordinary Differential Equation (ODE), and observing the intersection of two graphs when using integration to determine the area bounded by two curves).

1.5 CAS-enabled Technologies in Assessments

Just as teaching students mathematics with CAS calculators can be done either by grafting on CAS as an "add-on" or by integrating it more thoroughly into the unit structure, considerations taken into account in designing examination questions also vary. One approach could be that the types of questions that used to be asked in examinations where CAS was not permitted be retained (Lehning, 2002). Alternatively, the nature of questions asked can be adapted to allow for the capabilities of CAS (e.g., questions where interpretation of graphical CAS output is useful in problems where students would more typically have solved the same questions algebraically).

Questions that show actual output generated by a CAS calculator or computer software can also be asked, to test students' understanding of the link between output produced by a CAS and the type of answer they would usually expect to obtain. For example, requiring students to see the link between an algebraic solution produced by a CAS, when it is not in the usual form the student would have expected to obtain by-hand (Tonisson & Lepp, 2015), or requiring students to interpret a graphical solution to a problem, such as a slope fields diagram relating to solving a first order ODE.

1.6 Aims and Research Questions

The aims of this research are to:

- See how the complexity and difficulty level of Calculus examination questions can be captured by analysing the discourse of these examination questions, both in cases where CAS technology is permitted to be used by students in the examination and in cases where its use is not permitted in the examination but where there are outputs present in some of the examination questions that have been generated using CAS.
- See how students taking a Calculus course use CAS calculators when solving a series of Calculus questions of a comparable standard to those asked in examinations, including some where errors are commonly made and/or the output

produced by CAS technology is different to what they would have obtained if solving the same problem by-hand. Analyse the nature of the students' resulting discourse, with a focus on how they reflect on their answers in task-based interviews and how they integrate describing their use of the CAS technology into this discourse.

- Contextualise the analyses of the task-based interviews with use of a questionnaire which explores how Computer Algebra Systems (CAS) technology is used and perceived by undergraduate Calculus students at the time of the study.

To address these aims the following research questions have been formulated.

Research Questions

1. How can a commognitive framework be applied to effectively capture the complexity and difficulty level of written answer and multiple-choice examination questions asked in undergraduate Calculus units where use of CAS technology is available?
2. (a) How effectively do undergraduate Calculus students use their CAS calculator and use and interpret CAS output, especially when it is in a different format to what they would obtain by the methods of working by-hand they have been taught in class?

   (b) What can we learn from a commognitive analysis of task-based interviews of students in relation to how they reflect on their answers in this situation?
3. To what extent do undergraduate Calculus students use CAS and when do they believe its use to be most beneficial?

1.7 Nature of the Study/ Methodological Approach

The study is set in a multi-campus Australian university that offers undergraduate mathematics units to students, many of whom are preparing to be secondary school teachers, while others are taking the mathematics units as part of a degree in other, non-mathematical, disciplines such as Arts. These mathematics units, which consist mostly of Calculus topics, are situated in different learning environments at two of the university's campuses: at one use of CAS is permitted in the final examinations but not in the mid-semester tests and use of CAS calculators (and examples of output from CAS software) is also embedded into the course notes and is taught in class, while at the other campus, units covering similar topic areas are taught in a CAS-inactive learning environment and CAS calculators are not permitted in the final examinations. Examinations, mid-semester tests and learning materials from a selection of these units at the campus where CAS is used in the learning materials and permitted in the final examinations were chosen for analysis in this study, because this learning and assessment environment provided the opportunity to analyse and compare the discourse of test and examination questions in an environment where CAS could potentially have a noticeable impact. The mid-term tests, which were multiple-choice with CAS calculators not permitted but with CAS screen output sometimes included in individual questions, were also included in the analysis, as CAS-free assessments of this type are common in both types of learning environment and therefore were needed to provide a full picture of the different types of discourse and visual mediators present in such assessment tasks. Analysis of multiple-choice mid-term test papers from a Calculus unit at a second university, where CAS was permitted in the tests, has also been carried out, to see the possible impact of CAS on multiple-choice questions when its use *is* permitted in this style of test.

Selected questions from the test and examination papers just described have been analysed qualitatively using a commognitive analysis framework adapted from that of Morgan and Sfard (2016). In addition, responses of four student participants who separately solved a series of problems and reflected on their answers during task-based interviews were analysed qualitatively using commognitive analysis. The results of student surveys on the extent of use and their attitudes to CAS were also analysed to contextualise the results of the assessment question and task-based interview analyses.

1.8 Outline of the Thesis

This first chapter details the nature and purpose of the research study and sets out the research problem, the background for the study including the use of CAS calculators in undergraduate Calculus examinations, and how this links with students' overall approach to solving questions in such examinations and their attitudes to the CAS technology in general.

In chapter 2, the theoretical background to the study is presented. This begins with an outline of the conceptual frameworks underpinning commognitive analysis, including communicational theory (Sfard, 2008) and social semiotics (Halliday, 1978). It is followed by an overview of the epistemological realisations that commognition has contributed to educational research. Sfard's commognitive analysis framework (2008) is described in detail, followed by a description of how other researchers have further modified and extended this framework. Finally, this chapter includes reference to studies that have applied a commognitive analysis framework to analysing mathematical discourse in a variety of settings, including in textbooks, examinations (both the questions and students' answer attempts), classroom problem solving group studies and in task-based interviews.

In chapter 3, ways of capturing the variation in the discourse of Calculus at undergraduate level are reviewed, along with how this discourse and the associated Calculus topics are affected in the transition from high school to tertiary level. The nature of the graphical and symbolic visual mediators produced by CAS are then considered, with an emphasis on how they can affect the learning and discourse of Calculus. Literature on the use of Computer Algebra Systems technology in examinations is then reviewed, including the extent and nature of its use in senior high school and university examination assessments, its impact on the discourse of examination questions and the type of questions it is suitable to ask, and ways of classifying or otherwise analysing the impact of CAS. Literature relating to practical factors in running such examinations will also be considered. Finally, research on students' attitudes to using CAS is considered, including looking for any relationships between students' attitudes to use of CAS and how they use it when solving mathematics problems.

Chapter 4 begins with an overview of the study, including the aims and associated research questions. A summary of the methods that were used to analyse the data is then provided. The situational context of the study and the relevant characteristics of the participants are then described, with the latter including reference to the mathematical educational background and expected level of familiarity with CAS technology for these students from their previous studies of mathematics in high school. An overview of the types of data that were collected follows, including the types of test and examination scripts (e.g., written answer or multiple-choice, CAS active or CAS inactive, year level, topic areas covered), and the background learning environment (CAS active or CAS inactive). The protocols for the task-based interviews and the student questionnaire administration are provided. The adaptation of Sfard's (2008) commognitive framework that was used in this study is described in detail, linking it to the relevant parts of other commognitive frameworks in the existing literature on commognition, and outlining my new additions to the framework, which relate to discourse on the use of technology in the examinations.

Chapter 5 is the first of two results chapters. In this chapter, I present a commognitive analysis for a selection of Calculus examination questions that vary in the use of technology and the extent and type of written discourse involved. The questions vary as to whether they are short answer or multiple-choice, include visual mediators in the form of CAS screen output, CAS technology is/is not permitted to answer them, and in the characteristics and extent of written discourse present. In applying the commognitive framework to each question, the different aspects of mathematisation and subjectifying the discourse in the framework are described, with a focus on how the presence and use of technology in each case affects the discourse and level of complexity of these different types of examination questions, as well as the degree of student autonomy in answering the questions.

Chapter 6 is the second results chapter. In this chapter, I present the analysis of the task-based interviews using a commognitive analysis framework, which has the same components as those outlined in the previous chapters, to analyse both the written answers and the interview transcripts for these students. Because the students were interacting with their calculator screen environment in attempting to solve the mathematics problems provided to them, an additional aspect of the commognitive framework, derived from the work of Ng (2016), concerning dynamic versus static diagrams, was especially relevant to this analysis. These results are contextualised by the responses and analysis of the questionnaire data, which looked at the attitudes of students from the same cohort regarding their perspectives and their use of CAS, their perceptions of the utility of CAS, and their preferences for CAS or by-hand methods and access to CAS in assessment.

Chapter 7 brings together and discusses the key findings of the results from the previous two chapters, in light of the existing research literature on commognition, Calculus, assessment and students' attitudes towards CAS.

Chapter 8 concludes the thesis. The implications which follow from these results for mathematical and educational practice in the field of university mathematics assessment using CAS technology are discussed. This chapter also points out limitations of the study, together with consideration of the advantages and challenges in using commognition to analyse the types of data considered in this thesis. The wider theoretical contributions of the study are outlined, as well as providing overall conclusions and recommendations for further research following from the results of this study.

CHAPTER 2

THEORETICAL BACKGROUND

2.1 Overview

In chapter 2, I present the theoretical background to the study. Chapter 1 gave an overview of the role and impact of Computer Algebra Systems (CAS) technology in mathematics examination assessments, together with an outline of the aims and research questions of this thesis. As seen in Chapter 1, CAS technology can have an impact on the nature of examination questions asked, due to fewer procedural steps being required to answer many types of Calculus questions when CAS technology is available. However, little research has been done on how it specifically affects the *discourse* of such examination questions, which is to be investigated in this thesis, using a commognitive analysis framework.

The chapter begins with an outline of the theoretical perspectives underpinning commognitive analysis, including communicational theory (Morgan & Sfard, 2016; Sfard, 2008), social semiotics (Halliday, 1978), and considerations of human thinking as a form of communication. Sfard's commognitive analysis framework itself is then described in detail. Situated studies that have applied a commognitive analysis framework to analysing mathematical discourse in a variety of settings, including in textbooks, examinations (both the questions and students' answer attempts), classroom problem solving group studies and in task-based interviews then follow. Some of these researchers (e.g., Alshwaikh, 2016; Thoma, 2018) have not only applied the framework but also modified and extended it.

Commognition was developed by Sfard (2008) to challenge and expand on traditional visions of human development, including the development of human thinking and learning. Commognitive analysis connects thinking and communication, so the first section of this chapter starts with an overview of theories of thinking, followed by a second section on theories of communication. Then follows a section linking thinking and communication and lastly sections looking at the theory of commognition and its applications.

More specifically, the second section, 2.2, of this chapter examines developments in understandings of what constitutes human learning and includes an overview of behaviourism, cognitivism and participationist theories on learning, and relates these to epistemological issues about learning. This is relevant to the analysis of examination questions, as they are designed to assess human learning (in mathematics topics). Section 2.3 on human communication focuses on discourse, particularly in relation to the participationist model of thought/learning, as this is the most applicable to this research on commognition. The associated issues of the use of, and transplanting of, metaphors in discourses and objectification, as well as the distinction between monological and dialogical discourses are discussed. Both these types of discourse can be present in mathematics examinations and have different implications regarding, for instance, the relationship of the examiner and the examinees. Section 2.4 looks at theories that have linked thinking and communication, especially in terms of discourse. An overview of Sfard's Commognitive framework (Sfard, 2008) is then presented in Section 2.5, as this underpins the theoretical framework and the commognitive approach to analysis of the data from the study. The derivation of commognition (Sfard, 2008, 2020) from communication and cognition is a different, but useful, way of talking/writing about thinking, namely, as communicating. Links to the works of Wittgenstein (1921/2009, 1961) and Vygotsky (1978) are presented, before proceeding to an outline of Sfard's actual commognitive framework and its extensions and applications (section 2.6). Section 2.7 overviews the notion of commognitive conflicts, which are an important potential source of error when students are navigating between different types of mathematical discourse, in which the same words or symbols can signify different meanings. The chapter is concluded in section 2.8.

2.2 Development of Human Learning and Thinking

The development of human learning is related to a key question in educational research which is: what type of changes occur when learning takes place? Learning and changes in theories of what learning is and how it takes place are relevant to this research thesis, as examination papers and tests are assessing students' learning. Below is an overview of some of the main theories of thought and learning from the past century that have been considered relevant to addressing this question. Those addressed here are behaviourism, cognitivism and participationism.

Behaviourists do not believe in the validity or usefulness of the concept of human consciousness with John B. Watson, who was one of the most influential theorists of behaviourism claiming in his book, *Behaviourism* (Watson, 1924), that "consciousness is neither a definable nor a usable concept" (p. 3). Following on from this, behaviourists believe that behaviour (which includes learning) can be studied in a systematic and observable manner, without requiring knowledge of a person's internal mental states. Classical behaviourists also believed that every human response is elicited by a specific stimulus, as a consequence of "the relation between publicly observable behaviour and publicly observable variables in the environment" (Moore, 1999, p. 43).

Following on from this, neo-behaviourists argued that all learning (and behaviour) can be described in terms of conditioning, where "operant conditioning" (Skinner, 1937) is a type of learning in which changes in an individual's behaviour occur as the result of responding to the consequences of the individual's previous behaviours of a similar type. Later Hull (1943) called this instrumental learning and he believed that human behaviour is an interaction between organism and environment, in which the environment provides the stimulus and a response is given by the organism.

While behaviourists acknowledged that thinking exists but identified it as a behaviour, cognitivists argued that the way people think affects their behaviour, believing it necessary to consider mental functioning in order to be able to properly understand behaviour (Chomsky, 1959). Cognitivists argue that learning is changes to the contents of the mind, rather than changes in how likely people are to respond to a stimulus in a particular way, with learning "concerned not so much with what learners do but with what they know and how they come to acquire" (Ertmer & Newby, 2013, p. 51) that knowledge. Cognitivists see the process of learning as involving "transferring information from our environment into our long-term memories" (Bourne et al., 1986, p. 91), a process of *acquisition* of content. A central aspect of cognitivist theory is Piaget's concept of *schemata* in cognitive development, as cognitive structures used in the construction of knowledge, that allow an individual to retain, group and organise aspects relating to particular objects and experiences. In turn, each schema has a logical connection "with all the other schemata and itself constitutes a totality with differentiated parts" (Piaget, 1936, p. 7).

There are differing viewpoints among cognitivists, as to factors which contribute to how knowledge is acquired. Chomsky (1957, 1965) believes acquisition of knowledge occurs due to humans being born with a universal grammar, which is a basic understanding of how to communicate, giving them an innate capacity to quickly acquire and develop language in similar ways (Chomsky 1957, 1965). In contrast, Piaget (1970) rejects the notion of humans being born with such an innate capacity, believing that acquisition of knowledge results from a gradual process of construction, influenced to a greater extent by events in the environment a person is interacting with. As a part of this, Piaget studied cognitive development in children, believing this to be the only practical way to gain understanding of how knowledge is acquired (Bodner, 1986). Piaget concluded that cognitive development in children occurs in four stages: from initially acquiring understanding of the world through sensory and motor interactions with physical objects, to then being able to use symbols in representing objects and events, followed by developing the ability for logical thought about these and lastly, from about age twelve onwards, being capable of scientific reasoning, including the ability to think abstractly.

However, one limitation of the acquisitionist approach of cognitivism in general is that it is focused exclusively on how an individual acquires and processes knowledge, without taking account of the dynamics of historical change in influencing and changing human actions. In contrast, sociocultural theory looks at the contributions that society makes to the development of individuals, including how they learn, by viewing mind as mental functioning “inherently situated in social interactional, cultural, institutional and historical context[s]” (Wertsch, 1991, p. 86). In turn, sociocultural theory has led to the evolution of participationist theory, where participation in collective activities leads to the development of individualised ways of doing, rather than the opposite trajectory of development as advocated by acquisitionists (Sfard, 2008). Advocates of participationist theory (see Lave & Wenger, 1991; Rogoff, 1990) are wary of broad or generalised claims about ‘cognitive invariants’; whether across cultural borders, different historical periods or even transferred by an individual from one situation to another. However, this theory does not deny the existence of *any* cognitive invariants. Advocates of participationism claim that “in those processes of learning that are unique to humans, the learner becomes a participant of well-defined historically established forms of activity” (Sfard, 2020, p. 95).

Participationists see learning as “the development of ways in which an individual participates in well established communal activities” (Kieran et al., 2002, p. 23). In particular, participationists are most interested in the ongoing *interactions* that lead to the development of these activities, rather than individual characteristics that account for patterns of visible behaviour (Kieran et al., 2002). This therefore does not preclude there also being some knowledge acquisition, as suggested by cognitivists. All this means that while the focus for acqusitionist researchers is to identify unchanging indicators of learning across contexts, participationist researchers focus on the activity itself and "its changing, context-sensitive dimensions" (Kieran et al., 2002, p. 24).

Jean Lave was one of the first, in the late 1980s, to be critical of the then widely accepted cognitivist discourse "for all the weaknesses typical of all objectified discourses" (Sfard 2008, p. 77), where keywords are used as if they signify objects that exist independently of this discourse. Lave and Wenger (1991) called for terms such as knowledge acquisition and learning transfer to be rejected and instead learning be thought of "as *legitimate peripheral participation* in communities of practice" (p. 94). Also relevant to participationist theory is activity theory which grew out of the work of Vygotsky (1978) and associates such as Leontiev (1977/2014). Leontiev emphasised the importance of focusing on the activity of participants in learning, thus seeing the activity of the subject as an interconnecting link between object and subject, presenting a "subject-activity-object" pattern" (Leontiev, 2014, p. 160).

Implications of this participationist model of thinking for learning and discourse in mathematics will be discussed in section 2.4, when looking at the connections between thought and communication, suffice to say at this point that the contribution of participationist theory has been to suggest that thinking is an inherently individual form of human doing which has developed from a patterned collective activity (Leontiev, 1977/2014; Sfard, 2020), that is, human communication. Before proceeding to examining interconnection between thought and communication, I overview the development of human communication to establish a foundation for section 2.4.

2.3 Development of Human Communication

To understand the particular view of the development of human communication that underpins Sfard's commognition (2008), it is necessary first to understand what she means by discourse, how that construct aligns with what I am meaning when writing about discourse in this thesis, the role of metaphor and metaphorical projection in bringing about discursive change, and finally the notion of object and the process of objectification. These ideas will now be dealt with in turn in the coming sub-sections.

2.3.1 Discourse

In general terms, ‘discourse’ can be defined as “written or spoken communication or debate” or, in linguistic terms, as “a text or conversation” (Stevenson & Waite, the Concise Oxford Dictionary, 2011, p. 409). While the term ‘discourse’ has historically been defined differently depending on the context, the work of Foucault (1981) informs the basis of the definition that is relevant to this research. His definition of discourse includes types of communication and representation activities that are governed by a set of rules, both explicit and implicit, that both enable an activity and, at the same time, impose limitations on it. The work of proponents of Critical Discourse Analysis, such as Fairclough (1993), are also relevant here, as they view language as a social practice. Fairclough (1993), for example stated that

> viewing language use as social practice implies, first, that it is a mode of action … and, secondly, that it is always a socially and historically situated mode of action, in a dialectical relationship with other facets of 'the social' (its 'social context') - it is socially shaped, but it is also socially shaping, or *constitutive.* (p. 134)

Following on from these ideas, Sfard (2008) defines discourse by the delimiting nature of “different types of communication … that draw some individuals together while excluding some others” (p. 91). Different communities of communicators share objects, the kinds of mediators used, and the rules they follow as participants (Sfard, 2008, p. 93), for example, educational researchers form a discourse community but so do mathematics education researchers, educators and mathematicians and these discourse communities overlap to more or less extent. In her research, Sfard uses the term *discourse* to denote instances of communicating, whether in the present or developing over time, “whether with others or with oneself, whether predominantly verbal or with the help of any other symbolic system.” (Sfard, 2001, p. 28). Thus, any society consists of several “partially overlapping communities of discourse” (Sfard, 2008, p. 91). To become a member in the wider community of discourse, one must participate in the communicative acts of any collective that is one of these communities of discourse with their shared objects, mediators and rules to be followed (Sfard, 2008).

In the research community of discourse, for example, there are smaller communities which use what are called monological and dialogical discourses which shape how researchers present their narratives about the world and human activity such as communicating, understanding, and thinking. The Russian philosopher, Bakhtin (1986, p. 163), in an essay, *Toward a Methodology for the Human Sciences*, wrote about the notion of “monologization” where multiple other human voices are obliterated in utterances (text or talk) and only one remains, what Hays (2008) labels as “the author’s unitary truth” (p. 69). The ideology of a monologic work “inevitably transforms the represented world into a voiceless object of [its] deduction” (Bakhtin, 1984, p. 83) producing "an objectified world, a world corresponding to a single and unified authorial consciousness" (Bakhtin, 1984, p. 9). The narratives of monological discourses are seen as “depending on the world itself rather than on the human storyteller” ... [and] “are also fully alienated” (Sfard 2008, p. 66), that is, using discursive forms as if they occur by themselves without any participation by people. For example, the statement “the equation of a straight line is $y = mx + c$, where m is the gradient of the line and c is the y-intercept” is monological discourse. By contrast, in a dialogical discourse text or talk is always part of an ongoing conversation between two or more people in some activity and “the speaker... is oriented precisely toward ... an actively responsive understanding...[and] does not expect passive understanding ... [duplicating the speaker’s] idea in someone else’s mind [instead expecting] response, agreement, sympathy, objection, execution, and so forth” (Bakhtin, 1986, p. 69). A dialogical narrative is offered by its authors as one of several possibilities; but it is expected to be convincing enough to be heard and later even endorsed by the discourse community (Sfard, 2008, p. 66). An example of dialogical discourse, between a teacher and her students in relation to finding the equation of a straight line, would be as follows:

Teacher: Suppose we want to find the equation of a straight line and we know the values for two points on the line. How would you suggest we do this?

Student A: We could plot the points and draw the line through them, and use this to estimate how steep it is and where it meets the y axis.

Student B: We could calculate the slope and substitute in one of the points to solve for the y-intercept.

Student C: We could enter the values for the two points into our CAS calculator and use the linear regression command.

Teacher: Let’s now have a discussion about what you all see as the relative advantages or disadvantages of these different approaches.

Stradling the two "worlds" (i.e., the discursive communities of mathematicians and mathematics education researchers), I now see that, to progress my understanding in these communities, I have to become a member of both by participating in the communicative acts of both of these communities of discourse with their shared objects, mediators and rules to be followed. In addition, taking a dialogical approach to research discourse will help me and others "realise that some of the objects that populate" text and talk in the research and work communities in our lived experiences "are but discursive constructs and, as such, may be removed or redefined" (Sfard 2008, p. 92).

2.3.2 Metaphor

Sfard recognises the importance of metaphors in communication and she defines their general nature as "the action of 'transplanting' words from one discourse to another" (2008, p. 39). More importantly, metaphors allow us "to organise new experiences in terms of those with which we are already familiar" (Sfard, 2008, p. 40), facilitating how we make sense of those experiences. Reddy (1979), in his seminal publication on metaphor, showed us "how words characteristic of one discourse may take us in a systematic way to another, seemingly unrelated one" (Sfard, 2008, p. 39) by using words in the transport discourse to figuratively project to communication discourse. Metaphor was now seen as more than just a literary device, rather a mechanism constitutive of new discourses. We now not only realised that new knowledge comes from old knowledge but that the mechanism for doing this was metaphorical projection (Sfard, 1998; 2008). The effectiveness of metaphors in heralding or bringing about discursive change is due to familiar words still being able to be used with the old rules that seem in agreement with the new context. Thus, "once the metaphorical term is introduced, the rules of its use are gradually modified, resulting in a whole new set of language games" (Sfard, 2008, p. 41). In terms of changed metaphors for learning, this would mean the need to learn changed "rules of the game" (Voigt, 1998) for both learning and assessment of learning. For example, transitions could be potentially made, by use of metaphor, from standard written classroom mathematical discourse to that involving commands used with a particular type of CAS technology, allowing the production of endorsable narratives around effectively integrating the use of that type of technology into a mathematics subject.

Implications of the *metaphors for learning* that guide our work in teaching, assessing and research were of central concern to Sfard. She argued that "implications of a metaphor are a result of contextual determinants" at least as much as they are a result of the metaphor itself (Sfard, 1998, p. 5). She identified the divide between cognitivist and participationist thinking and theorising about learning in the late 1990s as meaning that educational research was "caught between two metaphors" which she referred to as the "acquisition metaphor" and the "participation metaphor" (p. 5). According to the acquisition metaphor, learning is the acquisition of something and the goal of learning is individual enrichment. The student is the recipient or consumer of whatever is to be acquired and the teacher is positioned as a provider, facilitator or mediator of learning. Knowledge or concepts become an individual possession which can be made public and knowing is having or possessing whatever was to be acquired. In contrast, the participation metaphor frames learning as becoming a participant and its goal is community building. The student is seen as a peripherical participant or apprentice whereas the teacher is the expert participant in the community, a preserver of its practices or discourse. Knowledge is an aspect of practice or discourse or activity within the community and knowing is belonging, participating, or communicating in it (Sfard, 1998). Sfard (1998) further states that "all our concepts and beliefs have their roots in a limited number of fundamental ideas that cross disciplinary boundaries and are carried from one domain to another by the **language** we use" (p. 5). Not being satisfied with either metaphor of learning, Sfard (2008) proposed combining the terms cognitive and communicational to coin her own neologism, the adjective commognitive, rather than give a new meaning to the existing English word "communication".

However, Sfard (2008) cautions that the introduction of a new word metaphorically can bring a false sense of security in our sensing we understand as we cannot guarantee others are using it in the same way. It is thus important in what follows in this thesis to give operational meanings of new words aligning with how they will be used. This is particularly important in the field of commognition which is replete with new words (e.g., commognitive conflict or communicational action) or "old" words (e.g., object, alienation) operationalised in ways that at times are not expected. To mitigate this, I have provided a glossary in alphabetical order at the beginning of the thesis for quick reference for the reader, which also has cross-links to parts of the thesis where these words are first introduced.

2.3.3 Objects and objectification

The notion of an "object" is central to any discussion about discourses (Hindess & Hirst, 1977), including mathematical discourses of the types to be found in examination papers, but defining what is meant by "object" can be surprisingly difficult. With respect to material objects, such as a handheld calculator, an examination paper or a university, or intangible mental activities, Sfard (2008) notes that "people are said to act on, or to be somehow directed or constrained in their action by, an entity that, even if perceptually inaccessible, is implied to have an independent existence of sorts" (p. 43). When we as observers of actions describe our impressions of what we observe, we discursively turn actions into objects. On observing what two students do in a task-based interview we say, for example, "two interviewees constructed similar conceptions of the derivative function". The entity to which we point with the word "conception" is not directly observable. Once we start talking about "conception" as an object we have reified the original action. If we start presenting "conceptions" in an impersonal way as if they have an existence of their own separate from human participation, we lose sight of the metaphorical nature of the original use of "conception" in our statement. "All the objects around which commognitive narratives revolve must be understood as metaphors originating in discourses about material objects" (Sfard, 2018, p. 226), that is *metaphors of objects*. Metaphors of objects, in this sense, are more than substitutions for "things", they also are the initial creators of these "things" (Sfard, 2008). Discursive objects differ from *realisations*, as realisations are able to be perceived and accessed. Realisations are external representations of the word or words used to stand instead of the object in response to seeing or hearing the word or words. For example, "intersection of the curves $y^2 = 4x$ and $y = x - 2$" can be realised as a graphical representation showing the two curves intersecting. There can be more than one realisation for the word or words being used. The intersection of the curves could be realised in a table of function values, for example. Realisations are subjective and can differ from person to person (Antonini et al., 2020).

O*bjectification*, as just seen in the example in the last paragraph, is the process whereby new objects are constructed: a noun begins to be used as if it stands for a pre-existing entity in the world, independent of human involvement. For example, in describing Calculus calculations, saying, "The derivative for $x\sqrt{2x+1}$ is $\frac{x}{\sqrt{2x+1}} + \sqrt{2x+1}$", is an example of objectified discourse. In contrast, describing the same calculations in terms of what a person *did* with the function in performing the calculations as, “When Vida differentiated $x\sqrt{2x+1}$, she got $\frac{x}{\sqrt{2x+1}} + \sqrt{2x+1}$”, is an example of dialogical or subjectified discourse. Both these types of discourse are commonly present when talking about mathematics in mathematics classrooms, including in examination papers, making this distinction important for my current research.

Understanding the nature of the process of objectification is also essential. Sfard (2008) describes how this process involves both *reification* and *alienation*. Reification occurs when we substitute talk about actions with talk about objects, which we saw in the conception example above where a noun was introduced to describe actions we had observed. Alienation occurs when phenomena are presented in an impersonal manner, as if they could happen without human involvement as seen in the first statement in the example about the derivative for $x\sqrt{2x+1}$ where the actor, Vida, and her action of differentiating in the second statement has been removed and replaced by the noun “derivative” becoming the subject of the sentence.

2.4 Human Thinking as a Form of Communication

Communicational theory (Morgan & Sfard, 2016) recognises different areas of human knowledge including mathematics, as *discursive activities* (Kieran et al., 2002; Gutiérrez et al., 2010). This could be seen as following on logically from the work of psychologists and philosophers who believe it is not possible to separate thought from its expression. For example, Wittgenstein, who originally had accepted the idea that words were a naming of the things in the world, opposed the viewpoint that thinking is an “incorporeal process” that is the cause of speech and which gives it meaning, which it would be possible to separate from speaking (Wittgenstein 1921/2009). In many instances “the meaning of a word is its use in the language” (Wittgenstein, 1953/2003, p. 18). Meaning then becomes “an aspect of human discursive activity”- a point noted by Sfard (2008, p. 73) when developing the notion of commognition.

The participationist model of learning, inspired by the writings of Vygotsky (1978, 1987) and others, also suggests viewing human thinking as a form of communication. Vygotsky (1978) believed that all human processes have an inherently social nature, stating that

> Every function in the child’s cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (inter-psychological)

> and then inside the child (intra-psychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relationships between individuals. (p. 57)

Thus, the development of an individual is a process involving carrying forward higher mental functions from the social to the psychological plane of the individual.

This all relates to epistemological issues around the nature of learning. Sfard (2008) then developed communicational theory following on from these works of Vygotsky and Wittgenstein. Communicational theory "views language-based communication as central to all human activities" (Morgan & Sfard, 2016, p. 99) and is discussed further in the next section on commognition.

2.5 Sfard's Commognitive Framework

What is Commognition? In short, Communication + cognition! Sfard (2008) connects cognition and communication noting that they are "different manifestations of basically the same phenomenon" (p. 83). It is from this connection that she has developed the term *commognition*, which incorporates concepts from both social semiotics (Halliday, 1978; Hodge & Kress, 1988) and communicational theory (Sfard, 2008), which are discussed further in subsections 2.5.1 and 2.5.2, respectively.

2.5.1 Social semiotics

Semiotics is the study of signs, symbols, and signification. Its focus is on *how* meaning is created, rather than on what it is. Following on from this, social semiotics "is concerned with meaning makers and meaning making [and it investigates] the media of dissemination and the modes of communication that people use and develop to represent their understanding of the world" (Bezemer & Jewit, 2009, p. 1) and in relating to other people.

Social semiotics is a key component of commognition, as it opposes a common *content-form duality* assumption in discursive research which suggests that the form of communication (e.g., the way of teaching or testing mathematics) can be changed while keeping the content (e.g., the mathematical 'concepts' being taught or tested) intact. That is, there is an assumption of the separability of form and content. Social semiotics does not assume this duality, because "language and other communicational modes are *functional*, not representational" (Morgan & Sfard, 2016, p. 100). In other words, analysis of communication is focused on "what is *achieved* by the text within a particular *context*", rather than uncovering the author's intentions or determining "some absolute '*real meaning*' of the words that they refer to" (p. 100).

"Social semioticians also consider communication to be inevitably multimodal, involving various means of communication including language, images and gestures" (Alshwaikh, 2016, p. 166). As communication in mathematics is also multimodal including "language, diagrams, graphs and other forms, algebraic notations and gesture" (Alshwaikh, 2016, p. 169), this makes social semiotics a valuable tool for analysing mathematical documents, including examination papers. For example, a Calculus question on finding the volume of a solid generated by revolving a curve about the x-axis would present multimodal communication, involving written language describing the problem, algebraic notation giving the equation of the curve, and a graphical diagram to assist students to visualise the problem.

Halliday introduced social semiotics into linguistics in *Language as Social Semiotic* (Halliday, 1978) and also through his later work on systemic functional grammar and linguistics (Halliday, 1985). Halliday's (1978) theory of language emphasises the multifunctionality of language and sees any text as simultaneously enacting what he calls the 'ideational', 'interpersonal' and 'textual' functions of language (p. 134), with these three functions commonly referred to as 'meta-functions'. The *ideational meta-function* is concerned with people construing experiences and the 'reality' of the world by means of language, which is in opposition to "the traditional cognitivist perspective that language is a (more or less imperfect) means of representing pre-existing conceptual structures" (Morgan & Sfard, 2016, p. 100) and is concerned with clauses as representations. The *interpersonal meta-function* is concerned with the social world, especially in construing the identities and relationships of the participants in the communication and is concerned with clauses as exchanges. By focussing on the interpersonal function of language use, we can delve into "how participants position themselves and others within a social practice" (Morgan & Sfard, 2016, p. 100). This has application to the analysis of mathematics examinations considered in this thesis as it allows us to look at how mathematics examinations "position students with respect to mathematics and towards other participants of mathematical discourse" and to look at the "kinds of mathematics activities are students expected to engage in" (Morgan & Sfard, 2016, p. 100) and at how much autonomy they have when performing these activities. The *textual meta-function* is concerned with "construing the role of the text itself as part of a social practice" (Morgan

& Sfard, 2016, p. 100), such as the text on examination papers conveying how the presence of CAS technology should affect what an examinee can be asked in a written examination paper.

2.5.2 Communicational theory and discourse

Communicational theory is a theory for studying human cognition, which is used in Sfard's commognitive analysis. As described in Sfard (2001), one of its core premises is that "thinking may be conceptualised as a case of communication, that is, as one's communication with oneself" (p. 26). Taking this perspective, Sfard (2008) views thinking as dialogical engaging in exchanges with ourselves, informing, arguing, interrogating through asking questions and then waiting for our own responses.

In contrast to other areas of science such as chemistry and biology, "where the discourse and its objects are separate entities", Sfard (2008) describes mathematics as "a multilayered recursive structure of discourses-about-discourse, and its objects therefore are, in themselves, discursive constructs" (Sfard, 2008, p. 161). "Benzene" and "vertebrates" are names that are given to things that have a physical existence in the world whereas "function", "derivative" etcetera in mathematics have no separate physical existence, instead mathematics produces or creates the things it talks about. Sfard argues that claiming mathematics as a discourse is quite different from saying that mathematics is a language. The latter disguises this highly recursive structure. In turn, "if learning mathematics is conceptualised as a development of a mathematical discourse" (Sfard, 2001, p. 28) then, as we saw in section 2.3.1, this involves an initiation into the wider community of mathematical discourse, participating in the communicative acts of the collective of the mathematics classroom which is one of the communities of mathematical discourse with their shared objects, mediators and rules to be followed (Sfard, 2008).

2.5.3 Sfard's commognitive analysis framework

Commognitive analysis is a discursive approach to doing mathematics, due to its focus on language and communication, with learning viewed as communicating in the discourse of a specific community. In this study, the community is mathematics learners in a tertiary undergraduate Calculus-based mathematics course. As described in Sfard (2015), the works of some postmodern philosophers have also linked discursive activities to scientific research with statements such as "scientific knowledge is a kind of discourse" (Lyotard, 1979, p. 3) and have also linked it to "school-type learning" (Sfard, 2015, p. 132). In the commognitive approach, doing mathematics is the active engagement with mathematical discourse. In turn, mathematical discourse, like other discourses of "different communities of communicating actor" (Sfard 2008, p. 93), has four interrelated characteristic features: keywords (vocabulary and syntax), visual mediators, routines and endorsed narratives. Mathematics, regarded as a form of discourse, is thus characterised by four commognitive constructs: word use, visual mediators, routines and endorsed narratives.

Word use, in this context, focuses on the specialised language of mathematics and so includes number words, names of operations such as addition or subtraction or mathematical commands (e.g., find, solve, integrate, show) or processes such as differentiation and naming geometric figures, as these are recognised by all, even those not participating in a mathematical community as distinctive of mathematical vocabulary used in communication. Use of these words in mathematics is "usually governed by explicit definitions" which are often quite different from when these same words are used in colloquial discourse (Morgan & Sfard, 2016, p. 101), a point made by many others (e.g., Halliday, 1978). For example, the word *integrate* has a different meaning in the context of a mathematical community of Calculus discourse than in colloquial discourse.

Visual mediators in mathematics are physical objects which participants in mathematical discourse communities use in order to clarify what they are talking about. These mediators have as their raison d'être being a means of communication. They include "different types of symbolic artefacts", such as "written words, algebraic ideographs, diagrams, graphs, and various iconic drawings" (Morgan & Sfard, 2016, p. 101). For example, an equation such as $y = x^2 - 5$ is a symbolic visual mediator, while a graph of this function is a graphical visual mediator. Alshwaikh (2016) extended descriptions and classification of visual mediators, including a focus on "geometric diagrams", such as those involving triangles or circles, which can assist with geometric proofs. Alshwaikh's (2016) contribution to identification of visual mediators of this type will be discussed further in section 2.6, when considering applications and extensions of Morgan and Sfard's (2016) framework.

Routines are the types of action demanded of particular actors in the community. Viirman and Nardi (2021) focus on students' engagement with graphing routines in a set of mathematical modelling tasks, including determining where there is evidence of the students using *exploration routines* and where the students are using *ritual routines*. In commognitive analysis, routines are classified as ‘explorations’ if the completion of their performance produces or substantiates an endorsable narrative (Sfard, 2008, p. 224). As an example of what could be considered an ‘exploration’, Viirman and Nardi (2021) suggest conjecturing the trend in a dataset from a graph. Exploration routines can be further classified as being *construction*, *substantiation* or *recall routines*. By contrast, rituals are classified as routines that do *not* produce or substantiate an endorsed narrative (nor a change in objects) and that instead have as their primary goal “creating and sustaining a bond with other people” (Sfard, 2008, p. 241). Considering the overall goals of explorations, rituals and deeds helps in distinguishing between them. As described in Thoma (2018), a ritualistic participation in a discourse is a “matter of rote implementation of memorised routines” whereas an explorative participation involves “construction and substantiation of the narratives about mathematical objects” (pp. 49 – 50). Deeds are performance routines that involve a set of rules for a patterned sequence of practical actions that produce or change objects (Sfard, 2008) and are therefore always *practical routines*. As the focus in my thesis is on *discursive* routines, only rituals and explorations will be analysed. These are usually discursive routines because they involve learners interpreting a task situation as requiring a *communicational action*, leading them to become a participant in a particular type of mathematical discourse (Lavie et al., 2019).

Heyd-Metzuyanmin et al. (2016) further distinguish between rituals and explorations, writing that manipulating mathematical symbols is often the focus when participating in rituals while, in contrast, explorative participation includes producing *objectified* discourse in talking about mathematical symbols. While ritual participation has an emphasis on human action, explorative participation is focused on investigating mathematical objects and producing or substantiating endorsed narratives about them, with the corresponding mathematical discourse about these objects often *alienated* from human activity. For example, in looking for indications of the use of ritual or exploratory routines in students’ problem-solving work on mathematical modelling tasks, Viirman and Nardi (2018) analysed how the students spoke about mathematics, with instances where they spoke about it in terms of manipulation of symbols suggesting ritual engagement, while instances where they spoke about properties of mathematical objects suggested explorative engagement. To further assist in distinguishing when the students were using these different types of routines, they also tried to determine the students’ aims in using each routine. This was done by, for example, looking in the students’ discourse for any evidence of their demonstrating an understanding of a routine's relevance to the solution of a problem, which would be expected with explorative engagement, or whether they appeared to be performing a routine because they believed, from interaction with the research team, that they should do so, which would suggest ritual engagement. They also suggested as other indicators of ritual routine use "a strong reliance on external sources for substantiation, rigid rule following and mimicking previously encountered routines without regard for relevance to the problem at hand" (p. 3). However, Viirman and Nardi (2019, 2021) also argue that there is not necessarily always a clear division between exploration routines and ritual routines and that in some cases students can start with ritualistic behaviours in solving certain tasks and these then evolve into explorations, even during the course of the same problem-solving session. According to Lavie et al. (2019), students use rituals when participating in unfamiliar discourse. However, in further learning, their routines are expected to "undergo gradual de-ritualisation until they eventually turn into full-fledged explorations" (p. 2), possibly explaining Viirman and Nardi's observation as students being in transition.

Endorsed narratives are any 'stories' which are "considered by a mathematical community as a useful and reliable description of what this community regards as the 'mathematical universe', populated by 'mathematical objects'" (Morgan & Sfard, 2016, p. 101). In an undergraduate mathematics classroom, such an endorsed narrative could be: "the area of a triangle is equal to half the length of its base multiplied by its height." One aspect in determining the nature of such narratives is the extent of alienation of the mathematical discourse, which "construes the role of human agency in the origin of mathematical knowledge" (Morgan, 2016, p. 128). Alienated discourse is present when phenomena are presented "in an impersonal way, as if they were occurring of themselves, without the participation of human beings" (Sfard, 2008, p. 295). The extent to which mathematics is construed as involving material action or as relational processes, which describe only atemporal objects and their properties, also provides information about how stories about mathematical objects are presented in the text of examinations. In other words, we are determining the extent to which mathematical objects (including those integrated into CAS technological tools or screenshots) are involved in material processes or relational and existential processes. Material processes can be identified through statements which are "construing mathematics as an active process" (Morgan, 2016, p. 129) which takes place in time, with *actions* being carried out which involve mathematical objects. Such statements will sometimes identify a human agent, but they can also contain *alienated* discourse. For example, “if the curve $y = x^2$ is revolved about the x-axis, a circular paraboloid is produced.” In contrast, relational statements involve only *properties* of mathematical objects and *relationships* between them. For example, “the curve $y = x^2 + 5$ is a parabola with a vertex at $(0, 5)$.”

Word use, *visual mediators*, *routines* and *endorsed narratives* are all *mathematising* aspects of discourse, which analyse the nature of the stories told about mathematical objects. There are also *subjectifying* aspects of discourse, which Sfard (2008) describes as being “a special case of the activity of objectifying, which occurs when the discursive focus shifts from actions and their objects to the performers of the actions” (p. 113). This concept of *subjectifying* can be used in analysing the nature of the relationship between students and an examination author, the degree of autonomy the students have in answering each question and the nature and extent of the information they need to unpack in answering the questions. Each of these features of *subjectifying* aspects of mathematical discourse, from the framework of Morgan and Sfard (2016), will now be described in more detail.

Student-author relationship: The presence, or absence, of any personal pronouns such as the inclusive use of the pronoun *we* in the discourse of examination questions is one of the aspects considered in Morgan and Sfard’s (2016) framework. If present, it suggests a depiction of the student and the examination author working collaboratively, which is otherwise absent. For example, a Calculus question such as “if we wanted to construct a cardboard box with a square base and volume 900 cubic centimetres, what is the minimum surface area of cardboard we would need to use?” suggests the student and examiner working collaboratively in determining the minimum amount of cardboard required for them to construct the box. In contrast, while requiring use of the same mathematics to find the solution, a question phrased as “what would be the minimum surface area of cardboard required for a box with a square base and volume 900 cubic centimetres?” suggests no relationship between the examiner and examinee or with the context of the scenario presented in the question.

Whether the student is given specific instructions (an imperative) or is invited to consider mathematical questions is also included in their framework, as this also gives an indication of the nature of the student-examiner relationship.

Student autonomy: The other aspect of *subjectifying* the discourse of examination papers included in Morgan and Sfard's (2016) commognitive analysis is *student autonomy*[1]. The first component of this which they measure is designing a solution path the students could be expected to follow, including determining the *grain size,* which is the minimal number of independent decisions a student needs to make to solve a problem, "while designing a series of elementary steps necessary to solve [it]" (Morgan & Sfard, 2016, p. 113). In turn, the term "elementary step" is used to specify a step that "can be executed in a single operation, without further partitioning of the problem into smaller ones" (p. 113), with this being based on the assumption that the students sitting the examination have "reasonable mastery of the mathematical discourse at hand" (p. 113), based on the community of mathematical discourse they are from (e.g., first-year university students). The *student autonomy* section of their commognitive framework is also designed to record whether or not the students had any autonomy in choosing the form of their answer and in "choosing or constructing the mode of response" (p. 108), with the latter including determining whether they are required to produce any symbolic or graphical visual mediators as part of their answer. This *student autonomy* section of their framework also records the "complexity of utterances" using features such as sentence length and measures of grammatical and logical complexity in the examination questions. This measure is included in the student autonomy section as it is also "related to the number of decisions the examinee has to make while interpreting the text of the examination" (p. 110).

The doing of mathematics can occur within a particularly well defined and bounded context such as in the preparing for, and sitting of, Calculus-based examinations as commonly occurs in undergraduate university contexts thus where communicational acts (i.e., texts) are functional. For this thesis, it will be instructive to determine what is achieved by such texts particularly with respect to the impact of CAS technology. Morgan and Sfard (2016) have tabulated their commognitive analysis framework which includes the indicators of *mathematising* and *subjectifying* aspects of discourse that were outlined in this sub-section, together with associated questions to guide the analysis. This table will be

[1] I will apply this definition of autonomy in chapter 5, in analysing how much autonomy students are given in answering an examination question, rather than using alternative terms such as agency or agentivity. In turn, I will discuss agency separately, when analysing the extent and nature of alienation of the discourse of examination questions themselves.

presented in Chapter 4, the methodology chapter. *Mathematising* and *subjectifying* are kinds of discourses that are found in both examination papers and events such as interviews or problem-solving sessions, which will be relevant to my current research in analysing not only examination papers, but also students' task-based interviews.

2.6 Applications and Extensions of Sfard's Commognitive Framework

A commognitive framework can be used to analyse "the mathematical discourse of tasks in order to examine their mathematical content, level of difficulty, guidance and support, the complexity of language, the use of diagrams and the non-mathematical context in the tasks" (Tetaj, 2021, p. 643). Because all mathematical tasks contain some form of mathematical discourse, commognitive analysis has been used to analyse a wide range of different types of tasks in different settings from primary, secondary school and tertiary mathematics classrooms in a growing number of countries world-wide. This includes analysis of Graduate Certificate of Education Ordinary Level and Graduate Certificate of Secondary Education mathematics examination questions (e.g., Morgan & Sfard, 2016), students' written answers to end of year, first-year university mathematics examination questions (Thoma, 2018; Thoma & Nardi, 2018), second-year University students coursework mathematics tasks during their first encounter with group theory (Ioannou, 2018), university Biology students' group work on graphing tasks involving mathematical modelling in an interactive classroom environment, including discussions between the lecturer and students as they worked on these tasks (Tetaj, 2021; Viirman & Nardi, 2021), and mathematical modelling problems in a mathematics course for engineers (Rogovchenko, 2021), amongst others. The commognitive theoretical framework has also been applied to mathematical discourse in school mathematics textbooks (Alshwaikh, 2016), the role of paper-based and digital mediations of thinking about Calculus in secondary school Calculus classes (Ng, 2016, 2019), interviews of university students preparing for their final examination in Abstract Algebra (Ioannou, 2018b), secondary school students' written discourse about their experiences in a dynamic interactive digital environment about functions (Antonini et al., 2020), a questionnaire for university students in a first-year Calculus course about their understanding of tangent lines (Biza, 2021), a comparison of native-English and native-Korean speaking university students' discourses on infinity and limit (Kim et al., 2017), the content knowledge about mathematical modelling of pre-service secondary mathematics teachers enrolled in an undergraduate functions and mathematical modelling subject (Park, 2017), Grade 8 and 9 students' thinking about linear equations (Roberts & le Roux, 2019), characterising how and when a way of proving develops in a primary mathematics classroom (Shinno & Fujita, 2022), as well as commognitive responsibility shift and its visualising in computer-supported one-to-one tutoring for secondary school mathematics (Lu et al., 2020), discursive shifts in the learning and teaching of Calculus at university level (Nardi et al. 2014) and upper secondary level (Tasara, 2018).

With the growing output from tertiary mathematics education researchers, it would seem the Commognitive approach (Sfard, 2008) has been gaining ground as a theoretical framework for university mathematics education research studies in recent years (Nardi et al., 2021). As seen above, the commognitive research so far has mainly provided nuanced, micro-level accounts of mathematical experience, although studies of discursive shifts which go beyond snapshot accounts are starting to appear. Nardi et al. (2021) see a potential in commognitive research in university mathematics education (UME) that is yet to be fully realised for "generating theoretically robust evaluations of UME longitudinal pedagogical interventions" (p. 5). They propose that time is ripe for "change research" (Reinholz et al., 2020) in university mathematics education and that the commognitive theoretical approach is a strong theoretical lens that would ensure the rigour to drive a reform agenda (Nardi et al., 2021).

What then has the growing corpus of commognitive studies to offer with respect to the mathematical topic of central interest in this thesis, for assessment in technological environments and for extensions of Sfard's commognitive framework (2008) or tools to be used in the commognitive analyses to come in this thesis? So as not to double up, findings of relevance from commognitive studies related to Calculus, and assessment in technological environments will be presented in Chapter 3. The remainder of this section will address extensions of Sfard's commognitive framework and tools used in analysis in some of the studies mentioned above that I have deemed of relevance to my study to be reported in this thesis.

Alshwaikh (2016) applied commognitive analysis to Palestinian secondary school geometry textbooks. He was able to extend the locus of analysis from English to Arabic and he offers more detailed and refined classifications and descriptions of visual mediators, especially "geometric diagrams" as these are frequently present in geometry textbooks. His adapted commognitive framework more generally focuses on the "verbal and diagrammatic modes and the relationship between them" (Alshwaikh, 2016, p. 169), while by contrast the main focus of Morgan and Sfard's (2016) commognitive framework was on language. This is highly relevant to my research as use of graphs and other diagrams is especially widespread when CAS technology is available, due to the additional production, representation and assessment opportunities afforded by such technology. Furthermore, diagrams including graphs are more likely to be a focus in Calculus examinations and tests, especially with use of CAS and its inherent multiple representations.

Alshwaikh (2016) structured his scheme around Halliday's (1978) three meta-functions (ideational, interpersonal and textual) of language (see section 2.5.1), to enable integrating "new tools for the analysis of visual elements drawn from multimodal social semiotics" (Alshwaikh, 2016, p. 169). As part of the section of his framework that corresponds to the ideational meta-function, Alshwaikh (2016) distinguishes between *narrative and conceptual diagrams*, a distinction that was originally made in Kress and Van Leeuwen (2006). Narrative diagrams "present mathematics as an activity that may be seen as involving humans" (Alshwaikh 2016, pp. 169 – 170). Thus, such diagrams suggest human construction or interaction with the diagram itself over time. These diagrams can include vectors, other arrows or dotted lines for this purpose. By contrast, in conceptual diagrams human actions are absent and such diagrams instead indicate "atemporal objects or relationships" (Alshwaikh, 2016, p. 170). This distinction between narrative and conceptual diagrams is important in commognitive analysis, as it relates to the issue of objectification of mathematical discourse, with conceptual diagrams contributing to objectification of the discourse.

In addressing the interpersonal meta-function, Alshwaikh's adapted framework includes, based on work by Kress and Van Leeuwen (2006), a distinction between "demand contact" and "offer contact" between the viewer and visual text: "In written mathematics text a demand contact usually asks for information, generally using a question", while "offer contact, in contrast, provides information … Demand and offer contact may be represented in (geometrical) diagrams by visual cues such as question marks or labels" (Alshwaikh 2016, pp. 170 -171). This is relevant to my current research as CAS-active mathematics examinations sometimes include diagrams which are produced by a person using a CAS calculator or computer software, and so it is of interest to what the diagram invites the student to do, as well as the nature of the information it presents to the student.

Ng (2016, 2019), through her study which compared Calculus students' communication in two secondary mathematics classroom environments, has expanded on Sfard's (2008) view of visual mediators. Ng explored the role of paper-based and digital mediations of thinking about Calculus. In doing so she distinguished between 'static' and 'dynamic' environments in mathematical settings, where a static environment contains "static visual representations, such as those found in textbook diagrams" and in examination papers, while a dynamic environment, as exploited by the use of dynamic geometry environments (DGEs) (Ng, 2016, p. 122), enables students to directly interact with the screen of a technological tool such as a calculator or computer to change values of parameters. Further, "the dragging modality offered by DGEs enables users to interact with parameters in embodied ways and to observe change of functions dynamically" (p. 119). This classification of dynamic representations has particular relevance in task-based interviews, where students have access to CAS calculators allowing dynamic transformations, as will be the case in the task-based interviews in the study this thesis is reporting. It is also a useful classification in analysis of written assessment tasks, as it classifies the type of diagrams which are present in actual mathematics examinations as static visual mediators/diagrams.

Antonini et al. (2020) also use a new construct for dynamic visual mediators which they call *dynamic interactive mediators (DIMs)* to capture mediators which both change over time (i.e., they are dynamic) and respond to a person's manipulations (i.e., they are interactive). This seems to be exactly what Ng (2016, 2019) was referring to as dynamic visual mediators. Antonini et al.'s elaboration of the notion of dynamic interactive mediator within the Commognitive theoretical frame (Sfard, 2008) leads to discourse also being viewed as able to happen between a human or a group of humans and a dynamic interactive mediator, as a discourse is a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with reactions" (Sfard, 2008, p. 297).

A second new construct introduced by Antonini et al. (2020) is *transitional discourse.* According to these authors, transitional discourse provides "important entry points to mathematical discourse for students who are not yet experts, but newcomers to the community of mathematicians” (p. 5). This discourse involves informal constructs which are used in a similar manner to those an expert in the discourse would use. These informal constructs can be juxtaposed with constructs from the endorsed narratives of mathematics used by experts, so are seen by Antonini et al. (2020) as playing an important role in teaching and learning, which in commognitive theory means being initiated into a discourse community (see section 2.3.1). This construct has potential for the implications from the task-based interviews, so will be considered in later chapters of this thesis if the construct of colloquial discourse is not adequate for my purposes.

The work of Thoma (2018) has also provided a useful structural tool for commognitive analysis and presentation of analysis. She produced commognitive analysis summary tables which had three columns, two showing the commognitive analysis of a particular task and the third aligning rows of this analysis to the perspective of the lecturer in her study on assessment, as she had interview data from the lecturers who had set the examinations she was analysing. This general type of tabular structure will be provided for the commognitive analyses in this thesis, where in this case the third column of the summary tables will analyse the role of CAS technology (where applicable) in applying each part of the commognitive analysis framework to examination questions.

The use of commognitive analysis in mathematics education research shifts the focus of research from the process of changes in learners to transformations in mathematical discourse as new participants are initiated into an ever-evolving discourse community.

2.7 Commognitive Conflicts

Commognitive conflicts are an important source of potential error or confusion that can occur when interacting with two (or more) different types of discourse. These occur when “the encounters between interlocutors use the same mathematical signifiers (words or written symbols) in different ways or perform the same mathematical tasks according to differing rules” (Sfard 2008, p. 161). This aspect has not been previously identified or analysed for users of CAS in the context of a commognitive analysis framework. Such commognitive conflicts will be investigated further in chapter 6.

Kontorovich (2021) points out that the phrase “encounters between interlocutors” has been the stimulus for many researchers to seek and find commognitive conflicts between two or more participants in interpersonal communicating (e.g., Kontorovich et al., 2019; Nachlieli

& Elbaum-Cohen, 2021). However, Sfard's definition of commognitive conflict as a "situation that arises when communication occurs across incommensurable discourses" (2008, p. 296) allows for the possibility of *intrapersonal commognitive conflicts* according to Kontorovich (2021). Kontorovich (2021) explored this through a written questionnaire targeting knowledge of square roots among eleven students enrolled in a mathematics pathway course to studies at the university. The focus was on where the same student generated apparently conflicting responses, that is, inconsistencies within their own discourse. These intrapersonal commognitive conflicts can occur in task situations that an observer might perceive as similar, but which, for the student doing the task, could evoke the use of different mathematical signifiers, the use of the same signifiers in different ways, or the carrying out of actions in accord with different rules. Rather than the student construing the various parts of the questionnaire as several invitations to do the same thing (i.e., take the square root), what may seem as a conflict within the student's discourse could be a reasonable set of differing actions to take in situations that the student *construed* as being different.

Viirman and Nardi (2018, 2021) identified interpersonal commognitive conflicts that occurred while biology students were working on mathematical modelling tasks. Where these conflicts sometimes occur, is due to different intentions and understanding from students and the instructors when working on solving the mathematical modelling problems. Thoma and Nardi (2018) give examples of intrapersonal commognitive conflicts in students' written first-year university examinations, where they focus on commognitive conflicts between university level mathematics discourse and the school discourse that the students are typically still more confident with from their previous studies. Implications from their findings of relevance to this thesis are that such conflicts are a key part of students' transition from school to university mathematics as they learn about the distinct aspects of discourses in different areas of mathematics and that instructors being aware of these types of conflicts could facilitate a smoother student transition from upper secondary to university mathematics. Identification of such commognitive conflicts from examination scripts and task-based interviews will be important in the coming study in this thesis.

The study of commognitive conflicts is also relevant to this research in that intrapersonal commognitive conflicts can potentially occur in a setting where students are learning with Computer Algebra Systems technology, in solving problems where some of the notation used in standard algebraic mathematics classroom discourse can be interpreted by the technology as signifying something different.

2.8 Conclusion

I commenced this chapter by outlining different theories of learning, including behaviourism, cognitivism and participationism. Of these theories, the participationist view of human thinking as a form of communication and participation in a community of discourse, was seen to be influential in the development of Sfard’s (2008) theory of commognition and her associated use of the term *discourse*, in which communication can occur with oneself or with others. Sfard’s theory of commognition combines aspects of communicational theory, which identifies knowledge of mathematics (and other fields) as *discursive activities*, with social semiotics, which focuses on how meaning is created, considers communication to be multimodal (as is often the case in mathematics), and takes into account the positioning of participants who are engaging in discourse within a social practice.

Upon then reviewing the components of Morgan and Sfard’s (2016) associated commognitive analysis framework, I could see that applying this framework, in viewing mathematics as a discourse, of the type argued by Sfard (2008, 2020), would allow me to obtain a deep insight into the underlying structure of examination and test questions, both in how they mathematise, in telling stories which involve mathematical objects; and how they subjectify the discourse, in positioning students in relation to the examination or test and giving an account of the expected nature of their actions. The use of a commognitive framework identifies that changing the textual form of a question does not only have the potential to alter the level of difficulty of a question; but also, can change the ways in which students will interact with the text and position themselves within the examination environment, in deciding their approach to answering the question. When technology is used, the effect of the variation in discourses between different examination questions potentially becomes, if anything, even more pronounced, as it allows for asking a wider range of questions, including allowing time for questions of greater grammatical and algebraic complexity, due to the assumed time-saving advantage of CAS technology. The types of visual mediators, both provided in the text of an assessment item and that can be produced by the students when CAS technology is assumed, are more diverse, and extensions of Sfard’s framework by Alshwaikh (2016), Ng (2016, 2019) and Antonini et al. (2019) are valuable here, in allowing us to elicit features of these diagrams that will influence how an assessment item speaks to students and how students are positioned in relation to the questions and their expected responses. Identifying if any commognitive conflicts are present, when students are interacting with the discourse of CAS, can also provide valuable insight into the impact of CAS on their mathematical discourse.

A further literature review of Computer Algebra System technology, the teaching and learning of Calculus and assessment, especially in university mathematics service courses, will be reported in the next chapter. The results of the commognitive analysis of the test and examination papers will follow in Chapter 5, before the commognitive analyses of the task-based interviews will be reported in Chapter 6 and contextualised by the results of a qualitative analysis of questionnaire data. Chapter 7 will be a discussion of the findings in light of the literature and the research questions and Chapter 8 will present the conclusions from the study and implications for practice in tertiary mathematics teaching and further research.

CHAPTER 3

LITERATURE REVIEW: CALCULUS, CAS TECHNOLOGY AND ASSESSMENT

3.1 Overview

In Chapter 2, the theoretical background to, and applications of, Sfard's commognitive analysis framework (Alshwaikh, 2016; Antonini et al., 2020; Morgan & Sfard, 2016; Ng, 2016, 2019; Sfard, 2008) to be applied in the thesis were discussed. This chapter reviews literature relevant to teaching, learning and assessment of undergraduate Calculus units and the use and role of technology, especially Computer Algebra Systems (CAS) technology in assessment and in solving Calculus problems.

Section 3.2 outlines how the topics typically taught in undergraduate Calculus are affected in the transition from high school to tertiary level, with section 3.2.1 reviewing research about this in relation to the variation in discourses that is present and how this can influence students' learning and discourse at tertiary level. Section 3.2.2 investigates how the teaching of Calculus can vary within university departments and even within individual subjects, depending on the instructor, and at how a commognitive framework can be applied to analysing the associated Calculus discourse at tertiary level. The transition from year 12 to university level mathematics is important, as it indicates the communities of discourse the students are coming from and transitioning into.

Section 3.3 looks at the graphical and symbolic visual mediators produced by CAS and at how these can affect the discourse, teaching and learning of Calculus. This includes examining the issue of differences between required inputs and outputs of CAS technology, compared to the traditional written discourse of mathematics. Section 3.4 reviews literature on the extent of use of CAS technology in examinations and considerations in setting and evaluating the nature of examination questions in a CAS-active environment. In section 3.5, research on students' attitudes to using CAS is reviewed, together with links between students' attitudes to use of CAS and how they use it in practice. Finally, section 3.6 brings the chapter to a conclusion, outlining the main threads in this literature review and how the remaining chapters will progress the thesis.

3.2 Calculus and Topics in Secondary-Tertiary Transition

In this thesis, I am particularly interested in content areas such as Calculus and other topics in undergraduate mathematics where CAS technology has an effect, especially in first year university. When mathematics is recognised as a form of discourse, it becomes clear that the learning of mathematics frequently involves learners needing to make substantial discursive shifts, particularly as they transition from one educational level to another (see, e.g., Ghedamsi & Lecorre, 2021), from pre-school to primary school to lower secondary school to upper secondary to tertiary education (Jacinto & Carreira, 2013). The teaching of mathematics involves the facilitating of these discursive shifts and dealing with the tensions that come from conflicts that arise from changing language use (in both mathematics and interaction with others and objects in the classroom) and expectations for learning and doing mathematics in different environments (e.g., a dynamic interactive digital environment, see Antonini et al., 2020) and the tools (e.g., CAS calculators, Wolfram Alpha) that are allowed in different spaces such as in learning but not assessment. I am interested in the transition from year 12 to first-year university as this gives the “historically situated context” and “communities of discourse” from which most of these students are coming, that is, their *assumed precedent search spaces* (Lavie et al., 2019; Viirman & Nardi, 2021).

Calculus is one of the fundamental courses in undergraduate mathematics (Ghedamsi & Lecorre, 2021) and is a mandatory class for many majors in universities including those in the sciences, engineering and students preparing for secondary mathematics teaching. Science students, for example, learn Calculus in their first and second years. As Calculus concepts are a basis for many more advanced mathematics topics, it is important to the success of many undergraduate students at university. Calculus topics typically taught in undergraduate mathematics units include differentiation, integration and differential equations (Bressoud et al., 2016), together with associated topics such as functions and limits. The Calculus topics taught in undergraduate Calculus are often first introduced in the final years of high school (Bressoud et al., 2016; VCAA, 2016). It is therefore informative to examine research literature about what occurs in the teaching and learning of Calculus and similar topics in the transition from upper secondary to first-year university (in section 3.2.1), before examining critical aspects of the teaching and learning of Calculus and similar topics at university level in section 3.2.2, particularly those being foregrounded by Commognitive research.

3.2.1 Researching the secondary-tertiary transition in mathematics

In transitioning from upper secondary school to undergraduate mathematics, students experience numerous changes, especially in mathematical cultures (Biehler, 2019; Corriveau & Bednarz, 2017) and discourses (Pinto, 2019; Sfard, 2014). In the commognitive approach, learning mathematics is conceptualised as a change in the learner's mathematical discourse. As we saw in chapter 2, Sfard (2008) distinguishes between *object-level learning,* which extends a learner's knowledge of existing words, routines, visual mediators and endorsed narratives; and *meta-level learning,* which produces changes in the meta-rules of a discourse. We are interested in changes in both of these at the secondary-tertiary interface.

Several topics at this interface that occur on both sides of the interface in a variety of mathematical contexts, are subject to different definitions and are expressed in several representational forms. For example, at secondary level, a pointwise (evaluating functions and their derivatives at individual points) and global (working with the general algebraic equation and overall graph) perspective of functions is most commonly used. In contrast, at tertiary level a transition is required to a view of functions that considers their behaviour on an interval (Hong & Thomas, 2015). The latter perspective is necessary to enable examination of mathematical concepts such as continuity of functions. Another such topic is tangent lines, which illustrates the need for us to examine the precedent search spaces of undergraduate students to fully contextualise the discourse communities the students in my study participated in and the commognitive conflicts that could arise for them in assessment and classroom situations. Biza has engaged in several studies of tangent lines at the upper secondary level (Biza et al., 2008) and at the secondary-tertiary transition (Biza, 2017, 2019, 2021; Biza & Zachariades, 2010). Her work also has transitioned from a cognitive lens, in looking at how students' "existing cognitive structures" (Biza et al., 2008, p. 55) can be extended or reconstructed (Harel & Tall, 1991), consistent with the works of cognitivists such as Piaget (1970), to more recently using a commognitive lens (Biza, 2017, 2019, 2021). For the latter, the discursive characteristics of the responses of students entering university to a questionnaire on tangents and associated background high school textbook content involving tangents were analysed in detail.

The tangent line appears in Euclidean Geometry, Algebra, Calculus and Analysis and there are differences between parts of the mathematical discourse about tangency in each of these areas. For example, in Euclidean Geometry a tangent line to a circle has only one point in common with the circle and the circle is always on the same side of a given tangent line. In contrast, in Calculus the tangent to the graph of a function at a point is defined as a line that has that point in common with the graph and which has its slope equal to the derivative of the function at that point. This means there are some functions where some tangents will also touch the graph or cut through it at additional points or even coincide with a greater part of the function, which is not possible for tangents to a circle. As Biza (2021) states, in these different topics "students are asked to navigate across definitions, properties, perspectives and terms without necessarily being aware of their underpinning differences" (p. 2).

The influence of students' previous experiences and the educational context in which they have learnt, and are learning, about topics such as tangents is important to my work about understanding factors that could influence how students might position themselves in relation to solving university level Calculus problems and that can contribute to shaping their resulting discursive activity. Biza (2021) identified students' previous learning experiences with tangents as *precedent events*, that will therefore affect how students react to different task situations involving tangents. Students' experiences of tangents at school therefore play a big part in generating the *precedent search space* that a given student will access when encountering tangents again in task situations, including examinations, during their studies of mathematics at university.

Biza (2021) investigated the influence of students' experiences with tangent lines at secondary school level on their subsequent work with, and understanding of, tangents several months after completing year 12, and after their participation in a university admission examination. She views learning as a longitudinal process and introduced the notion of the *discursive footprint* of tangents as "the discursive characteristics of how tangents are dealt with in different mathematical domains in the course of students' studies" (p. 1). To investigate this, she analysed the mathematical discourse of the textbooks typically used by school students in Greece, classifying key parts of their discourse on tangents based on whether the discourse came from Geometry, Calculus, Algebra, Analysis or combinations of these content areas. She then looked for the replication of parts of this discourse in the participants' answers to her questionnaire. This was to identify manifestations of the discursive footprint of tangents in these participants' work. The notion of *discursive footprint* can be applied to other areas of mathematical discourse, with Nardi et al. (2021) describing the concept of *discursive footprint* as "a pedagogically potent descriptor, and evaluator, of students' work on a specific mathematical topic" (p. 2).

The differences in discourses at school and university between the different mathematical topics where tangents are present can, in turn, lead to commognitive conflicts between university level discourse about tangents and parts of the school level discourse. This can occur when, for instance, students apply concepts about tangents that are applicable to one field of mathematics (e.g., Euclidean Geometry) in a different context (e.g., Calculus), where some of them do not apply or have a different actual meaning (Biza, 2019, 2021). Similar commognitive conflicts between school and university level discourse can occur in other areas of mathematics, with Thoma and Nardi (2018) identifying instances of commognitive conflicts between the discourses of Algebra and Set Theory, when analysing mathematics examination scripts of students in first-year university. For example, there are different requirements when carrying out a substantiation routine to prove an equality, depending on whether one is working in the discourse of Algebra or the discourse of Set Theory (Thoma & Nardi, 2018). Thoma and Nardi (2018) also identified commognitive conflicts that can occur between the rules of school and university discourses, based on how stringent the requirements contained in the rules of the discourse are. Thoma (2018) gives the example that when requiring students to prove a statement is true, in some cases at school level providing an example that satisfies the statement would suffice, while it would be expected at university level that a proof would need to be much more rigorous and general, with the provision of merely an example not being accepted as an endorsable narrative.

More generally, research on the learning culture in Calculus in the secondary-tertiary transition has found that a shift from mostly procedural thinking to a deeper level of more advanced understanding and formal thinking is required (Borba & Villarreal, 2005). In turn, changes in mathematical culture from secondary level to tertiary level also arise in part due to differences in the environment in which the students are leaning, including having new teachers with their own distinct teaching practices, a different social environment, and with the students themselves having more autonomy in how they approach their own learning (Ghedamsi & Lecorre, 2021). The areas where Ghedamsi and LeCorre (2021) found changes in academic expectations, when compared to secondary level, included learning expectations for approaching problem-solving tasks carrying out formal mathematical reasoning in proofs, the nature of the semiotic representations to be used and the types of technical methods taught. One concern identified was that there was not enough focus on mathematical reasoning in the high school mathematics textbooks they considered. This could add to the challenges for students in making the transition from secondary level to tertiary level mathematics.

An aspect of the changing nature of semiotic representations from secondary to tertiary level is in how letters and other symbols are used as mathematical signifiers, which frequently changes at this transition. For example, Corriveau and Bednarz (2017) observed the transition from secondary to tertiary level coinciding with “a fracturing of the meaning of symbols” (p. 2). A part of the reason this occurs is the wider range of different types of mathematical objects that need to be considered in tertiary mathematics, with these also being used in a greater number of different contexts. However, Corriveau and Bednarz (2017) also observed that university level teachers are less likely to associate a given letter “with a specific attribution” (p. 15). This occurs because of the wider range of types of mathematical objects that symbols, especially letters, represent at tertiary level where they are used in more and different contexts and the greater variation in the letters used even in naming the same features of mathematical objects. This contrasts with the secondary mathematics setting Corriveau and Bednarz (2017) observed in the high school textbooks and teaching they analysed, where they found that “intentionally, and over a long period of time, the secondary teachers made consistent use of certain symbols” (p. 9). An example is that they found secondary teachers and textbooks consistently used a, b, h and k in the same context in relation to functions, with these letters corresponding to horizontal dilations/contractions, vertical dilations/contractions, horizontal translations, and vertical translations of functions, respectively. This means that the secondary teachers were more frequently choosing specific symbols to use which were dependent on the context. In contrast, tertiary teachers were more varied in their use of symbolism, often choosing their own assignment of specific letters to signify particular functions, constants, limiting values etcetera, rather than using a universal discourse of particular symbols signifying a predetermined meaning in a given context. In turn, this affects students’ understanding and symbolic discourse.

The findings of this section indicate that there are several types of changes that occur during the secondary to tertiary transition, including the verbal and symbolic discourse of mathematics which is used, the level of academic expectations on students, and the mathematical cultures they encounter. Differences in discourse and mathematical cultures also occur *within* the tertiary level, as discussed in the coming sub-section 3.2.2.

3.2.2 Differing contexts and discourses in undergraduate Calculus teaching and learning

At tertiary level, the mathematical culture and the nature of the community of discourse students experience will partly depend on the type of department and the course in which they are learning mathematics. For example, Biza (2021) observes that mathematics departments will typically teach their students about derivatives using a “tangent line perspective”, whereas Engineering departments will commonly use a “rate of change perspective” (p. 2). However, both of these perspectives originate from the discourse of school mathematics.

Pinto (2019) found variability in the discursive content of tutorial classes about the definition of the derivative and its applications depending on the instructor (teaching assistant) taking the classes, despite these classes being taken concurrently in the same (Real Analysis) unit, and each instructor being provided with the same lesson plan. Pinto found that the classes taken by the instructors were similar in relation to content that required object-level learning, but that they clearly had different goals in relation to meta-level learning, indicating the need for “a shared and explicit discourse on meta-learning at [tertiary] level” (p. 1). Biza et al. (2016) report on research findings which suggest that how lecturers approach teaching is also influenced by their research practices and their academic background, such as what type of qualification in mathematics they have. These aspects also contribute to their own precedent search space which they are accessing when teaching. Biza et al. (2016) also present findings that the teaching practices of mathematics graduate teaching assistants are influenced by their “beliefs about teaching and learning of mathematics” (p. 4).

Studies applying a commognitive framework to analyse aspects of the discourse used in Calculus at tertiary level have included those analysing lecturers’ and other instructors’ discourse, students’ discourse and the interface between the two. For example, in using a commognitive framework to investigate how university Mathematics instructors teach the concept of function in lectures, Viirman (2013) used videotaped lectures given by teachers at three Swedish universities to Engineering and Computer Science students. The aim of the study was to classify their discursive activities, with a focus on providing “as exhaustive a list as possible of the routines of the teachers” (p. 524). He found similarities in the teachers' discourse in word use, visual mediators and narratives, but that there were both similarities and differences regarding routines, prompting him to analyse the nature of the routines they used in more detail. Both construction and substantiation routines were found to occur in the discourse of all the teachers, but there were also distinct differences within these general categories, with the frequency and nature of substantiation routines also differing greatly between teachers, indicating that "the pictures they paint of what it means to be doing mathematics differ" (p. 524). The most common construction routines the teachers used were those for constructing definitions of mathematical objects, together with associated examples. Several types of definition construction routines were identified by Viirman (2013), where these were distinguished by whether properties of, or connections between, examples or counterexamples were used to lead into constructing such definitions, whether new concepts were introduced through the use of the definitions or whether a mathematical object only needed to be defined by giving it a name, after previously being constructed in a different context. Viirman (2013) found variation in which of the above definition construction routines the different teachers used in constructing the definition of function. Nardi et al. (2014) continued to explore this work, further analysing how the lecturers in the study constructed and talked about the function object in their discourses. They concluded that this way of analysing the nature of the lecturers' definition construction routines demonstrates how actions initially perceived as similar can be systematised to show the variation that occurs in discursive patterns in lecturers' practices.

Viirman (2015) extended his 2013 study further, introducing a classification of routines used in the lecturers' discourse which they use to instruct students according to whether these routines were used to explain concepts to students, to motivate them or to pose questions to them. He found that all three of these types of routines were used by the seven lecturers in the study, but that a number of different sub-categories of routines were used in different ways and to a different extent by the different lecturers. He also found that, in introducing students to new mathematical content, those teachers who made the most frequent reference to established mathematical facts in relation to this were also less likely than other teachers to use metaphor, applied examples and everyday non-technical language. While this classification of routines was only developed in the context of learning about functions, it is an example of a way in which a commognitive framework can be applied to analysing some of the teaching practices of mathematics lecturers, which as Viirman (2015) also acknowledges, could be further considered and extended in relation to teaching practice in mathematics at other universities, and in other subject areas.

While the above studies focused on differences in the nature of the *routines* used by instructors, another aspect of discourse that can also vary, and which is affected by the educational context, is that of *visual mediators*. These include graphical visual mediators, which are frequently present in written Calculus discourse, in a variety of contexts. These can be used to help improve conceptual understanding in many areas of Calculus, for example, in interpreting the concept of a derivative, stationary points, points of inflection and integrals (Hong & Thomas, 2015). Slope fields diagrams can assist in interpreting the solutions of first order differential equations (Raj, 2006), while graphing solution curves for first-order (Villate, 2006) and higher order differential equations can also assist conceptual understanding. Teaching with visual mediators is most effective when the links between them and the associated concepts and any relevant symbolic visual mediators are also made clear.

To investigate this using a commognitive framework, Park (2015) looked at the use of, and extent of, connections and transformations being made between symbolic and graphical visual mediators, in a tertiary, Calculus-related context. She used a commognitive framework to analyse the discourse of three first-year university Calculus instructors on the concept of the *derivative*, focusing on how they addressed the derivative as a point-specific value using the concept of the limit and as the derivative of a function on an interval. Park found that, although the instructors represented the limit using symbolic notation and also provided a graphical illustration of the associated tangent line, they did not explicitly provide for their students the *connections* between these symbolic and graphical visual mediators, in relation to the concept of derivatives. Park (2015) noted that the only connection used in the instructors' discourse was the word 'slope', and its use was also inconsistent "when illustrating and talking about lines, secants and tangents" (p. 248). Graphical visual mediators showing the derivative as a function were also limited to illustrations of *constant* values of the derivative. These features of the instructors' word use and visual mediators (and the lack of direct linking of symbolic and graphical visual mediators) were also found to consistently occur in the associated routines and endorsed narratives in their discourse about the derivative. Park (2015) also observed that this lack of connections being explicitly made between the graphical and symbolic visual mediators associated with the derivative appeared to be related to difficulties that students also typically have with the concept of a derivative. This includes the associated misconception some students have in describing the derivative as a “tangent line”, as found in Park (2013), where she reported on the results of a commognitive analysis examining the discourse of Calculus students when they were learning about the concept of the derivative as a function. In turn, as was seen in section 3.2.1, tangents are a topic which also appears in other contexts, with different associated properties (Biza, 2017). Park makes reference to the value of using commognition (Sfard, 2008) to analyse the data, which enabled identification of a lack of connection between aspects of the endorsed narratives of the teacher about the derivative and the students’ understanding of what the teacher was trying to convey. As in Biza (2019), these results demonstrate how using a commognitive framework can identify some of the challenges in navigating between different definitions and representations of the same concept in different contexts, including tangents and the derivative.

Limits are another concept that is related to Calculus and which occurs in different contexts, where commognitive analysis on the discourse and students’ learning of limits adds further insight. Gucler (2013) applied Sfard’s commognitive framework (2008) to analysing and directly comparing characteristics of the discourse on limits of a university instructor and his students. This study found that although the students “used similar visual mediators, words, and metarules to endorse particular limit-related narratives” (p. 451), the way in which they did so was less coherent than that presented by their instructor, and sometimes involved their attempting to substantiate a narrative of a limit as a *process* using properties that should be associated with a different narrative; that of a limit as a *number*. Even by the end of the course, he found that the majority of the students did not objectify a limit as a number. This situation has parallels to the problems Park (2013) identified in students’ discourse about the derivative, with students in that case struggling to move from the concept of a derivative at a point to that of the derivative as a function. In both these studies, instances were found where students made the mistake of using some of the same features of a part of the discourse of a concept (limits, the derivative) that the instructor used, but in the wrong context within the discourse, in an attempt to substantiate a different narrative with subtle but important conceptual differences. Gucler (2013) concluded that using commognitive analysis was helpful in identifying the tangled relationship between different aspects of the discourse on limits, which in this case also highlighted the nature of breakdowns in communication between the lecturer's and his students' discourses on limits.

The above studies demonstrate how using a commognitive framework can enrich understanding of the sometimes subtle differences in discourse that occur in the teaching of Calculus and its associated topics at tertiary level, both in cases such as described in Biza (2019), Park (2015) and Gucler (2013) where the same topics (tangents, derivatives and limits, respectively) have different properties depending on the context and where lecturers are actually teaching the same topics in the same units but making different use of routines (Nardi et al., 2014; Viirman, 2013, 2015) or visual mediators (Park 2013, 2015). In turn, this discourse affects the nature of students' precedent search space that they draw on when approaching learning about concepts and solving Calculus problems themselves. The finding that students tend to have different, less coherent discourses in some Calculus topics than those of their instructors (Gucler, 2013; Park, 2013) is consistent with students being apprentices in the discourses (Sfard, 2008) of the tertiary level mathematical content they are being taught.

When technology such as CAS calculators or software is introduced into the teaching and learning of Calculus subjects, this can also have a large impact on the discourse and learning of Calculus (Sangwin, 2019). Section 3.3 now follows, reviewing literature about the features of the visual mediators and symbolic discourse of CAS calculators and software and how these are used in the teaching and learning of Calculus, together with their effects on its associated discourse.

3.3 CAS Technology and its Effect on Calculus Learning and Discourse

Calculus is a topic area where CAS technology can have an especially large impact (Sangwin, 2019), although CAS also has uses in other areas of mathematics typically taught to undergraduates including, for example, Linear Algebra (Caridade et al., 2015; Diaz et al., 2011), Analytical Geometry (Neeves, 2018), Optimisation/Operations Research (Jarvis et al, 2018, 2022) and Discrete Mathematics (Ivanov et al., 2017). Use of CAS can occur not only in the type of topics taught in first-year Calculus such as differentiation and its applications including Taylor polynomials (Varbanova, 2017), integration and limits (Hong & Thomas, 2015), but also in more advanced associated topics such as Vector Calculus (Craig & Akkaya, 2022). In turn, when CAS is used, it has an effect on the associated discourse used in Calculus, as will be described in Sections 3.3.1 and 3.3.2. The first of these sections provides an overview of some of the types of graphical *visual mediators* that can be produced by different types of CAS and how they are subsequently used to assist in teaching and learning about Calculus topics. Section 3.3.2 will then review literature on the nature of the *symbolic written discourse* of CAS and how this is related to other types of Calculus discourse. Students are already learning new parts of the traditional written discourses of Calculus, together with the additional spoken discourse used in Calculus which includes new words for symbols (e.g., with the use of spoken terminology potentially including *y-prime*, *y-dash*, *f-dash x*, *ddx* etcetera when talking about derivatives, and the use of colloquial abbreviated terms such as *diff* when talking about the associated mathematical process of differentiation). The symbolic written discourse of CAS will be reviewed with regard to the nature of its syntax and outputs, with an emphasis on how these sometimes differ from what is required when working by-hand using the endorsed discourse of Calculus and associated topics such as Algebra and Functions. This section will also investigate how effectively students manage these differences and implications for the discourse students engage in when using CAS. How CAS can affect teaching and learning overall will be reviewed in Section 3.3.3, focussing on how this can affect the emphasis on different types of learning, which in turn also has an effect on the types of discourses used.

3.3.1 The use of graphical visual mediators produced by CAS

One important aspect of CAS is its visualisation capabilities to produce graphical visual mediators, with these having the potential to improve students' own visualisation skills and consequently their understanding of Calculus concepts (Bressoud et al., 2016). For example, CAS calculators and software can show a visual representation of the integral of a function, including the corresponding area bounded by the curve and the x-axis, which can also be applied to finding the total area bounded by the curve and the x-axis in a given interval or the area bounded by two curves. For example, Thomas and Hong (2005) demonstrated use of a TI-89 CAS calculator in teaching students to integrate $cos(x)$ from 0 to π, combined with producing a shaded graph on the calculator. Sotaro et al. (2021) showed an example of using the TI-NSpire CX CAS calculator to find the area bounded by two curves. Figure 3.1 shows an example of using the current version of Wolfram Alpha to find the area bounded by the function and the x-axis; between $x = 1$ and $x = 5$.

Generation of direction field diagrams and particular solution curves associated with first order differential equations is possible with most models of CAS calculators and some types of CAS software including Maxima (Villate, 2006) and Maple (Matos, 2014). For example, Figure 3.2 shows a slope field diagram generated by the TI-NSpire CX CAS calculator, for the differential equation $y' = \frac{1}{2}(1 + y^2)$, together with a particular solution curve that satisfies the initial condition $y(0) = -1$. This is another example where CAS can be used to assist with conceptual understanding, in allowing for multiple representations, in this case of solutions of a differential equation, with such visual representations helping develop students' conceptual understanding of the behaviour of such equations.

Figure 3.1

Use of Wolfram Alpha to find the Area Bounded by a Curve and the x-axis

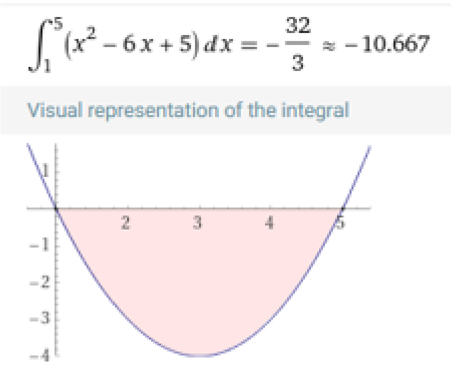
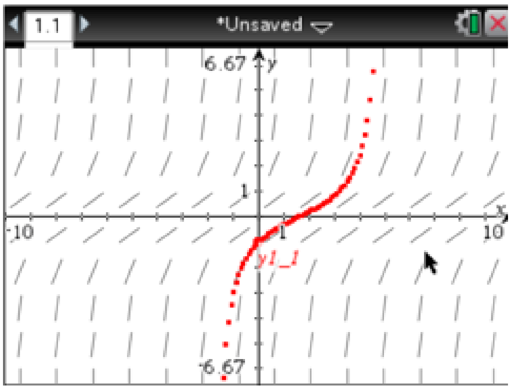


Figure 3.2

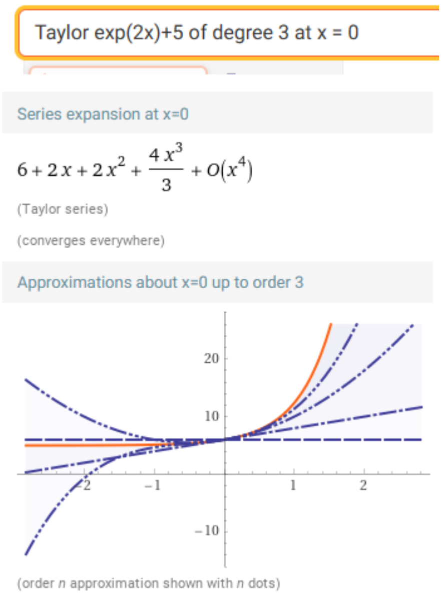
Slope field Plot for the Differential Equation $y' = \frac{1}{2}(1 + y^2)$ Using TI-NSpire CAS Calculator, Together with the Particular Solution Curve at $y(0) = -1$



Visualisation using CAS can also assist in conceptually understanding Taylor polynomials at a point as approximations to a function near that point that become increasing close to the function as the degree of the Taylor polynomial approximations increase. This can be done by first determining one or more Taylor approximations (which all CAS packages and calculators have a command for) and then plotting the resulting approximations, together with the curve of the original function. An example of this using Wolfram Alpha is shown in Figure 3.3, where Wolfram Alpha's **Taylor** command will plot the curve and all its successive Taylor approximations at a given point, up to the specified degree. Therefore, in this example, where degree 3 has been specified, the curve $y = e^{2x} + 5$ is plotted (with solid lines) and the successive Taylor approximations $y = 6$, $y = 6 + 2x$, $y = 6 + 2x + 2x^2$ and $y = 6 + 2x + 2x^2 + \frac{4}{3}x^3$ about $x = 0$ are plotted with dashed lines, visually demonstrating the property that the higher degree approximations are a better fit to the curve. Varbanova (2017) also demonstrated an example of using CAS software to plot Taylor approximations together with a curve, but applied in their case to visualising Taylor approximations associated with the solution curve of a differential equation.

Figure 3.3

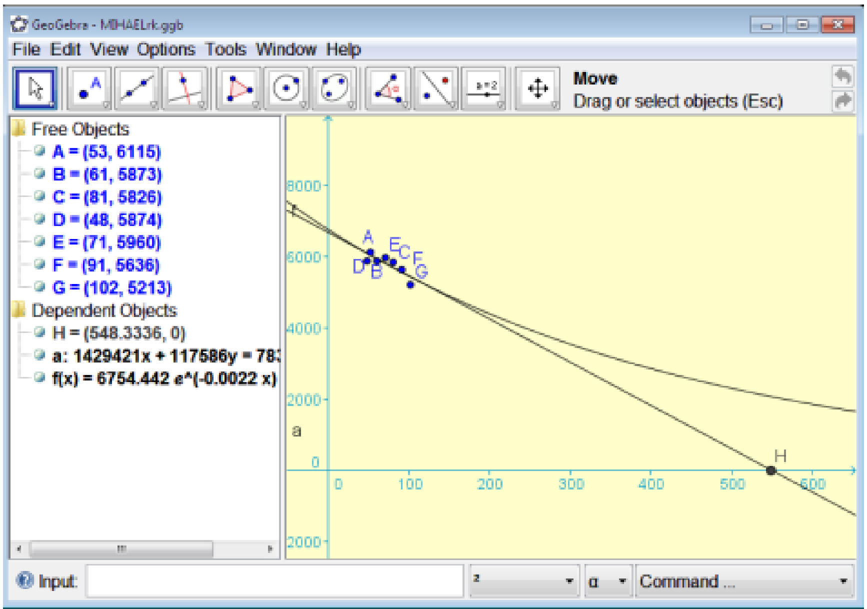
Plot from Wolfram Alpha, Showing the Graph of $y = e^{2x} + 5$ *together with Degree 0, 1, 2 and 3 Taylor Polynomial Approximations about x = 0*



The graphing capabilities of CAS can also assist in producing visual mediators for help in understanding and finding solutions to mathematical modelling problems. For example, Budinski and Takaci (2011) reported on a study where GeoGebra was used for a mathematical modelling problem which involved finding an equation and associated curve of best fit to model changes in the size of the population of Ruski Krstur over time. This involved fitting possible solution curves to the data set provided, with a plot of their raw data and the linear and exponential curves of best fit shown in Figure 3.4. The students who took part in the study were then required to choose the most suitable curve to model the data, based not only on the visual fit to the data but also on the practical constraints of the problem (e.g., population size cannot be negative), which led to their choosing the exponential curve $f(x) = 6754.442e^{-0.0022x}$ and then using the equation of this curve of best fit to help in determining the differential equation associated with this solution curve. Several of the students interacting with this task identified the practical constraint of non-negative population size, leading them to correctly choose the exponential curve of best fit over the linear curve of best fit. This example demonstrates how the visualisation properties of CAS can be used to assist students' conceptual understanding in the context of real-life modelling problems.

Figure 3.4

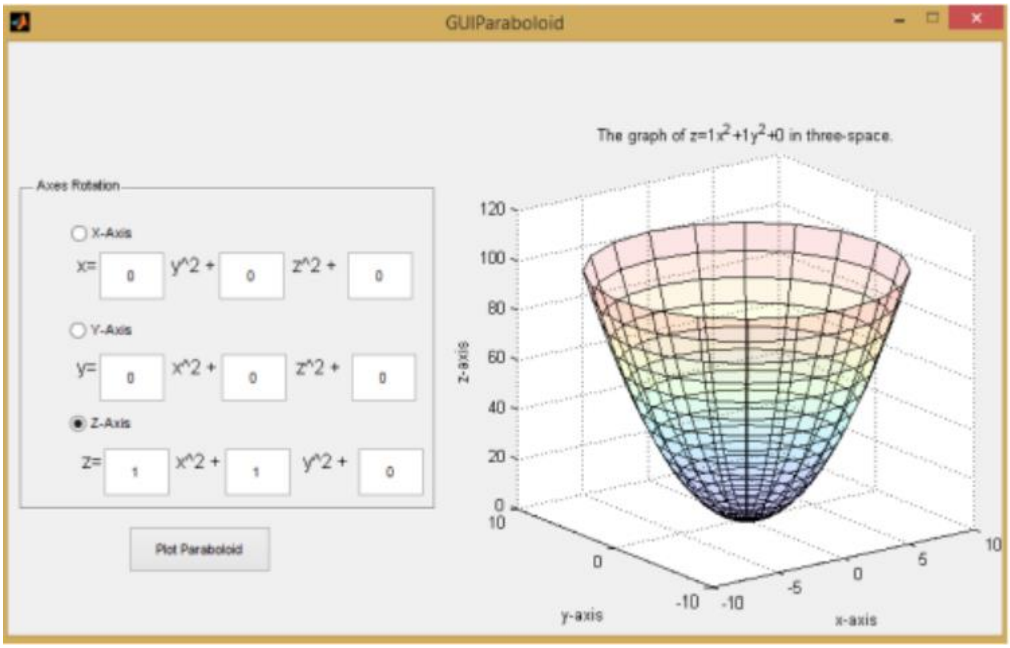
Plot of Raw Data and the Linear and Exponential Curves of Best Fit (Budinski & Takaci, 2011, p. 109)



Visualisation of three-dimensional surfaces can also improve students' conceptual understanding and is possible using CAS software such as GeoGebra (Wassie & Zergaw, 2018) and Matlab (Majid et al., 2013), with applications to Calculus topic areas students often have difficulty with, such as visualising quadric surfaces (Abd Halim et al., 2019). The programming capabilities of Matlab also make it possible to create graphical user interface programs within it, with Abd Halim et al. (2019) describing an example of using this to enable students to input different values of parameters, in graphing quadric surfaces. An example of a paraboloid (which is a type of quadric surface) produced in this way using Matlab, from Abd Halim et al. (2019), is shown in Figure 3.5. Visualisation with different types of coordinates (e.g., polar coordinates) in two- and three-dimensional space is also possible with CAS software, including Mathematica (Nieto & Ramos, 2021).

Figure 3.5

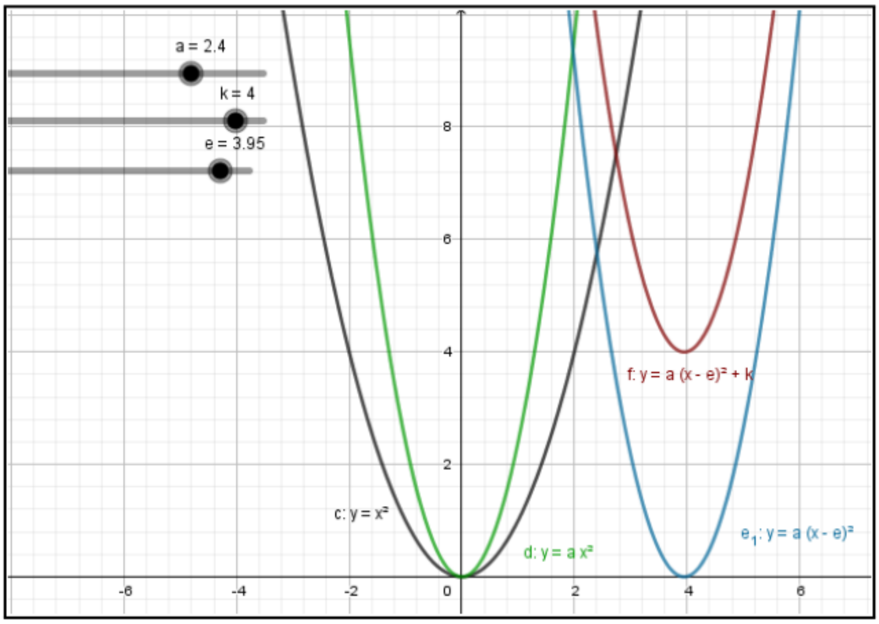
Plot of a Paraboloid Using Matlab (Abd Halim et al., 2019, p. 11)



Many CAS packages now also include a Dynamic Geometry aspect, for example, GeoGebra (Craig & Akkaya, 2022; Wassie & Zergaw, 2018) and Mathematica (Nieto & Ramos, 2021). This allows students to be able to change values of parameters using a slider to visualise how the nature and behaviour of functions change, with this helping to facilitate “good classroom discussion” (Leigh-Lancaster & Stacey, 2022, p. 14). GeoGebra has an increasing number of dynamic tools being added to it (Craig & Akkaya, 2022) and can be used in a wide range of areas of Calculus, to help in developing conceptual understanding through use of dynamic graphical visual mediators. For example, Figure 3.6 from Wassie & Zergaw (2018) shows how GeoGebra can be used to develop students’ understanding of quadratic functions. They demonstrated this by graphing several parabolas where the parameters *a*, *e* and *k* can be varied using the sliders shown near the top left-hand corner of Figure 3.6, to see the corresponding effects on the dilation, horizontal shifting and vertical shifting, respectively, on graphs of the form $y = a(x - e)^2 + k$. GeoGebra can also produce visual mediators representing Riemann sum concepts, by subdividing a curve into *n* subintervals (where the user can vary the values of *n*) and graphing and calculating the resulting area under the corresponding rectangles as a Riemann approximation to the area bounded by part of the curve and the x-axis (Caglayan, 2016).

Figure 3.6

GeoGebra Plot of Quadratic Functions Where the Parameters a, e and k can be Varied using Sliders (Wassie & Zergaw, 2018, p. 73)



While graphical visual mediators produced by CAS can assist students' conceptual understanding, students need to look at them carefully in case they are in a different format from what they are expecting. For example, in Mathematica and Wolfram Alpha, the axes will sometimes not intersect at (0,0), which can make such a graph more difficult for students to interpret (Meagher, 2012) and can lead to students writing them down incorrectly or identifying features such as the y-intercept incorrectly if they just assume the axes will intersect at the origin. An example of this type of graphical representation is shown in Figure 3.7, for a graph of the function $y = x^3 + 9x^2 + 26x + 24$. This is the first of two plots produced by Wolfram Alpha when it is requested to graph this function. While this graph clearly shows the three x-intercepts of the function, if not looked at carefully by a student, it could lead them to conclude that the y-intercept is negative (when in fact it is equal to 24).

The axis limits on graphical visual mediators produced by CAS will also sometimes need to be adjusted, in order to clearly see all the relevant parts of a graph and an understanding of other features of the function and its associated graph is also important. For example, in the case of rational functions, sections of the graph may appear to be "missing," requiring students to adjust the Window settings, in order to see those parts of the graph.

Figure 3.7

Wolfram Alpha Plot of a Cubic Function, with the Axes Not Intersecting at the Origin

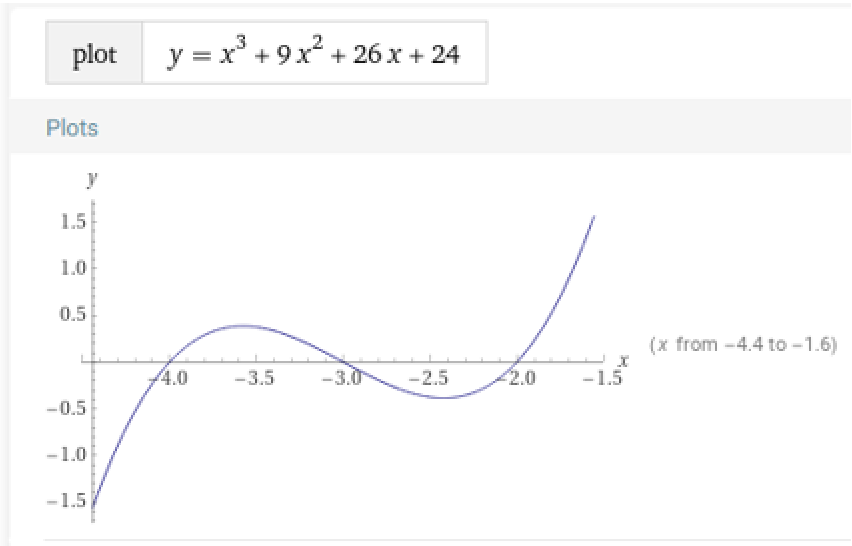
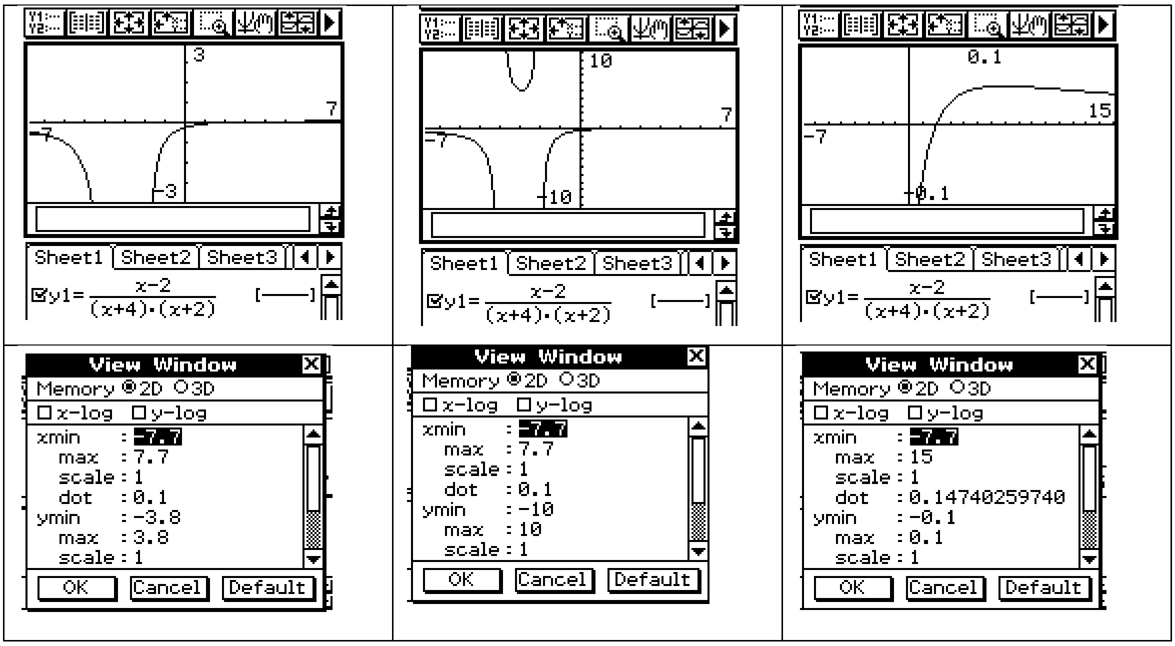


Figure 3.8 shows three images of the rational function $y = \frac{x-2}{(x+4)(x+2)}$, produced by a Casio Classpad 330 CAS calculator, together with the corresponding **window** settings for each image. This is a rational function which has vertical asymptotes at $x = -4$ and $x = -2$ and a horizontal asymptote at $y = 0$. Such asymptotes are also not shown, and therefore need to be determined and drawn by the user (typically using dotted or dashed lines). The graph on the left is the default version of the graph of this function, produced by the calculator, where the section of the graph between the asymptotes at $x = -4$ and $x = -2$ is not visible. The middle graph has had the y-values of the window expanded to range from -10 to 10, which now indicates the behaviour of the function in the section between $x = -4$ and $x = -2$. The graph on the right has had the y-values of the window reduced to -0.1 to 0.1, which now makes it clear how the function behaves when x is greater than 0, where the graph is very close to the x-axis, and how it approaches the horizontal asymptote $y = 0$ from above as x gets large. The information from these three

images could be combined to produce a pen and paper drawing of the graph which is reasonably accurate. This demonstrates the importance of having an awareness of the general form a graph is expected to take, with Quesada (2011) giving a similar example and highlighting the importance of knowing "the number and type of asymptotes that a rational function has" (p. 49 which can otherwise "easily contribute to a student missing a branch of the function" (p. 49).

Figure 3.8

Casio Classpad CAS Calculator Plots of a Rational Function, With Different Window Settings



All of these graphical visual mediators contribute to the discourse of CAS and its potential to enhance learning. For example, Leigh-Lancaster and Stacey (2022) commented that the increased visualisation options provided with sliders and other visual features of modern CAS allow it to be used frequently in an explorative manner, to get an initial idea or insight into data or the nature of a problem before proceeding with the solution process itself. The next section reviews literature which considers the associated symbolic visual mediators present when using CAS, both in the required inputs and resulting symbolic outputs, and the effect these can have on mathematical discourse.

3.3.2 The effect of CAS on mathematical discourse in terms of CAS symbolic outputs and syntax

The symbolic and associated written mathematical discourse of CAS can affect both the teaching and learning of CAS. One reason for this is that the syntax/inputs to CAS often differ from the way the same expressions would typically be written by-hand, in the mathematically endorsed discourse of Algebra, Functions and Calculus. This means that students using CAS "have to be aware of possibilities and constraints, of possible differences between mathematical and CAS functioning, of symbolic notations and internal algorithms" (Thomas & Hong, 2005, p. 298). This can present challenges for students in learning to use the technology effectively, with Sangwin and Ramsden (2007) identifying the difficulties presented by needing to mediate between "traditional mathematical notation" (p. 921) and what is required with CAS syntax. In looking at different types of CAS, including Mathematica, Maple, Axiom, Derive and Maxima, Sangwin and Ramsden (2007) identified differences in the nature of the syntactical input required, including that all these systems except for Derive and Mathematica "were strict in requiring an explicit multiplication sign * in algebraic expressions" (p. 925) (which is also still the case in more recent versions of these programs). Sangwin and Ramsden (2007) explored first-year mathematics students' ability to manage Maple's CAS syntax, finding that omission of the * sign was the most common error these students made in entering the required syntax for solving a variety of mathematics problems. For example, the expression *day* would automatically be interpreted in such CAS packages as the name of a variable (or constant), whereas in the traditional discourse of algebra it could represent $d * a * y$, if d, a and y had each been defined to represent individual variables or constants. One consequence of expressions involving two or more letters (with no multiplication sign between them) being interpreted as new variables, is the use of such multi-character variable names can potentially help some students keep meaning of expressions in sight when working in a CAS environment (Pierce & Stacey, 2004), as is also possible when working by-hand with algebra. The different requirements in how multiplication of letters and variable names can be entered using different types of CAS, illustrates the importance of being aware, not only of the syntactical requirements of the type of CAS being used, but also of the need to keep track of the letters used to define all variables and constants being used in solving a mathematics problem (which is also important when working by-hand using algebra). This is especially the case if students are working concurrently with more than one type of CAS, as sometimes occurs at tertiary level, or if they are using different types of CAS to those which they are familiar with from past experience).

Another difference in syntax compared to the usual discourse of mathematics occurs with the Mathematica CAS package, with the requirement to enclose the argument of a function in square brackets (Abell & Braselton, 2021), as opposed to the parentheses used in the standard discourse of functions and in most other CAS packages. One advantage of Mathematica requiring this, described by Sangwin and Ramsden (2007) is that, unlike the situation with most other CAS packages this also allows for “a multiplication sign to be omitted” in algebraic expressions “wherever this would not be ambiguous” (p. 925). In practice, this means that in cases where a letter is followed by a bracketed expression in parentheses, Mathematica will interpret this as algebraic multiplication, while most other types of CAS software would interpret this is as a function. For example, in most other CAS programs, an expression such as $c(a + b)$ would be interpreted as a function c with argument $a + b$ whereas in Mathematica, this expression would signify c multiplied by $(a + b)$, consistent with the usual discourse of algebra. This demonstrates the importance of having an awareness of the *context* in which an algebraic expression is being used (e.g., functions, algebra etcetera) and the nature of the required symbolic discourse to be input into the specific type of CAS being used.

In the case of using CAS calculators or menu-driven CAS software, it is also important to be aware of the correct menu to use in a given context, according to the rules of the CAS. For example, on TI-NSpire CAS calculators, the **derivative** command only produces a *partial* derivative, meaning that when carrying out implicit differentiation for instance, it is actually not the correct commend to use and a separate **impDif** command is instead appropriate in such cases.

In addition to their difficulties with omission of the required * sign when using most types of CAS, incorrect use of parentheses in CAS has also been identified as a common source of difficulty for students (Pierce & Stacey, 2004; Sangwin & Ramsden, 2007) in part, again, because of differences between conventions when entering expressions in CAS and when writing them by-hand. Some versions of CAS also require functions to be formally defined (Pierce & Stacey, 2004) before they are able to be used; an additional discursive requirement that students need to be aware of when using those types of CAS.

All of the syntactic requirements of different types of CAS just described involve the symbolic *discourse* of CAS, and demonstrate the need for students to engage with both the endorsed discourse of Algebra and Calculus that they are familiar with from working by-hand and the, at times different, rules of discourse associated with any types of CAS they are using in their studies. The syntactic and associated written discourse of CAS has also been evolving, with, for example, the most recent versions of Mathematica including provisions of suggestions for syntax and "an auto-completion function for commands" (Nieto & Ramos, 2021, p. 150), to assist users, potentially making students' transition to engaging in its discourse smoother. Another recent addition to Mathematica, with important implications for how students (and others) engage with it discursively is the option to write commands in "free form", in which Mathematica itself interprets and translates the request the user makes in English into the language of Mathematica (Nieto & Ramos, 2021), although this raises the associated question of what types of written discourses Mathematica has been programmed to recognise as commands. The relative complexity of syntax for different CAS programs can be one of the aspects considered by instructors, with, for example, Dimiceli et al. (2010) seeing use of Wolfram Alpha to be advantageous in part due it requiring less syntax to be typed than many other CAS packages of that time, in order to obtain the solution to many Calculus problems.

A word of caution about the use of CAS syntax is given by Jankvist and Misfeldt (2015), who warn that when using some CAS commands, there is less opportunity for students to discover errors that they would have been more likely to identify if working by-hand. This is because if these errors were made in hand calculation, the associated methods would then be less likely to "make much sense in the translation of the problem" (p. 16). To illustrate, they give the example of the **deSolve** command being used incorrectly by a student on what is meant to be a separable differential equation, but which the student enters incorrectly into their CAS calculator, by entering x instead of y on the right-hand side, resulting in a different type of differential equation. Jankvist and Misfeld (2015) point out that "the **deSolve** command does not require students to distinguish between different types of differential equations" (p. 16), making such errors less likely to be picked up. They express the concern that, if students mostly rely on using CAS to solve problems of this type, then this could undermine their ability to acquire mathematical understanding through objectification and the ability to separate such mathematical objects from the associated mathematical processes. This highlights the importance of effectively integrating the discourse of CAS commands, such as **deSolve**, into Calculus teaching, so they are taught in the context of also understanding the nature of the solutions of different types of differential equations, as opposed to, for example, merely being taught these commands as processes to learn by rote.

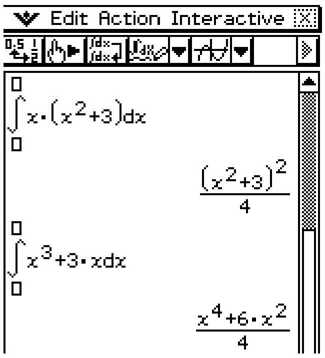
Programming with CAS software has also sometimes been taught to help in solving mathematics problems, such as, for example, with Matlab, Maple (Betteridge et al., 2022) and CAS calculators, such as the TI-Nspire CAS (Kadijevich, 2014). While the use of CAS for writing such programs is not a focus of this thesis, it is important to acknowledge as it increases the complexity and extent to which participants must engage in the discourse of the CAS technology when CAS is used in this way.

The differences in syntax and written commands when using CAS can have big effect on the classroom discourse of mathematics. This firstly occurs directly, due to the different rules of the discourse, in cases such as those described in Algebra and Calculus, where symbolic expressions need to be entered in a specific way, which may differ from that which students have typically used when working by-hand. Secondly, CAS-related commands (e.g., **deSolve**) are integrated into the, sometimes colloquial, discourse of how instructors and students talk about solving mathematics problems, which has a direct effect on their resulting written and verbal discourse. Further investigation of how students talk about CAS when solving Calculus problems is considered in this study. Likewise, the effect of the discourse of CAS on written examination questions is considered.

The symbolic *outputs* produced by CAS are also sometimes different from those produced in working by-hand, with the methods used by CAS to solve mathematics problems not always readily available for users to see, and where "techniques used by CAS typically do not reflect techniques used" when working by-hand (Kadijevich, 2014, p. 86). Subtle differences in the way an input expression is presented can also change the algorithm CAS uses, resulting in the answer being presented in different forms. For example, in describing integration of polynomials using a TI-89 CAS calculator, Thomas and Hong (2005) observed that "CAS deals with integration of certain factorised forms of functions that it can 'recognise' in a manner different from the standard format polynomial" (p. 218). They give the example that integrating the factorised expression $x(x^2 + 3)$ produced a different form of the answer to that obtained if integrating the equivalent expression $x^3 + 3x$ with, in this example, the arbitrary constant of integration in the answer equal to 9/4 in the first case (with the answer needing to be expanded to see this) and 0 in the second case. This also occurs with more recent models of CAS calculators, including the TI-NSpire and the CASIO Classpad; the resulting output of the latter is shown in Figure 3.9. In turn, this can cause confusion for some students, in examples such this, where the constant term in the resulting answers differs. This can also occur with the integrals of some trigonometric functions where, in addition to the constant of integration sometimes being expressed differently, the answer as a whole can also look very different, due to the identities that relate different trigonometric functions.

Figure 3.9

CASIO Classpad Calculator Output, Showing the Integral of the Same Function in Two Different Forms



The example in Figure 3.9 is also typical of CAS calculators, in that they omit the constant of integration, meaning they are presenting an antiderivative, as opposed to the integral of a function, which again can lead to student errors (although omitting the constant of integration is also a common error students make even when working by-hand). In seeing equivalence between different expressions or being able to effectively simplify some relatively complicated solutions produced by CAS, other mathematical skills are required, such as “algebraic insight and an understanding of the conventions and limitations of the technology being used” (Pierce & Stacey, 2004, p. 15). In some cases, in determining equivalence of two expressions, knowledge of functions including their natural domains or the identities that relate them (e.g., trigonometric identities) is also necessary.

In learning Calculus, it is also important for students to be aware that in some cases, CAS can actually produce incorrect or incomplete outputs. For example, Matlab can sometimes produce incorrect results for integrals of unusual functions, because “the deterministic routine **integral** samples its functions adaptively” (Betteridge et al., 2022, p. 8). Lehning (2002) also demonstrated cases, where Maple and Mathematica at that time gave incorrect answers for certain integrals. CAS will also at times not specify restrictions on a function's domain when simplifying expressions, such as the TI-NSpire CAS calculator typically simplifying $\frac{x^2}{x}$ to be equal to x, without specifying the constraint that $x \neq 0$ (Kadijevich, 2014), demonstrating the importance for students to know "when to use its affordances with care" (p. 86).

To investigate how students deal with CAS output when it is in a different form to that obtained when working by-had, Tonisson and Lepp (2015) gave problems to a sample of 38 pairs of first-year university students, who were required to solve seven trigonometric equations both by-hand and using the CAS computer package, Wolfram Alpha. The students were then required to evaluate their confidence in their hand-calculated answers when comparing them to the answers produced using CAS and to judge whether or not they thought the answer they had produced and the corresponding answer produced by CAS were equivalent. The students' solutions of each equation and their opinions on equivalence/non-equivalence between their hand-calculated and CAS answers were then analysed quantitatively, in terms of how they rated their level of confidence in the correctness of each by-hand answer and how unexpected they rated the corresponding CAS answer to be. The students were also asked how they thought the two answers were related and to analyse any differences. Tonisson and Lepp (2015) found that in cases where students' answers looked very different from the answer produced by Wolfram Alpha, the students often made the wrong judgement about whether or not their hand-calculated answer was correct, and that they did not provide much analysis of why their answers looked different from those of Wolfram Alpha. This suggests that typically they "did not know how to determine equivalence/non-equivalence and quite often did it incorrectly" (p. 120). As a consequence, Betteridge et al. (2022) observed that when using CAS software, students should be aware of its limitations, should be able to identify when an answer given by CAS is incorrect and thus that they should take responsibility for evaluating the correctness of the answers produced by CAS.

CAS software will also sometimes produce additional lines of output that are not required to answer the question students are trying to solve. Pierce and Stacey (2001) gave the example of the CAS software, DERIVE, providing both real and complex solutions to equations, which caused confusion among some undergraduate Calculus students in Pierce's study (Pierce, 2001) of use of the DERIVE software, due to students having no experience of complex numbers and therefore not even understanding what the complex number notation signified. Software such as Wolfram Alpha also frequently produces additional lines of output not necessarily required by a student in answering particular types of questions, suggesting that how complex students will find such output will depend on the extent to which they are knowers of outputs of this type. One aspect that will affect this is how familiar the students are with the mathematical discourse which is present in such outputs, both in the symbolic visual mediators used and the use of any mathematical words. Their general understanding of the solution process will also determine how they interact with the output, in identifying which parts they need to consider in answering a given question.

These aspects of CAS, that is, requiring different written inputs and sometimes producing unexpected outputs in the form of symbolic visual mediators, have not been analysed previously explicitly using a *commognitive* framework. However, consistent with one of the main aspects of commognition, some researchers have previously applied *sociocultural* approaches to analysing the role played by *instruments* which include CAS calculators. For example, consistent with the commognitive perspective, Artigue (2002) describes mathematics as a human activity, with its resulting products being "dependent on the social and cultural contexts where they develop" (p. 265). When investigating the relationship between the outputs produced by CAS, relative to the context of work done by-hand solving associated mathematics problems in a classroom setting, Artigue (2002) identified that, when considering CAS, there is a greater diversity of ways in which mathematical objects are represented due to the algorithms in the software. The transformations between such objects are codified by the software, with CAS breaking "institutional norms by frequently producing very unexpected results" (p. 265). In turn, this results in difficulties when trying to establish the equivalence of expressions which, at times, "go far beyond what is usual for the classroom" (p. 265). This means that, when using CAS, students need more, rather than less, versatility mathematically and an ability to recognise mathematical expressions which are expressed in a variety of different forms (Leigh-Lancaster & Stacey, 2022), including requiring algebraic insight "to deal with the sometimes surprising answers provided by the machine" (Stacey, 2003, p. 40). Berger (2009) also cautions that "we cannot assume that students will be able to adequately interpret CAS output, even if they are able to generate the appropriate CAS commands" (p.

34) to produce the required output, demonstrating the importance of integrating these aspects of the discourse of CAS into teaching.

Changes in the type of technology used from high school to tertiary level also frequently occur and this includes with CAS technology, with students sometimes learning new types of CAS software packages during their study of tertiary mathematics. This also requires discursive shifts, due to the different ways in which different CAS variants present answers, as well as in the differing nature of the required inputs, as was described above. The ability to make connections between the symbolic and graphical outputs produced by CAS is also important, and this will be discussed now, in section 3.3.3.

3.3.3 Linking the different representations of CAS in teaching and learning

One of the most powerful aspects of using CAS is being able to *link* its different algebraic, graphical and numerical representations, with this allowing for "a variety of approaches for explaining concepts and for solving problems" (Leigh-Lancaster & Stacey, 2022, p. 17) in teaching and learning. However, this must be done effectively, to allow CAS to "become the springboard from which students' learning of Calculus is infused with, and enhanced by, the affordances of technology", rather than becoming an obstacle to be overcome (Meagher, 2012, p. 8). Which of these that occurs depends on the nature of the relationship that evolves between a student and CAS technology. Such teaching includes increasing students' awareness of the ways in which CAS outputs can appear, so that they do not end up spending too much time trying to reconcile CAS output with their expectations when it is in a different form. Meagher (2012, p. 11) described a study in which students were using Mathematica but where "little or no effort" had actually been made to teach it to them. She observed that, in this situation, the students spent more time on *Mathematica* "rather than doing mathematics" (p. 11), demonstrating that students need learning experiences with new software rather than being expected to just use and learn it by themselves.

In cases where CAS is used and taught effectively, it can assist with students' conceptual understanding and also allow them to be able to learn more complex concepts. For example, in students' learning, "CAS may also enable new types of dynamic representations and interactions with representations that challenge understanding, as well as promoting inter-representational thinking" (Thomas & Hong, 2005, p. 218). Dimicieli et al. (2010) describe how using the CAS software Wolfram Alpha promotes this inter-representational thinking, as it “will report all knowledge from its database in a variety of formats”, meaning that, at times, “one simple query can result in an equation, a graph, an exact answer and an approximate answer” (p. 1063). They give the example of how, when evaluating a definite integral, Wolfram Alpha will provide both the numerical answer and the graphical interpretation showing the area bounded by the curve and the x-axis, allowing students to “see connections in how the integral is set up and how the value of the integral changes based on the limits of integration” (p. 1066).

However, students do not always utilise the full multi-representation capabilities of CAS available to them. For example, Rogovchenko (2021) used a commognitive analysis framework to examine how a sample of fourth year Engineering students solved mathematical modelling problems involving differential equations, in a setting where they had access to Maple and Matlab CAS software. The set mathematical modelling problems were designed to encourage the use of *explorative* mathematical routines, but it was found that most of the students’ routines involving using CAS were *rituals*. While, at times, the students used it for numerical computations or to produce graphical visual mediators for data visualisation, they also did not take advantage of its mathematical symbolic capabilities for analytically solving the differential equations involved in the problem-solving process.

Use of CAS can change both the *pragmatic* and the *epistemic* value of topics being taught (Artigue, 2002; Stacey, 2003). For example, CAS can be put to *pragmatic* use in saving time with routine procedural tasks, allowing more time to consequently develop students’ conceptual knowledge through its multiple representation capabilities, which in turn would be using its *epistemic* value. In some cases, CAS can be used, in teaching, in both of these ways. One instance of this, in the context of teaching students about ordinary differential equations (ODEs), is described in a study by Matos (2014), where Maple CAS software was used in teaching a university unit on this topic. In this unit CAS was used in a *pragmatic* way to carry out the steps in the process of solving differential equations that the students would already be assumed to have mastered from their previous studies (e.g., some graphing, basic algebra, finding matrix determinants, working out partial fractions etcetera), that is, drawing on their precedent search spaces. Where the steps in solving the differential equations involved processes and concepts that were new to the students, these were done by-hand. According to Matos (2014), this approach is judicious CAS use, which “enables the instructor to work a larger number of more complex problems” (p. 1) due to the time saved, while ensuring that Maple did not become a *crutch.* The instructors also introduced the students to Maple's **dsolve** command, which can solve ODEs in one step, but for the purpose of checking answers already worked out by using more intermediate steps, with the students also being advised that the **dsolve** command might not always give answers in the same form as when working by-hand. However, CAS was also used for some *epistemic value* to assist with the students' conceptual understanding, as the unit also included use of Maple's **DEplot** command, which can produce solutions curves for ODEs and the associated direction fields, to allow the students to investigate and understand "how small changes in a model or equation can change the end result" (p. 4).

Another study which involved *epistemic* use of CAS to assist with conceptual understanding was Hong and Thomas (2015), where the teaching of a first-year Calculus unit took advantage of the multiple representation capabilities (algebraic, graphical and tabular) of CAS. The aim was to encourage students' "versatile thinking about functions, especially in relation to properties arising from a graphical investigation of differentiation and integration" (p. 183), using the TI-Nspire CAS calculator and Autograph dynamic graphing software. The calculator was used to tabulate values of h increasingly close to 0 to use in the formula for a derivative by first principles, to help students understand the concept that the derivative is the limit of this expression as h approaches 0. They were also shown how to visualise the derivative as the gradient of the tangent to a curve at a point by drawing a tangent to the curve with CAS at different points, thus providing a connection between symbolic and graphical representations. This was to improve their understanding of the relationship between the graph, the slope of a tangent line and the associated derivative. To further enhance students' understanding of this relationship, additional visual examples where a tangent line cuts the curve in more than one place or coincides with part of the curve could also be valuable, in addressing the common misconception of many students that tangents always touch a curve at only one point (Biza, 2021). In the integration module of the study by Hong and Thomas (2015), students were taught visualisation of Reimann sums with an Excel spreadsheet and graphing windows on the CAS calculator. The number of subintervals was varied so that they could see the improved approximation of the area under the curve when more, smaller, subintervals were used, and relate that to the definition of an integral as the limit of the corresponding Riemann sums. The study demonstrated how the different capabilities of CAS can be combined, with the aim of "encouraging students to develop *epistemic* mathematical techniques" (p. 198). Such epistemic techniques become increasingly important in the transition from secondary to tertiary level; a transition which Hong and Thomas (2015) describes as having "a number of epistemological gaps" (p. 183), including needing to move to a local or interval perspective of functions. Investigating differentiation and integration using graphical methods, such as those just described, can help students in developing this way of thinking about functions at tertiary level.

In addition to CAS being taught to have *pragmatic* or *epistemic* value in classes, the ways in which it is presented in textbooks (and course notes) can also appear in either, or both, of these forms. For example, Misfeldt and Jankvist (2018) examined Danish senior secondary mathematics textbooks in which CAS was used in a mostly pragmatic way, in "outsourcing" some or all steps in what is described as a "proof", in some cases producing only "a proof that may convince, not a proof that explains" (p. 379), also raising epistemological questions about what is considered a sound mathematical "proof". In other cases they found use of CAS to augment "the classical argumentation" by providing empirical and "techno-authoritarian" confirmation of the results obtained at particular steps in a proof (Misfeldt & Jankvist, 2018, p. 380). They also found instances where CAS was used after traditional completion of a formal proof, for verifying the correctness and validity of the proof. However, they highlight the importance of not allowing the main reasoning in the proof to be obscured by over-emphasis on the authority of CAS as the judge of the correctness of individual steps contained within such proofs, an example illustrating the importance of using CAS wisely.

The Calculus Reform Movement, which originated in the late 1980s, strongly supported the epistemic use of technology such as CAS in textbooks, to develop students' conceptual understanding, advising that "instructional content should be enhanced with multiple representations and should be supported with technology" (Sevimli, 2016, p. 2). Consequently, textbooks aimed to promote courses in which understanding and computation reinforced each other. However, while this movement has led to greater use of CAS and other technology in Calculus textbooks, the emphasis has not always been on *epistemic* uses of CAS, with, for example, Hughes-Hallett (2006) observing that the majority of textbooks of the time typically had "open-ended problems and extended applications" only appearing "as an add-on at the end of the chapter" (p. 45), rather than being an emphasis. Bressoud et al. (2016) observed that almost all textbooks at the time of their study now recognised the importance of including graphical and numerical representations of derivatives and integrals. More recently, Pogorelova (2022) conducted a detailed analysis of the first chapter, on functions, from the 2018 edition of Hughes-Hallett's Calculus textbook. Pogorelova (2022) concluded that, consistent with the objectives of the Calculus Reform Movement, the chapter emphasised "problem solving skills and multiple representations to help students visualise concepts," while also finding that chapter still had a "heavy emphasis on symbolic and algebraic representations," (p. 24) identifying that the latter could be due to the necessity for students to "acquire competency in mathematical language that serves as a foundation for future topics" (p. 24). The constantly changing and evolving nature of CAS technology (Buteau et al., 2014; Leigh-Lancaster & Stacey, 2022) provides opportunities for it to be further integrated into teaching and associated learning materials, with CAS evolving to provide solutions to an increasing "range of question types and algebraic structures" (Leigh-Lancaster & Stacey, 2022, p. 9), even though, in practice, this does not always occur.

3.3.4 Key impacts of CAS technology on learning and discourse

CAS calculators and computer packages have distinctive symbolic *discourses*, which are not always of the type engaged in when working by-hand. This can have an impact on the nature of the required algebraic expressions and commands to be input into the CAS, highlighting the necessity of an awareness of the context in which an expression is being entered (e.g., algebraic multiplication or a function). Studies considered in this review have shown that students sometimes struggle with this syntactical aspect of CAS. This is partly because the rules of discourse that students need to engage in

difficulty with. In helping students to build skills for managing the differences between the algebraic outputs produced by CAS and those generated by-hand, developing their algebraic insight was said to be important, as well as their understanding of any limitations in the answers presented by CAS in solving certain types of mathematics problems. CAS software will also sometimes produce multiple lines of output, some of which might not be familiar to students, potentially affecting the complexity of problems given to students where such outputs are provided in teaching/learning/assessment.

The capabilities of CAS calculators and computer packages to produce a wide range of graphical visual mediators can be utilised in teaching, to develop students' conceptual understanding of Calculus topics including differentiation, integration, differential equations, limits and applications of these topics (e.g., Taylor polynomials). The ability to link such graphical visual mediators with associated symbolic representations of mathematical objects is an important skill and, when CAS is used in this way, it is being assigned an epistemic role. CAS can also have a pragmatic role in teaching and learning when it is used to carry out mathematical processes that are time-consuming to do by-hand, including in parts of a solution process that might not be an emphasis in the topic being taught or to solve problems (e.g., factorising high degree polynomials, working out complicated integrals) that are beyond the skills of the student cohort. Use of CAS can also eliminate the need for some procedural steps that would usually be required when working by-hand, which, combined with its multiple-representation capabilities, have the potential to allow more time for teaching additional content or for teaching content associated with the topics already covered which assists with conceptual understanding, as for example, in the case of linking graphical and symbolic representations. As with the symbolic outputs produced by CAS, an awareness that graphical outputs will not always be in the form expected if working by-hand is important, as in the cases of axes not intersecting at the origin. An awareness of the general properties of the functions being graphed is also important, as was demonstrated in the case of rational functions, in order to ensure that key pieces of information are not missed when interpreting such graphs or reproducing them by-hand.

The mathematical capabilities of CAS calculators and software, and the nature of their associated discourses are important considerations when setting examination assessments where CAS has a role. Having clear objectives as to the extent to which CAS is expected to have pragmatic value and the extent to which it should have epistemic value is also important in devising suitable assessment examination questions. The use of CAS technology in assessments will now be discussed in section 3.4, with a focus on its use in test and examination assessments.

3.4 Use of CAS Technology in Examination Assessments

When setting examination tasks where CAS (or other digital technology) is permitted, the question arises of how the examination tasks will change and what students do mathematically during such examinations (Fahlgren et al., 2021). Aspects to consider include the types of questions that can be asked, the discourse of such questions, how to judge their level of difficulty and what impact CAS has on this. There are also practical factors to consider in running such examinations, such as what technology students will have access to when completing them.

It is important at this stage to establish that in this thesis, my focus is on what is called "assessment *with* technology" (Drijvers et al., 2016; Fahlgren et al., 2021), meaning assessment where students have access to CAS technology on calculators or computers, during the examination, which has the potential to assist them in answering some questions, or, in some cases, where CAS output from such technology is provided as part of written examination questions. This is as opposed to "assessment *through* technology" (Drijvers et al., 2016; Fahlgren et al., 2021), which I am not considering, where technology provides the testing environment, such as in online examinations and quizzes where students enter their answers into an online interface, and, in some cases, also have their answers automatically marked by computer software (which will in some instances also provide them with feedback). For the review of literature in this area, I will also consider studies such as Drijvers (2018) which involve examinations that are simultaneously conducted both with *and* through technology. I am also focusing on *summative* assessments such as examinations, which have the overall "aim of grading students' level of proficiency" (Fahlgren et al., 2021) in the mathematics topics being assessed.

Subsection 3.4.1 now follows, providing background to the use of CAS and other technology in examinations at secondary school and university level in Australia. Section 3.4.2 will then review literature on factors to consider in conducting CAS-active examinations. This includes the administration of such examinations, ways in which academics have classified and analysed the relative impact of CAS on different types of examination questions, the opportunities CAS presents for testing additional elements of students' conceptual knowledge, and considerations when CAS produces outputs in a different form to that obtained when working by-hand. Finally, the issue of CAS presenting more possible solution paths to many questions and implications of this for how students are required to present their answers in examinations will be reviewed. The focus will be on studies which investigate these aspects for examinations on Calculus topics, especially at tertiary level.

3.4.1 Use of CAS and other technology in assessment

The extent of technology use, including CAS use, varies at secondary level, even within Australia, and this is reflected in the different requirements in the different states about what technology is permitted in final year secondary school examinations. For example, in Victoria (Leigh-Lancaster & Stacey, 2022), Western Australia and Tasmania; a CAS calculator is expected in senior secondary school mathematics examinations (with, in those subjects which include Calculus topics, the students also sitting a second, CAS-free examination). In these states, a natural incorporation of the technology into the secondary school curriculum and assessments has progressed over the years; a situation resembling that foreshadowed by Madison (2001) and Artigue (2002). In Victoria for instance, where my study is situated, CAS calculators have been the assumed technology in examinations in the core units 'Further Mathematics' (both examinations), and 2-hour technology active examinations in both 'Mathematical Methods' and 'Specialist Mathematics' (since 2010) in year 12, with these latter two units also having a one-hour technology-free examination (Leigh-Lancaster, 2010; Leigh-Lancaster & Stacey, 2022; VCAA, 2016). This contrasts with the situation in some other Australian states, where only graphics calculators (South Australia, Queensland) or only scientific calculators (New South Wales) are permitted.

In contrast, while use of CAS technology is relatively common in regular university level teaching (Jankvist et al., 2021) and, in such cases, is often permitted in coursework such as assignments (Buteau et al., 2014), permission to use CAS calculators (and even graphing calculators) is much less frequent in university level summative tests and examinations (Kemp et al., 2013; Selinski & Milbourne, 2015), creating "a sharp discontinuity between high school and college calculus" (Thomas et al., 2017, p. 111). CAS use in examinations in Australia at tertiary level is far less widespread than seen at school level (Kemp et al., 2013; Kissane et al, 2015). Similar restrictions on CAS usage in examinations apply in many, but not all, overseas countries. Recent examples outside Australia where CAS is permitted in at least some examination assessments include in Denmark and Germany (Jankvist et al., 2021), the Netherlands (Craig & Akkaya, 2022) and the UK (Betteridge et al., 2022).

3.4.2 Practical considerations in use of CAS in assessments

When setting examinations where use of CAS is permitted, the environment in which the examinations take place, the type of questions it is most beneficial to ask, and the type of answers students might be expected to produce, all need careful consideration. Several aspects of the examination environment need to be taken into account, including what technology students may be permitted to access in such examinations and practical considerations in ensuring this occurs. Factors that can be taken into consideration in setting the actual examination questions in a CAS-active environment include the relative impact that CAS will have on the level of difficulty of such questions, whether it allows for asking types of questions that previously would not be possible, and how students might be required or expected to present their answers, when they are permitted to use CAS technology in an examination. The sub-sections that follow address these aspects, by reviewing a selection of the relevant literature.

3.4.2.1 Managing students' access to stored or online information in CAS-active examination environments

When considering examinations in which use of CAS technology will be permitted, practical concerns are often present about the types of information students will be able to access in the examination environment. A survey of CAS usage in universities in Western Australia in 2015 found that handheld CAS calculators were generally banned in mathematics examinations for units taught at each university (Kissane et. al, 2015). Kemp et al. (2013), in a survey of academics at 28 Australian universities, found that staff concerns about use of graphing or CAS calculators in examinations "did not seem to be related to the mathematical capabilities of the calculators in most cases, but rather to the capacity of the calculators to provide a form of text storage" (p. 6). This situation is not unique to CAS calculators, as data and text storage is also possible on some other, less advanced types of programable calculators, such as graphics calculators. Tablets and smart phones are also not generally permitted in university mathematics examinations at this stage (Thomas et al., 2017), due to their remote access capabilities.

To minimise the effects of students potentially storing data and other information on their CAS devices, one consideration is that an emphasis in CAS-active examinations on asking questions that require "critical thinking rather than simple computations" (Keimer et al, 2022, p. 7), would reduce the influence of any stored data and programs on how well students perform. That is, such examinations would focus on questions where accessing stored written information using the technology would not assist in answering them, due to a focus on formulating problems and interpretation of results, rather than rewarding recall or computation. For example, Craig and Akkaya (2022) describe use of open-book examinations in a Vector Calculus unit, where the examination questions focused on interpreting existing CAS outputs and solving or formulating problems, rather than setting questions that required a theorem or formula to be directly stated.

Another concern, with actual computer-based examinations, is the potential for students to access additional resources online which are prohibited during the examination, such as accessing other forms of CAS technology. One way in which staff have avoided this problem, is to set up the computers the students use, so that some features are disabled. For example, Betteridge et al. (2022) describe mathematics examinations that were conducted where students had access to computers where use of Matlab was expected, but where the computers used in the examinations were “locked down so that only Matlab (and its help system) could be used” (p. 261). Drijvers (2018) examined the example of an examination given to high-achieving 18 year-old students in Finland where, in part of the examination, students could access laptop computers with a selection of different types of CAS software available, but, as in Betteridge et al. (2022), where each student’s computer was in a locked down mode so that internet access and access to any additional, unauthorised software was not possible (p. 53). Drijvers (2018) identified an advantage of such access to *multiple* CAS software options as solving “the issue of computer skills challenging the test validity” (p. 54), in that it is likely the type of CAS software individual students are familiar with from past learning will be one of the options that is available to use. In a study of the use of CAS technology in year 12 mathematics examinations in Victoria, Leigh-Lancaster and Stacey (2022) interviewed four consultants with experience in teaching and assessment. These interviews brought out that equity in teaching and assessing students with CAS should extend “beyond examination questions to having all models of devices well represented in textbooks and teacher support materials, so that all students can be taught about the range of capabilities of their own device” (p. 9). However, given the current large number of different brands and models of CAS, this might not be the most practical option. Matos (2014) advises that, in reference to concerns about students breaking examination rules by accessing free online CAS software such as Wolfram Alpha during examinations, that “if students are allowed to use laptops or tablets on exams, the instructor must be vigilant in keeping an eye on the computer screen, and being aware of how solutions on Wolfram Alpha look” (p. 5).

These concrete cases demonstrate that, even though the use of CAS technology needs to be implemented carefully in examinations, with likely restrictions imposed on unauthorised additional online resources, there are still a variety of ways in which such examinations can be effectively administered, meaning these concerns need not create an obstacle for the use of CAS in examinations.

3.4.2.2 Judging the impact of CAS on examination questions

A second consideration in setting CAS-active examinations is what impact the presence of CAS will have, when available for the students to use during such examinations. Kokol-Voljic (2000), in discussion with Bernhard Kutzler, produced a classification of individual examination questions in a CAS setting, according to whether solving them would involve *no CAS use*, *primary CAS use* (where use of CAS would be the main activity in answering the question) or *secondary CAS use* (in which CAS would have only a minor role in answering a question); combined with further classifying these questions which involved CAS use as either requiring only superficial knowledge of the tool or “in-depth” knowledge of it. MacAogáin (2000) classified examination questions as either *trivial with CAS* if they involved direct CAS use and typically reduce this to requiring no more than two or three steps, *easy with CAS* if CAS substantially reduces the difficulty but some additional mathematical knowledge/skills are also required (p. 142), *difficult with CAS* if CAS could help in answering them but the questions would still be considered “difficult”, and *CAS proof* if CAS would be of little or no use in answering them in the required way. While these classifications of the impact of CAS on examination questions can be helpful in decision-making on the types of questions to include in examinations and judging their impact, one limitation is that there can be some subjectivity about which category some examination questions fall into, for example, in cases where CAS has some impact on the solution process and where some conceptual understanding and/or working by-hand is also required. This was also identified as a limitation by Kokol-Voljc (2000) and Flynn and McCrae (2001). As an alternative that avoids this ambiguity, Hong et al. (2000) devised a simpler classification which identified questions as either *calculator positive* or *calculator neutral*, based on whether there was any perceived advantage in using a CAS calculator to answer a question. In relation to the impact of CAS, Brown (2003) classified questions as *CAS required* if students were not expected to be able to solve them without help from CAS, *CAS optional* if use of CAS could be helpful in answering a question but where its use would not be considered to make the question much easier to answer, *CAS neutral* if CAS “has no potential to contribute to the solution of the question” (p. 156) or *CAS excluded* if the question is such that CAS cannot be used directly to answer it.

Which classification is most useful/helpful partly depends on the *purpose* of using it in classifying examination questions in relation to CAS. For example, after considering the schemes described above, Oates (2011) chose to use the relatively simple scheme of Hong et al. (2000) in his study which involved developing a taxonomy regarding the integration of technology and its impact on the teaching, learning and assessment of tertiary mathematics. His taxonomy included a section to capture key aspects of the extent and nature of implementation and use of CAS technology in assessment. He chose the framework of Hong et al. (2000) because his study was “primarily concerned with whether CAS-availability has had any effect on the nature of questions, as opposed to the degree of that advantage” (p. 170).

In addition to considering, as described above, the relative impact of CAS in how much it might assist in answering examination questions (as compared to not being used), another important question is how to test the students to best measure their learning of the subject and their associated understanding of the underlying concepts of what they have been taught. This raises questions such as how to test students so that they are given opportunities to demonstrate their mathematical abilities. Drijvers (2018) observes that use of digital tools can trivialise some procedural parts of solving mathematics problems, raising questions about the importance of students demonstrating such skills. However, he also identifies that allowing students to do basic procedural work using CAS could save them time and therefore also allow for setting questions that test their “higher-order thinking skills” (p. 47).

One option would be to modify some questions so that CAS can be used, but where some intermediate steps or understanding of the nature of the problem must also be demonstrated by the students in answering such questions. For example, in solving a second-order, non-homogeneous, Ordinary Differential Equation with constant coefficients, the instructor could first ask the student to classify it, then to provide the auxiliary equation and *then* to solve the actual non-homogeneous equation itself (Matos, 2014; Tobin & Weiss, 2011). However, if the use of CAS technology is designed to promote innovative assessment, then it can be argued that “the nature of the tasks should differ from those usually used in traditional tests” so as to test students’ skills and abilities of a type that are not usually assessed (Fahlgren et al., 2021, p. 71). Fahlgren et al. (2021) regard tasks in which “technology is (almost) necessary to solve a task and which require reasoning not assessed in normal tests” as *transformative* tasks. They state that “digital technology could increase the opportunities to assess important mathematical competencies (e.g., problem solving and reasoning) and hence to increase the focus on practising these abilities" (p. 72), providing the opportunity for transformative assessments if suitable tasks are used.

The level of difficulty of mathematics problems involving CAS can be influenced not only by the number of steps required to solve such problems (MacAogáin, 2000; Flynn & McCrae, 2001) and the extent to which CAS could play a role in solving them (Flynn & McCrae, 2001; Kokol-Voljc, 2000; MacAogáin, 2000), but also by the amount of additional *written information* in the question that the students need to process. "Word problems" (such as those involving applications) are among those question types which students generally find the most difficult to solve (Verschaffel et al., 2020). Questions involving parameters are also often found relatively difficult by students (Leigh-Lancaster & Stacey, 2022; Stacey, 2003). Following on from this, the complexity of the discourse of the examination questions themselves is an important consideration, as discussed in Morgan and Sfard (2016). This is one factor that can have an effect on the level of difficulty of an examination question in a CAS environment and so will be in focus in the study to follow in this thesis.

The use of CAS in examination questions also provides opportunities for questions where students need to use or interpret visual mediators in the form of CAS output. For example, questions on direction fields diagrams for first order differential equations which are sometimes used in class exercises (Hyland et al., 2021) can also be asked in examinations in which knowledge of first-order differential equations is being tested, with CAS output which shows such direction fields being part of such questions. Such questions will be considered for the examination questions to be analysed in this thesis. Another example of the use of graphical visual mediators in test and examination assessment occurs in Hong and Thomas (2015), where in a mid-term test in a first-year Calculus subject taught with CAS encouraging linking of multiple representations of derivatives and integrals, one question presented students with two graphs showing functions and asked them to sketch the graph of the derivative of each of these functions, as shown in Figure 3.10. A question in the final examination for the subject provided the graph of a function, and asked students to make a rough sketch of an antiderivative, subject to the initial condition $F(0) = 0$. This question is shown in Figure 3.11.

These two questions were set to test if students "were able to demonstrate versatile thinking" (p. 199) in solving these problems. In answering the first question, 35.7% of the students successfully used an algebraic approach in combination with recognition, from the graphical visual mediators provided, of the general properties and associated algebraic form that each of the functions must have. In contrast, 55.9% of the students approached the question by considering visually the relationship between the behaviour of the original function and the associated derivative function, using local thinking or interval thinking about the general properties of the derivative function. In both these solution methods, the students obtained the correct answer by demonstrating versatile thinking in “constructing a method based on principles they [had] learned” (p. 193). For the second question, 28.7% of students used a mostly algebraic approach, which required first producing a general algebraic expression which satisfied the properties of the graph shown. However, at the end of the solution process, less than half these students obtained the correct result. In contrast, the other students who attempted the question used interval-based visual thinking to answer this question. In doing so, they first correctly located the two stationary points of the antiderivative function. They then looked at the graph of the original function, to determine the nature of the gradients of tangents to the curve in each interval separated by the stationary points. This allowed them to see where the antiderivative function would be increasing and where it would be decreasing, with the majority of them going on to correctly sketch the graph of the antiderivative function. These two examination questions from Thomas and Hong (2015) illustrate examples of how providing graphical visual mediators produced by CAS can allow testing of mathematical skills in a different way, that emphasises applying mathematical concepts and moves away from time-consuming procedural calculations. In such cases, CAS technology is being “used as a lever potential”, in “shifting the focus from any technical aspects” to the “problem solving and reasoning” (Jankvist et al, 2021, p. 102) aspects of answering such examination questions.

Figure 3.10

Examination Question Where Students are Required to Sketch Antiderivative (Hong & Thomas, 2015, p. 193)

a) b)

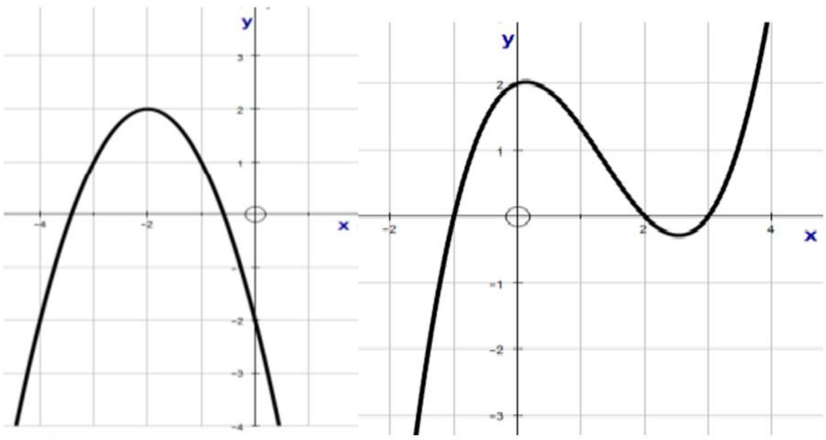


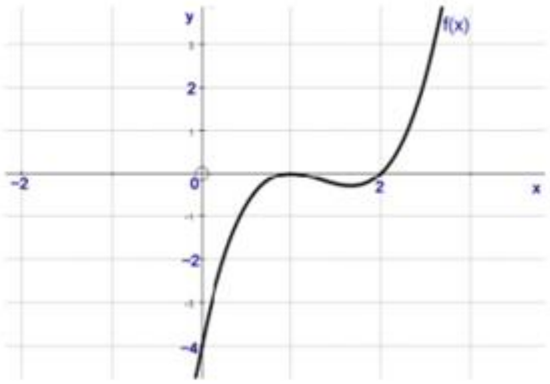
Fig. 10 The test questions on graphical derivative. Sketch the derivative for the given graphs

Leigh-Lancaster and Stacey (2022) describe another potential benefit of the use of CAS in assessment being that it requires students to "become more versatile in recognising mathematical expressions in a variety of formats" (p. 13). They suggest that assessment of this could include the presence of multiple-choice questions in which the answer options differ from that presented by the CAS devices being used by the students.

Figure 3.11

Examination Question Requiring Sketch of Antiderivative (Hong & Thomas, 2015, p. 194)

The graph of a function is shown in the figure. Make a rough sketch of an antiderivative F, given that $F(0)=0$.



Whenever technology is permitted in a unit's examinations, careful consideration needs to be given to its impact on the type of examination questions that should be set to best test students' knowledge of the unit. For instance, in the context of graphing calculators in examinations, Kemp et. al (1996) commented that appropriate use of such calculators "will not happen by accident, but rather needs be a conscious part of the examination design process" (p. 15). This is also applicable to the situation when CAS calculators and software are used in examinations. As described in Jankvist et al. (2021), an important question to consider in setting CAS-active examinations is what students are actually being assessed on: for instance, whether on their "mathematical learning outcomes" including problem solving and reasoning, on their technical "ability to use CAS in questions to solve the given mathematical tasks," or on both of these aspects (p. 101). The use of digital technology such as CAS in assessment invites consideration of the extent to which the type of technology used might affect "what students do (mathematically) during the assessment" (Fahlgren et al., 2021, p. 71).

Different CAS variants sometimes producing answers in different forms also needs to be taken into consideration, in cases where there are different CAS options available to the students taking an examination. For example, Ball and Stacey (2005) give an example of a multiple-choice question from a past Victorian Certificate of Education (VCE) Year 12 Mathematical Methods CAS examination from 2002, which involved differentiating $y = log_e cos(2x)$. In this case, a TI-89 CAS calculator would give the answer as $-2\tan(2x)$, matching up immediately with one of the answer options, whereas the other two brands of approved CAS calculators at that time would give answers in a form that required simplification in order to obtain that result. We need a means to expose inequities of this type in possible solutions, to allow students be taught effectively with CAS. This can potentially include the provision of support to students with different versions of CAS, so that they are aware of such differences and can cope with them in an examination setting. Commognitive analysis appears to be a means to help identify situations of this type; to hopefully expose variations in discursive patterns in examination questions and their possible solution paths which initially would be considered similar. In regard to the different possible solution paths, the *grain size* of a question (Morgan & Sfard, 2016), which is the minimal number of decisions required to answer it, is one useful measure in a commognitive framework, which would immediately identify differences in a question like the one described from Ball and Stacey (2005), depending on the model of CAS used to answer it.

In relation to the current year 12 mathematics examinations in Victoria, where CAS calculators are expected to be used, Leigh-Lancaster and Stacey (2022) observed an increase in how difficult and demanding some examination questions have become. They identified this as being "because they are more abstract and have greater use of parameters, and the breadth, depth, and variation in what can be asked has increased" (p. 8), with students also needing to have "the capacity to handle a broader range of mathematical processes symbolically" (p. 10). This has included a greater focus on questions which vary the values of parameters, replacing previous questions which contained numerical coefficients. Leigh-Lancaster and Stacey (2022) also reported an increase in the amount and variety of questions with a "problem-solving" or "explorative structure" (p. 10), attributing this to the increased variety of approaches to solving problems that is available when using CAS. On the examinations, this has included questions which present students with a written scenario and require them to develop an equation that fits it and to then solve the equation using CAS. They also found an increased number of examination questions involving mathematical applications of differentiation, such as questions involving tangents and normals.

Researchers have also previously raised the importance of evaluating the implications of an increased number of conceptual questions of these types. For example, Stacey (2003) advised that question design devices used to avoid questions becoming trivialised by CAS (e.g., the introduction of parameters) can increase their conceptual difficulty level, with Oates (2011) concurring that designers need to consider "the implications of fewer instrumental and more conceptual questions" (p. 169). Leigh-Lancaster and Stacey (2022) outline the motivations for question design for the current VCE examinations as "to test mathematically important content, to encourage students to learn mathematics deeply, and to test equitably considering the several brands and models of devices that students can use" (p. 9).

While studies described earlier in this section measured or described how CAS could affect the *solutio*n process, another important aspect of the impact of CAS on examination questions is the effect it has on the *discourse* of the questions themselves, which includes the complexity of the written text and also the complexity of any CAS outputs that are present as part of the question content. A commognitive framework (Morgan & Sfard, 2016) would assist with this, in providing an in-depth, qualitative record of the written complexity of the questions and any associated visual mediators. A commognitive approach is also useful in that it identifies that “changing the textual form does not simply make the mathematics more or less difficult but also changes the way in which a student may engage with the text and the possible ways of thinking about the mathematical content” (Tang et al., 2012, p. 1).

In applying a commognitive framework to analysing the discourse of the General Certificate of Secondary Education (final year school) mathematics examinations in the UK from different years, Morgan and Sfard (2016) found overall changes in the discourse of the examinations across the different time points (years) that they had selected. Some of the changes they found were associated with *mathematising* aspects of the discourse, and in the nature of the stories told about mathematical objects in the examinations. For example, they found examination questions typically had reduced grammatical complexity in more recent years, affecting the amount of information students would need to unpack in answering such questions. There were also changes in *subjectifying* aspects of the discourse, in what could be determined from analysing the examination questions about how the students sitting it would be expected to participate in mathematical discourse, including how much autonomy they would be given. In the more recent examinations, questions typically could be solved with a lower minimal number of single step decisions required than in earlier examinations. They also saw a reduction in the amount of autonomy students were given in “deciding about the problem solving trajectory” (p. 108) and the format in which some of their answers could be presented.

One important aspect of the commognitive analysis framework of Morgan and Sfard (2016), concerns the extent of students’ autonomy in how their answers to examination questions are permitted to be presented. While the examinations they analysed did not involve students using CAS technology, this is also a very important aspect to consider when CAS technology is permitted in an examination, as the use of CAS can have an especially large influence on the options available for obtaining and presenting answers to some mathematics problems. This will now be reviewed in section 3.4.2.3.

3.4.2.3 Considerations in how students might present answers when using CAS

For many Calculus questions, CAS can have an impact on the types of solutions students can present, both in the types of methods used and in the number of intermediate steps shown. When setting examinations, it is important to be aware of the possible types of answers students might provide. In relation to this, Brown (2003) explored the different solution methods available when a Texas Instruments CAS calculator was used to solve a variety of upper secondary level Calculus examination questions, finding that in some cases, both algebraic (symbolic) and graphical solutions were possible. For example, in determining the maximum height of a curve, he demonstrated that this could be done graphically, by plotting the curve on the calculator, locating the approximate location of the maximum from the graph and then using the **Maximum** command within the calculator's graphing window. He also indicated how the **fMax** command, present on Texas Instruments CAS calculators, could also be used directly to find the maximum value over a specified interval. By contrast, if finding the maximum value by-hand, the method would involve using differentiation, algebra and a slope test. Brown (2003) also found that there were instances where tabular methods could help in obtaining the answer. Due to the multiple-representation capabilities of CAS, determining the forms of an answer that will be deemed acceptable in assessments including examinations, will sometimes require consideration of whether "visual arguments," such as those involving graphical visual mediators, can be used and, if so, in what ways (Kadijevich, 2014), in questions that students would typically have been expected to only solve algebraically (when working in a technology-free environment).

In recent years, the capabilities of CAS calculators and computer packages have further evolved, meaning they "afford an increased variety of solution approaches with CAS" (Leigh-Lancaster & Stacey, 2022, p. 8). One example is that the increased visualisation capabilities of recent forms of CAS, which includes sliders, dynamic displays and simulations, have led to the inclusion of examination questions that encourage explorative use of slider options to vary parameters and consequently "to get a sense of the question and then a solution" (p. 10). A consequence of the range of symbolic and visual capabilities of CAS is that its availability in examinations also can potentially lead to the use of examinations where the examiner no longer "controls the solution strategy, to one in which the student controls the solution strategy and consequent form of the solution" (Brown, 2003, p. 178). This highlights the importance of examiners being aware of the different possible solution strategies that students might use, so that they are either “prepared to reward them equally, or alternatively constrain the question to one solution method” (Brown, 2003, p. 178). In the case of algebraic solutions, this includes requiring an awareness of the variation in the form such algebraic solutions could be presented in depending on the models of CAS permitted for use, suggesting some description should be included, in such cases, of what constitutes an “acceptable solution.”

At the tertiary level, the above points highlight the importance of instructors writing such CAS-active assessments both being aware of the variety of answer options that could be considered acceptable, and also making it clear to students sitting examinations as to when answers will be accepted if they are written down directly from using CAS and when students should also show intermediate steps or justifications for the answers they obtain. This latter point has also, at times, been a source of uncertainty for students in determining the amount of working they are expected to show in some examination questions where it says that “appropriate working must be shown”, an instance of which is referred to in Leigh-Lancaster and Stacey (2022).

With the availability of CAS having such a large impact on aspects of the learning and assessment of Calculus topics, it raises the question of what students’ attitudes are to CAS and its use in learning and assessment, as well as when they are most likely to use it in practice. This will now be reviewed in section 3.5.

3.5 Students’ Attitudes Towards and Use of Computer Algebra Systems

Section 3.5.1 will review literature which investigates the extent to which students perceive different aspects of CAS as useful, including multiple representation and algebraic capabilities, the potential to save time in solving some problems and the effect on their understanding of concepts and techniques. I will also report on studies of students’ attitudes towards affective aspects of CAS use, including students’ anxiety, confidence, liking of CAS, and its perceived usefulness. Studies which have explored associations between some of the above variables with each other, or with other variables, will also be considered. This will include reviewing studies relating students’ attitudes to CAS to their preferred “thinking” style, the type of instruction they received and their attitudes towards use of CAS in examinations. Section 3.5.2 will review studies that have investigated the extent and nature of students’ use of CAS, including how they integrate CAS use into their problem solving approaches. In section 3.5.3, the relationship between students' attitudes toward CAS and how they use it in practice will be explored.

3.5.1 Attitudes of students towards CAS in teaching and assessment

Stewart et al. (2005) investigated the attitudes to technology use in their mathematics course of 252 first-year and 88 second-year students undertaking science and engineering majors who were studying mathematics as part of these courses at the University of Canterbury, New Zealand. These students used Maple CAS software in their first-year mathematics unit that focused on Calculus and basic Linear Algebra, and Matlab in their second-year mathematics unit that included advanced Linear Algebra. Attitudes were determined based on their responses to a questionnaire in which they were asked about their attitudes to learning mathematics using computers, their experience of using computers in the course, and the value of the computers for specific content. Stewart et al. (2005) found that a higher proportion of the second-year students thought multiple-representation capabilities of computer packages, especially graphs, would assist their understanding of concepts. These second-year students were significantly more inclined to use CAS for visualisation, to want to use it to free up time for thinking about concepts, and to use CAS when stuck in solving a problem, although they also reported finding computer commands difficult to remember. In addition, they reported being less inclined to solve problems by-hand, preferring to use CAS for long or difficult calculations.

Funny (2019) in his investigation into how the use of the Integral Calculator app from the internet helped 38 engineering students learn integration techniques, found that students were reluctant to use their Smartphones for such purposes. Several students doubted the Integral Calculator app for their phone could help them understand the integration technique in focus. Reasons given were that the procedure used by the App was long, the steps were unclear due to skipped steps, and the different methods were different from their usual routine. Funny interpreted this as signifying students' waiting for the lecturer's explanation and wanting to use only a short, simple method, or shortcut to get answers rather than to understand the procedure.

Some studies also explore if there are relationships between different attitudes towards CAS or between attitudes towards CAS and other factors. Sevimli (2016) investigated how first-year university Calculus students' attitudes towards CAS technology differ depending on the learning environment (traditional instruction or using CAS LiveMath) and their way of thinking when solving problems. At the beginning of the study, the students were given a set of mathematics problems to solve and were consequently classified as an “analytical thinker” if they solved the majority of the problems using non-visual methods or a “visual thinker” if they solved the majority of the problems using visual methods. A questionnaire was administered to 43 participants after six weeks of instruction, which determined that, students in the CAS group were most likely to rate “conceptual understanding” as the instructional objective of the course, while those in the traditional instruction group were most likely to rate “procedural flexibility.” Students taught using CAS were most likely to select “using representation” and “visualization” as what they had obtained from the course. None of the participants in the traditional instruction group selected those options, instead mostly selecting the “algebraic thinking” option. Interviews were also conducted with four participants, representing the different combinations of instruction type and thinking. Overall findings were that students in the traditional group “place[d] more importance on procedural flexibility while students in the CAS group attach more importance to conceptual skills in terms of instructional objects” (p. 1). Of the four interviewees, those from the CAS group and the visual thinker from the traditional group had a “positive but limited” (p. 12) attitude to use of CAS, with an overall positive attitude to its use but concern that it could “affect procedural fluency negatively” (p. 12). In contrast, the participant from the traditional group who was analytical thinker had a negative attitude to use of CAS.

Ng (2003) developed a CAS Attitude Scale to evaluate pre-service teachers’ attitudes towards CAS, after they had completed a CAS-related training module. The questionnaire included questions relating to the respondents’ *anxiety, confidence, liking of CAS* and *perceived usefulness* of CAS. The 50 pre-service teachers who completed the questionnaire were university graduates, with 22 having at least three months of formal secondary school teaching experience. The results indicated that, on average, the teachers overall had a positive attitude to CAS on each of the four aspects measure. Significant, positive correlations were identified between each pair of aspects with the strongest association being between liking and usefulness (i.e., those with a higher liking score also tended to perceive CAS as more useful), followed by the association between anxiety and confidence (i.e., those who were less anxious about CAS tended to be more confident in their ability to use it).

Ng et al. (2005) in another study used *anxiety, confidence, liking of CAS* and *perceived usefulness* scales, but this time to investigate the attitudes towards CAS and achievement in mathematics of two classes of year 11 students. Students had access to a CAS Voyage 200 calculator during class in a CAS Intervention Program. Their attitudes to CAS was measured at three time points: after students in the treatment group were introduced to the functionality of CAS, at the end of a two-session workshop on using the CAS Voyage 200, and at the end of the CAS intervention program. The students were also tested at the end of the program, on a range of Calculus and Linear Algebra topics. Students were encouraged to reflect on their experiences of using CAS in journals, with the responses received indicating that almost all students found CAS useful in solving the mathematics tutorial problems given during the course. However, their opinions of the Voyage 200's graphing capabilities were more varied, due to problems such as low resolution sometimes affecting or distorting the shape of curves. They also referred to being "baffled" as to why in entering algebraic expressions a multiplication sign needed to be inserted; a similar situation to that described in Sangwin and Ramsden (2007). Unlike other types of CAS discussed earlier, the CAS Voyage 200 will actually produce an error message in such cases, for example, if writing $x(x - 3)$ rather than $x * (x - 3)$. However, they also wrote that encountering technical difficulties, such as with the CAS syntax, "had actually brought about a greater awareness of the details involved in writing mathematical expressions" (p. 64). On the questionnaires, the mean liking score increased significantly after the initial CAS workshop however students' average anxiety score was higher after the workshop but this was not a significant difference. After the workshop the students were more confident about using CAS and perceived it as more useful on average, although these differences were also not significant. After the intervention course however, all the attitude scores improved, with significantly lower mean anxiety, and higher confidence, liking and perceived usefulness scores indicating that students' attitudes towards CAS were more positive than before the intervention program and "it may also be inferred that the more experience students had with using the [CAS calculator], the less anxious they became" p. 67) As in Ng (2003), there were positive correlations between all the subscales, with, in this case, all of these correlations being statistically significant and with the strongest occurring between liking and usefulness.

Mohammad (2019) used a slight modification of a *Mathematics and Technology Attitudes Scale (MTAS)* originally developed by Pierce et al. (2007), which included a component measuring students' "attitudes towards learning mathematics with technology" (p. 348), in the form of CAS, on a five point Likert scale. This required students to rate (from strongly disagree to strongly agree) four aspects: whether they liked using CAS, would consider its use to be "worth the extra effort" (p. 348), found CAS made mathematics more interesting, and whether they perceived CAS would help them to learn mathematics better. The student cohort was 55 Norwegian upper secondary school students, who were mostly in 13th grade. Mohammad (2019) found their responses were quite varied, and selected a subset of 9 of these students, with three having a "low" score on the scale, three with "moderate scores," and three with "high scores" on it. These students were interviewed. One of the students with a high score on the "attitudes to learning mathematics with technology" scale also expressed support for the use of CAS to be compulsory in written examinations and none of the students in that category were explicitly against its use, whereas almost all the students with low or moderate "attitudes to learning mathematics with technology" scores expressed a negative attitude towards its use on national written examinations. While only a small sample size, this finding suggests a possible link between students who like CAS and perceive it as useful being more likely to be in favour of its use in examinations, a connection that could be explored further.

3.5.2 The extent and nature of students' use of CAS

Studies have also been conducted which survey students about the nature of their actual use of CAS or analyse it in more detail. Matos (2014) surveyed 11 students taking an undergraduate Differential Equations unit, in which Maple was taught, on the nature of their use of CAS, and found that all reported using Maple to help solve some of the problems presented in the unit and that overall, their survey responses indicated support of its use in the unit. The students were also surveyed about their use of the free CAS software Wolfram Alpha outside of class time, with only 11.1% of respondents reporting using it, potentially because of the emphasis on Maple in the actual unit.

Thomas and Hong (2005) analysed students' work during a workshop on the TI-89 CAS calculator, where the 8 students in their study had not used CAS before. The focus of their analysis was on identifying and classifying different ways in which CAS was used in students' problem solving approaches and the objectives the students had in using it in these different ways. They identified five different types of CAS use among the students: (1) using it directly for straightforward and complex procedures, (2) using it to check procedural by-hand work, (3) using it directly within a mathematical process, (4) performing a procedure within a more complex process [to possibly] reduce cognitive load and (5) using it to investigate conceptual ideas (p. 227). They found instances of each of these types of use of CAS among their students, concluding that “students were interacting with CAS representations both procedurally (most commonly with direct commands) and conceptually” (p. 229) with “some students already starting to integrate CAS into by-hand working, some of which was procedural in nature, but only a little of which was conceptual” (p. 227). The latter occurred when some students answered a question requiring testing for continuity by calculating the left-hand and right-hand limits, and in a few cases producing a graph of the function considered, to assist in visualizing the discontinuity that was in fact present in the graph (cf. Pierce & Stacey, 2001).

While the last two sections have considered students’ attitudes to CAS and use of CAS, respectively, some studies have related students’ attitudes towards CAS to their use of CAS, which will now be reviewed in section 3.5.3.

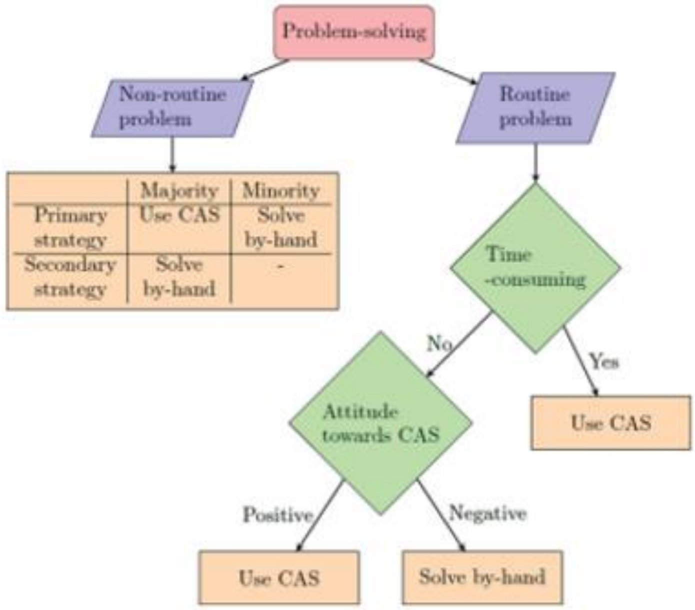
3.5.3 The relationship between students’ attitudes to CAS and their use of CAS

The study conducted by Mohammad (2019) of 55 Norwegian upper secondary students, included investigating if there was a link between the students’ attitudes towards CAS and how they used it in practice in problem solving. The students had at least three years of CAS experience, mostly in using GeoGebra, and some of them had previously experienced compulsory use of CAS on written examinations. The nine students who were interviewed about their attitudes to CAS were also given mathematics problems, which included several Calculus problems, to solve, using the CAS package GeoGebra. Mohammad (2019) found that, when students perceived a *routine problem* as time-consuming, they would generally use CAS. However, when they perceived a routine problem to not be time consuming, their CAS usage depended on attitude towards CAS, with those with a favourable attitude using it, while those with a negative attitude typically did not use it in solving such problems. The students were also given a *non-routine problem* to solve, in the form of a Calculus problem of a type they had not encountered before. Their pattern of CAS use was different from when they were solving routine problems, with the majority of those who attempted the non-routine problem initially working by-hand, but then switching to use of CAS as their primary strategy for solving it, while two other students only attempted the problem by-hand.

Figure 3.12 shows a simplified scheme for the students' problem solving with CAS or by-hand and how this relates to attitude to CAS use.

Figure 3.12

Student Scheme for Choosing to Use CAS in Problem Solving (Mohammad, 2019, p. 350)



Pierce and Stacey (2001) reported on Pierce's study of 30 Australian tertiary students who were taking an introductory Calculus course. Students were taught using the CAS package DERIVE both in classes and assessment, with lessons including graphical, algebraic and tabular representations of functions. They found the students frequently used graphical and algebraic representations of functions, but that they rarely used numeric representations in tables unless instructed to do so. Feedback about DERIVE from the course evaluation surveys was positive overall, with 65% of survey respondents agreeing or strongly agreeing that using DERIVE helped them understand mathematics, while 71% agreed or strongly agreed that DERIVE had helped them see patterns. The results also suggested exploratory behaviour by most of the students when using Derive, with 70% agreeing or strongly agreeing they tried out ideas using DERIVE while 76% agreed or strongly agreed they used DERIVE to try changing functions to see what happened. As part of a survey given to the students at the end of the course, they were asked which tool they would select most frequently for "speed, confidence and learning" (p. 41) for each of a selection of mathematics problems that they were presented with. The results suggested "students preferred to do examples that they considered to be easy by-hand and to move to CAS technology for more complicated, more time-consuming and less familiar problems” (p. 41). This preference for use of CAS for more time-consuming problems is consistent with the findings of Mohammad (2019) previously discussed, although in this case it has not been correlated directly with students’ overall attitude to CAS.

Cameron and Ball (2014) surveyed 7 Australian year 11 students in Victoria taking Mathematical Methods (CAS), and investigated whether their attitudes to CAS in terms of the same general aspects as in Ng (2003): *anxiety*, *confidence*, *liking of CAS* and *perceived usefulness*, influenced their use of CAS. They applied these categories to how students used it in learning, as opposed to Ng (2003), who related it only to how the participants felt about using it in their future teaching. They considered low anxiety, high confidence, liking (enjoying using) CAS and perceiving CAS to be useful to be positive attitudes towards CAS. The students were given a worksheet containing six Calculus exercises of types encountered previously, with two questions classified as “CAS-required” and the other four as “CAS-optional”, as per Brown’s (2003) classification. At each step of solving the problems, students were asked to record use of pen-and-paper or CAS. The students were then given a survey, adapted from the *Computer Algebra Systems Attitude Scale* (Ng, 2003). Overall, the survey showed the students expressed little or no anxiety when using CAS, with the exception of the one student who did not own a CAS calculator herself. Overall confidence levels were more varied than anxiety levels but still suggested most respondents were confident in their ability to use CAS. The two students with the lowest anxiety levels had the highest confidence levels, consistent with Ng (2003). Responses on the *liking* scale varied, with four students liking CAS overall but the other three having a slight dislike of it. On the perceived usefulness scale, all students’ overall responses were positive (on average). Although the findings indicated all respondents found CAS useful in *learning math*ematics, “this perceived usefulness was scored higher than the students’ responses” (Cameron & Ball, 2014, p. 149) regarding actually *using* CAS to do mathematics, for which two respondents gave a negative response. The students were then marked on the steps in each of the six Calculus problems, to measure the extent to which the students used CAS in answering each question. One of the students, who had a positive attitude to CAS with positive responses to each of the four aspects measured, had her answers analysed in detail. She made extensive use of CAS in solving the set problems, which Cameron and Ball (2014) concluded suggested that her positive attitude to CAS may have positively influenced her use of it.

3.5.4 Key features of students' attitudes towards CAS and relationship with CAS use

Past studies have considered different components of students' attitudes towards the utility of CAS, including its usefulness for saving time, understanding concepts and for producing multiple representations including graphs for visualisation. Students' attitudes towards affective aspects of CAS use have also been researched, including students' anxiety about using CAS, their confidence with CAS, their "liking" of it, and how useful they perceive it to be. The majority of the students included in these studies had positive attitudes towards use of CAS in relation to each of the above features. This included the majority of students having low levels of anxiety about CAS, which is considered an aspect of a positive attitude towards it. Studies which investigated if there were relationships between some of these aspects of attitudes towards CAS most commonly found positive associations in such cases, with students with more positive attitudes towards one aspect of CAS tending to also have more positive attitudes towards other aspects. It was also found that a higher percentage of students had a positive attitude towards using the multiple representation and visualisation capabilities of CAS if they were in second year (rather than first year), were visual thinkers (rather than analytical thinkers), or were taught with CAS (rather than with traditional instruction). The only study which indicated negative attitudes to CAS overall (Funny, 2019) was also the only study reviewed in which a Smartphone environment was used, rather than a calculator or computer environment. A different version of CAS to those used in the other studies was also used.

There are a variety of different ways in which CAS can be used in solving Calculus problems, with the classification developed by Thomas and Hong (2005) bringing out important aspects of this. When further reviewing students' attitudes towards CAS, by investigating the relationship between their attitudes to it and how they report using it in practice, how time-consuming students perceived problems to be had an association with how likely they were to use CAS for solving them, and that this interacted with their overall attitude towards CAS. In particular, Mohammad (2019) found that students in his study tended to use CAS whenever they perceived a routine problem as time-consuming, whereas if they did not perceive a problem as time-consuming they were most likely to use CAS if they have a positive attitude towards CAS but not to if they had a negative attitude towards CAS. This was consistent with the findings of Pierce and Stacey (2001), in relation to students in Pierce's study of undergraduate students also preferring to use CAS if they perceived problems as time-consuming. The results of the studies reviewed in this section also indicated students were more likely to frequently use CAS overall if they had a positive attitude towards it.

3.6 Conclusion

From section 3.2 I can conclude that some aspects of the discourse of Calculus typically change in the transition from secondary to tertiary level. This is in part due to the associated contexts in which the discourse is used (e.g., Geometry, Analysis) and partly because of the different academic expectations which are usually imposed on students at these different levels. There are greater instances of colloquial discourses used at secondary level and an expectation of more engagement in academic, scholarly mathematical discourses at tertiary level. These discursive shifts can potentially leave students moving in the “no man’s land” between these different discourse communities. Some of the subtleties of discourse can also vary by instructor, even within the same subject at the same university. Studies which have subjected university students’ and lecturers’ discourses to commognitive analysis have found that students’ discourses are typically more colloquial than instructors’ discourses and that common weaknesses and sources of students’ errors include struggling to link visual and symbolic characteristics of the same mathematical objects. Topics such as tangents, limits, derivatives and functions are areas where commognitive analyses have been conducted and these have highlighted some of these difficulties students have with these concepts.

Section 3.3 indicated that there are a diverse range of types of CAS technology that can, and have, been used in the teaching and learning of Calculus and their nature is also continually evolving with an increased use, for example, of Dynamic Geometry aspects of some CAS software. The different types of CAS also have associated distinct discourses, which vary according to the model of CAS used, and where both the required inputs and their corresponding outputs will at times produce verbal and symbolic discourses that have different rules from the standard discourses of Algebra, Functions and Calculus. The literature reviewed in this chapter has indicated instances where students have encountered these problems. In the previous chapter, it was seen that *commognitive conflicts* occur that are either *interpersonal,* when different interlocuters use discourse in a different way or 4r*intrapersonal*, when the same person uses discourse in an inconsistent manner. The findings of section 3.3 indicate the potential for such commognitive conflicts to occur between students' usual classroom discourse and that of the CAS technology they are using. This is an area that has been investigated in my study to be described in Chapter 4.

It was seen in section 3.4 that the use of CAS in university-level written examination assessment is relatively low and that there are various considerations when setting such questions, including the extent to which CAS replaces some by-hand steps and the opportunities CAS gives for asking, in some cases, questions that are actually more challenging and that test conceptual understanding, due to a reduced amount of time needing to be spent on routine procedural steps. As well as questions that permit use of CAS, as described above, CAS also offers the opportunity to ask questions that make use of CAS outputs. As was seen in the previous section, CAS calculators and software can produce graphical visual mediators such as slope fields diagrams, and these have the potential to be included in examination questions allowing for testing of students' conceptual capabilities to either describe what they see in such outputs or to link them with a symbolic representation of the same mathematical object, as has already sometimes been done in class exercises, such as in Hyland et al. (2021). A *commognitive framework* can be applied to analysing the discourse of examination questions in which CAS has an impact. This includes both for questions where students are permitted to use CAS, and those where written or graphical outputs generated by CAS are present. This application of commognitive analysis allows further insight into the discourse of such questions and the types of activities students could consequently be positioned to engage in when answering such questions. This has been considered in the study to follow.

Section 3.5 indicated that the majority of students in the studies reviewed had a positive attitude towards different aspects of CAS, and that this could also be related to whether or not they had been taught with CAS and their preferred problem solving approaches. In turn, findings suggested that students were more likely to choose to use CAS in solving problems they considered to be time-consuming and if they had a positive attitude towards CAS. Following on from these findings, the relationship between students' attitudes towards CAS and how they use it in practice for solving different types of Calculus problems has also been considered in the study to follow.

Chapter 4 will now outline the methodology and research process used for this study. It will also introduce the research questions, which follow on from the results of this literature review. The literature reviewed in this chapter and in Chapter 2 has identified that a commognitive framework provides a rich, in-depth way of analysing mathematical discourse in different settings. This includes, among others, in analysing the discourse of examination questions and students' written discourse when solving Calculus problems. However, a commognitive framework has not previously been directly applied to analysing the discourse of examination questions in a CAS-active environment. Chapter 5 will advance the thesis by reporting on my investigation of this, for a selection of university examination questions on Calculus topics. To then provide insight into the discourse of students when solving university level Calculus problems using CAS, Chapter 6 will report on my application of a commognitive analysis framework to analysing the results of task-based interviews with university student participants, who were given a set of Calculus problems to solve using a CAS calculator. The distribution of responses for students who were surveyed about their attitudes towards CAS will also be discussed. Chapter 7 will discuss the findings from Chapters 5 and 6, linking these with the existing research literature, while Chapter 8 will conclude the thesis by summarising the results of this study and considering the implications for future research.

CHAPTER 4

METHODOLOGY AND RESEARCH PROCESS

4.1 Overview

In Chapter 1, it was pointed out that Computer Algebra Systems (CAS) technology frequently has a big effect on how mathematics problems can be presented and solved, especially in Calculus topics (Hong & Thomas, 2015; Sevimli, 2016; Shahriari, 2019). It was seen that this is because it will often reduce the number of steps required to solve Calculus problems (Varbanova, 2017), where in some cases this reduction in steps is very large. It was also found that its multiple representation capabilities can assist both with the process of solving some Calculus problems and with conceptually understanding and interpreting the results (Leigh-Lancaster & Stacey, 2022).

In this thesis, in agreement with Morgan (2016), mathematics is "understood as a discursive practice, [which] includes not only the product of mathematical activity but also the processes that give rise to it and the values of the practice" (p. 122). The results of Chapter 3 indicated that the discourse of CAS is also often different to the written endorsed discourse of mathematics topics, including Calculus. Thus, the focus of my thesis is on the effect of CAS technology on setting and solving mathematics problems in undergraduate Calculus units. This has included investigating the impact of CAS on Calculus summative assessment from the perspectives of what is required of students, how a test or examination paper speaks to students and positions them as knowers and users of CAS technology and how they are likely to interact with such questions from their perspectives and their use of CAS. How students approach and talk about solving Calculus problems using CAS in task-based interviews was also explored. I also investigated their perceptions of the utility of CAS, and their preferences for CAS or by-hand methods and access to CAS in assessment.

Section 4.2 commences with a description of the nature of and theoretical background for my study, which was an instrumental case study. It then outlines the aims of the research and states the research questions being investigated. How Sfard's Commognitive framework (Morgan & Sfard, 2016; Sfard, 2008) has been applied is briefly overviewed as it is the theoretical framework for the study and is also the foundation for the analytical framework for both the assessment items and the task-based interviews. This section also overviews the research methods that were used to analyse the data under consideration in this study. Section 4.3 describes the context of the study and its participants. Section 4.4 provides a description of the data collection and the data instruments. The data analysis is described in Section 4.5, before Section 4.6 concludes the chapter.

4.2 Methodology

This research involved an instrumental case study (Stake, 1995, 2005), with the phenomenon under investigation being the impact of Computer Algebra Systems on assessment in undergraduate Calculus. Study of the case "is instrumental to accomplishing something other than understanding" (Stake, 1995, p. 3) the particular case being analysed, rather providing insight into an associated issue. This study was conducted to provide insight into the issue of how access to Computer Algebra Systems affects the level of difficulty and complexity of the questions asked on undergraduate Calculus test and examination assessments and students' answers to questions of this type, from a discursive perspective. The case was bounded to written undergraduate Calculus assessment items undertaken under test conditions, from subjects where CAS enabled technology had been taught, and to students taking a selection of these subjects. I have delimited it to include assessments from two Australian universities where I had access to such assessment items, during a particular time period, with the students drawn from one of these universities.

Stake's (1995, 2005) qualitative case study approach was chosen in preference to the widely used case study approach of Yin (2009), because Stake's framework aligns with a constructivist perspective, in which reality is influenced by humans' perspectives, with Stake (1995) agreeing that "no aspects of knowledge are purely of the external world, devoid of human construction" (p. 100). This is consistent with sociocultural theory's view of the situatedness of mental functioning and learning, which is also one of the fundamental principles underpinning Sfard's (2008) commognitive framework used in this thesis. By contrast, Yin's case study framework is consistent with a postpositivist theoretical orientation, in which reality is objective and consists of universal truths. Yin's conceptual framework is also more fixed in structure, with an emphasis on proving or disproving specific hypotheses, which was not the objective of my research and which would have limited the findings of this study. In contrast, in allowing for "discovery and interpretation [to] occur concurrently" (Boblin et al., 2013, p. 1268), while taking context into consideration, use of Stake's case study approach provided the opportunity for comprehensive analysis and interpretation of not just the assessment items themselves, but also associated aspects such as the students' work in the task-based interviews, and how the students had been taught in the associated written learning materials for each subject. A key part of the case study approach is the use of data from multiple sources, allowing a "holistic understanding of the phenomenon being researched" (Boblin et al., 2013, p. 1270). In this study, in addition to the examination papers, written learning materials and task-based interview data, questionnaire data on students' attitudes to CAS was also collected.

The first aim of this research was to develop a detailed insight into the effect of Computer Algebra Systems on the task difficulty and complexity of examination questions, by analysing the discursive characteristics of such questions. The second aim was to see how students taking the Calculus course under consideration used it in practice and talked about their answers, when solving a series of problems in task-based interviews, including some where the output produced by the CAS calculator was different to what they would have obtained if solving the same problem by-hand. The third aim was to contextualise the results of the task-based interviews, by surveying undergraduate students to determine how CAS technology was perceived by them.

The research questions addressed in this thesis are as follows:

1. How can a commognitive framework be applied to effectively capture the complexity and difficulty level of written answer and multiple-choice examination questions asked in undergraduate Calculus units where use of CAS technology is available?
2. (a) How effectively do undergraduate Calculus students use their CAS calculator and use and interpret CAS output, especially when it is in a different format to what they would obtain by the methods of working by-hand they have been taught in class?

   (b) What can we learn from a commognitive analysis of task-based interviews of students in relation to how they reflect on their answers in this situation?
3. To what extent do undergraduate Calculus students use CAS and when do they believe its use to be most beneficial?

In the sub-section 4.2.1, a brief overview of the application of Sfard's Commognitive framework (Sfard, 2008), which underpins the theoretical framework and the commognitive approach to analysis of the data from the study, is presented. Sub-section 4.2.2 then describes the types of analyses of the data that were used in this study. Sub-section 4.2.3 outlines ethical considerations and how they have been addressed in conducting this study.

4.2.1 Sfard's commognitive framework

As was seen in Chapter 2, Sfard's (2008) theory of commognition links cognition and communication, by identifying them as the same phenomenon manifested in different ways, leading Sfard (2008) to develop the term *commognition*. Commognitive analysis is a discursive approach to doing mathematics. In a discursive approach, the focus is on language and communication and learning is viewed as communicating in the discourse of a specific community, in our case, mathematics learners in a tertiary undergraduate Calculus-based mathematics course. In Morgan and Sfard's (2016) commognitive analysis framework there are two analytic parts: an analytic framework for the *mathematising aspects of examination discourse* (i.e., telling stories about mathematical objects) (Figure 4.4a); and an analytic scheme for *subjectifying aspects of examination discourse* (i.e., what examination questions indicate about the students and how they are expected to participate in mathematical discourse) (Figure 4.4b). The resulting mathematical communication can occur in a well-defined and bounded context such as in the preparing for, and sitting of, Calculus based examinations, as will be considered in this study.

As all mathematical tasks contain mathematical discourse, commognitive analysis has previously been used to analyse a diverse range of tasks in different settings, including analysis of mathematics examination questions (e.g., Morgan & Sfard, 2016; Thoma & Nardi, 2018), students' written answers to examination questions (Thoma & Nardi, 2018), mathematical discourse in school mathematics textbooks (Alshwaikh, 2016), and analysis of students' group work and discussion as they work on graphing tasks involving mathematics in an interactive classroom environment (Viirman & Nardi, 2021).

4.2.2 Research methods

Qualitative methods used in this study entailed the use of commognitive analysis for the questions on the examination and test papers and the task-based interview written problem scripts and transcribed interactions between myself as interviewer and the student during each interview.

Task analysis has been used to examine the nature of examination questions in undergraduate Calculus units, to help determine their level of task difficulty and task complexity. For the task analysis different existing schemes (e.g., Mueller & Forster, 2000) were initially used but as they did not produce analyses of the breadth and depth required, it was decided that the researcher would use a modification of an existing framework, Morgan and Sfard's (2016) analytical framework, that met her needs (see Section 4.5).

Task-based interviews are a means to gain knowledge about an individual's problem solving behaviour (Maher et al., 2014) both with respect to use of mathematical knowledge and technology use. Such an interview "is intended to elicit in subjects estimates of their existing knowledge, growth in knowledge, and also their representations of particular mathematical ideas, structures, and ways of reasoning" (p. 1). This necessitates attention to both task design to ensure the problems used will potentially elicit the mathematical and technological knowledge that is being targeted, and to the semi-structured interview protocol which allows for modifications in-the-moment based on the interviewer's judgement.

A questionnaire to capture how CAS technology and its use was perceived by undergraduate Calculus students at the time the study was conducted was considered necessary to address Research Question 3, and to provide context for the analysis of students' answers to the task-based interview questions used in addressing Research Questions 2(a) and 2(b). The questions on it were designed to give an indication of typical use of CAS and attitudes towards it by students from the same cohort as those who were given the task-based interviews and who sat examinations of the types being analysed. They were also designed to give an indication of how such students might typically position themselves in relation to the use of the CAS technology. For the student questionnaire response data, bar graphs showing a percentage frequency distribution summary of responses to individual questions were produced, for students in both year levels considered (first-year and third-year).

4.2.3 Ethical considerations

It is essential that all research be conducted ethically. The researcher recognises that she has a moral obligation to the members of her profession. The following actions have been completed:

- Ethics approvals, based on accepted informed consent procedures, were received from the Australian Catholic University's Human Research Ethics Committee 2014-297V for *Students' Attitudes to Use of CAS in Mathematics* and 2016-119E *Student Use of Computer Algebra Systems (CAS) in Undergraduate Mathematics: Casework.*
- Participating students received information about the aims of the research and the procedures (e.g., recording of interviews and collection of scripts in task-based interviews) and their right to withdraw at any time and gave written consent.
- Participants' identities have been protected. Information collected by the questionnaire was not identifiable. All participants were either not identified or were identified by codes such as Participant D in research reports and this thesis. Transcriptions of audio recording of the interviews were carried out by myself and no identifying details were included.

4.3 Situational Context and Participants of the Study

4.3.1 The site of the study

The site of the study was an inner-city campus of a multi-campus Australian University. This campus was chosen for the study because use of CAS calculators in lessons and the final examination was permitted and assumed in the undergraduate mathematics units offered to Arts/Education students. The researcher worked as a mathematics lecturer at another university in the city at the time.

4.3.2 The participants in the study

The participants in the study were in the first to third year of an undergraduate Arts degree, in which most students are being educated to become secondary school mathematics teachers. In the mathematics units under consideration in this thesis, these students were developing a broad knowledge of mathematics topics, including a wide range of Calculus topics in areas such as differentiation, integration, differential equations and applications.

4.3.2.1 Participant selection

The opportunity to complete a questionnaire on attitudes to CAS was offered to all first-year Calculus students at the beginning of a regular class in a first-year, semester-long, Calculus unit in the middle of the semester in 2015 and in late semester in 2016 and to all third-year students taking another Calculus unit in the middle of the semester in 2015. The respondents included 20 and 21 first-year students in 2015 and 2016, respectively and 9 third-year students in 2015, making a total of 50 students who completed the questionnaire (see Table 4.1). This data was used in addressing Research Question (RQ) 3.

Table 4.1

Participants in Student Questionnaire

Unit of mathematics	Year	Number of completed surveys	RQ
MATH104	2015	20	3
MATH104	2016	21	3
MATH310	2015	9	3

For the task-based interviews used in addressing Research Question (RQ) 2, an announcement was made on the unit websites of the first-year and second-year Calculus units, inviting students to participate. This resulted in 3 first-year Calculus students enrolled in MATH104 in 2016 and 1 second-year Calculus student enrolled in MATH203 in 2017 volunteering to participate. All four students were interviewed (see Table 4.2).

4.3.2.2 CAS calculator ownership of participants

It was expected that all the students in each part of the study would own a CAS calculator, as a CAS calculator was assumed in each mathematics unit they were studying, and the majority of the students had recently studied mathematics in high school in Victoria, where CAS calculators are required for pre-tertiary mathematics subjects, including in examinations. The models of calculator owned by the students included TI-Nspire CX CAS calculators and Casio Classpad 330 CAS calculators.

Table 4.2

Participants in Task-based Interviews

Unit of mathematics	Year	Number of participants	RQ
MATH104	2016	3	2
MATH203	2017	1	2

4.3.2.3 Previous use of CAS calculators by students in the study

Typically, students in the classes under consideration had recently graduated from schools in Victoria and such students had at least some experience of using CAS calculators, including in year 11 and 12 final year examinations. However, there were also some mature age students and students from interstate and overseas, some of whom would not have used the CAS calculator before. Therefore, it cannot be assumed that all students in the classes considered would have had prior experience of using CAS calculators before their tertiary studies. Those students who had used CAS calculators at school typically had been taught to use it for basic algebra, Calculus and graphing problems. CAS calculator skills required by the students in their tertiary mathematics and for the task-based interviews were entering a function, graphing, use of menus and understanding the logical structure of mathematical commands in order to determine what arguments[1] had to be entered into various calculator commands.

4.3.2.4 Mathematical background knowledge of participants

The formal pre-requisites for the first-year Calculus unit, as stated in the unit outline were: HSC Advanced Mathematics (not General Mathematics) (New South Wales) or Further Mathematics or Mathematical Methods (Victoria). However, the most advanced final year

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[1] Many commands on the CAS calculator require two or more arguments to be entered, where these are in brackets, separated by commas. For example, if we wanted to find the Taylor polynomial of order 3 which approximates $f(x) = \sin(x)$ about $x = 1$, the required CAS command would be **taylor($\sin(x)$, x, 3, 1)**, where this command has four arguments indicating respectively the function, the variable it is a function of, the order of the required Taylor polynomial and the value of x it is to be approximated about.

high school mathematics unit, which in Victoria was Specialist Mathematics, was not a pre-requisite. Some students, including mature age students or those from interstate or overseas, could also have equivalent recognised qualifications of a different type. The second- and third-year Calculus units had the first-year Calculus unit as a formal pre-requisite.

4.4 Data Collection and Instruments

Data collection was undertaken between 2015 and 2017. Data were collected in the form of documents (i.e., examination papers and tests, class notes, and unit outlines), responses to a multiple-choice questionnaire designed by the researcher, and student task-based interview responses (Maher et al., 2014). The interviews were also designed and conducted by the researcher.

4.4.1 Documents

Written examination papers and mid-semester tests from a selection of first-year, second-year and third-year Calculus units were collected for analysis. These were potentially from 4 different environments: (i) CAS active assessment with CAS active teaching/learning in the classroom, (ii) CAS active assessment with CAS inactive teaching/learning in the classroom, (iii) CAS inactive assessment with CAS active teaching/learning in the classroom and (iv) CAS inactive assessment with CAS inactive teaching/learning in the classroom. Because the focus of this study was on assessments provided to students who had been taught in a CAS active classroom environment, test and examination papers were only selected for subjects where students were taught in that setting (assessments of types (i) and (iii) above). These assessment papers were chosen to obtain a variety of different assessments and allow in-depth investigation of the phenomenon being studied. Table 4.3 gives an overview of the eleven written assessments collected, comprising five examination papers and six mid-term tests from five units. Figure 4.1 shows the categorisation of these papers according to the CAS active nature of the assessment and the teaching/learning environment. Within the category *CAS inactive assessment with CAS active teaching/learning in the classroom*, there was also a distinction between papers that included screenshots of CAS/graphing screens of mathematically enabled handheld devices or computer software (e.g., MATH314 midterm test, 2017, campus 1).

In addition to the assessment papers, class notes and unit outlines were collected from MATH104, MATH203, MATH310, MATH314 and HMS112P. These were expected to provide background information about the learning environment and to assist triangulation in the study. They proved critical to corroborating researcher interpretations during the commognitive analysis, as will be discussed in Section 4.5.

Table 4.3

Details of Examination Papers and Tests Collected

Mathematics Unit (date[s] of assessment instrument)	Type of paper	Type of questions	Topic	Written materials permitted	Technology permitted and assumed	Duration
MATH310[a] (2015)	Exam	Calculation and interpretation, short answer.	Differential & Difference Equations	No	CAS calculator	2 hrs
MATH104 (2016, 2017)	Midterm Test	Multiple-choice	Functions and Calculus	No	None	1 hr
MATH104 (2015, 2016, 2017)	Exam	Calculation and interpretation, short answer.	Functions and Calculus	No	CAS calculator	2 hrs
MATH203 (2017)	Midterm Test	Multiple-choice	Advanced Calculus	No	None	1 hr
MATH203 (2017)	Exam	Calculation and interpretation, short answer.	Advanced Calculus	No	CAS calculator	2 hrs
MATH314[a] (2017)	Midterm test	Multiple-choice	Differential Equations and Mechanics	No	None	1 hr
HMS112P (2010, 2011)	Midterm test	Multiple-choice	Calculus/ Engineering Mathematics	No	CAS calculator	1 hr

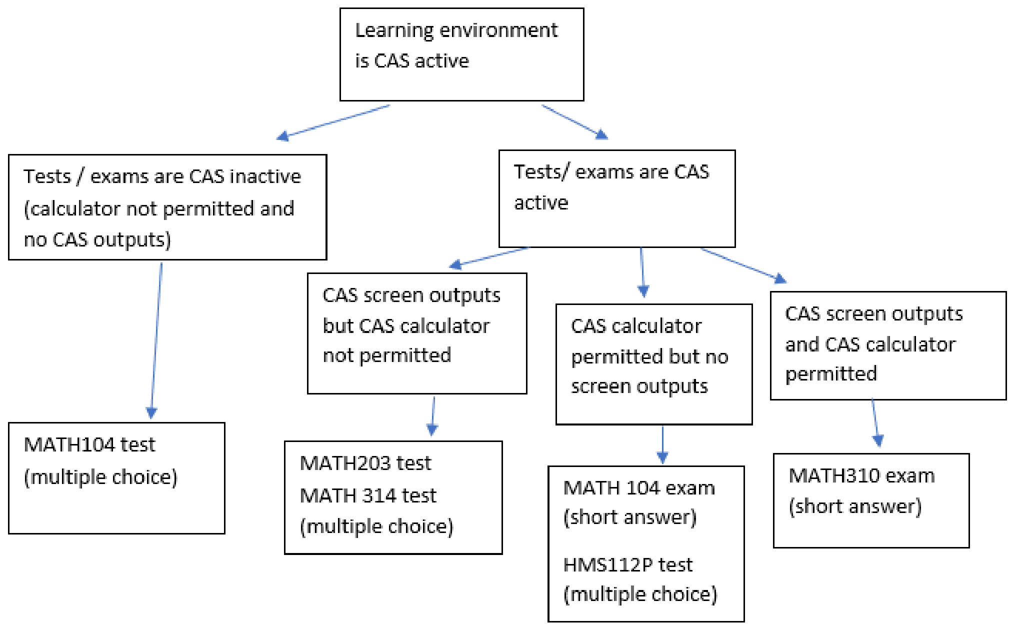
Note. [a] Contains screen from digital tools.

4.4.2 Student CAS questionnaire

A questionnaire was designed by the researcher to gather information about students' use of CAS, utility of CAS, preference for CAS or by-hand methods and access to CAS in assessment. There were two different formats to the survey questions: (a) closed questions where only one option should be selected and (b) questions where more than one option could be selected and where it could also be specified if any other options applied. Examples are shown in Figure 4.2. The full questionnaire is in Appendix A.

Figure 4.1

Selected Test and Examination Papers Classified by Technological Environment



4.4.2.1 Administration of student CAS questionnaire

The opportunity to complete the survey was offered to all students at the beginning of one of their regular classes in the unit (during the middle of the semester in 2015 and during the latter part of the semester in 2016). Each student who was present at that class was given a Participant Information Letter to read about the nature of the research and task, which also informed them that participation in the survey was anonymous and voluntary. The students who chose to complete it were then given approximately 10 minutes to individually complete the survey, remaining in the classroom in their usual seats. The surveys were then immediately collected by the researcher.

Figure 4.2

Example Question Types on Student CAS Questionnaire

Example 1 type (a) question

Do you believe use of CAS has improved your understanding of topics covered in this subject?

- ☐ Yes
- ☐ No
- ☐ Not sure

Example type (b) question

What type/s of CAS computer software did you use outside of formal class time? Tick all that apply.

- ☐ Wolfram Alpha
- ☐ Mathematica
- ☐ Maple
- ☐ MATLAB
- ☐ Other (please specify) ______________________________

Table 4.4

Categorisation of Questionnaire Questions

Category	Sub-category	Question
Use of CAS	Frequency of handheld use outside class	1
	Frequency of computer use outside class	2
	Type of computer CAS use outside class	3
	Enjoyment level in using CAS	7
Utility of CAS	Different purposes of use	4
	Improving understanding	5
	Utility mapped to specific topics	6
	Perceived usefulness of CAS	8
Preference for CAS or by-hand		9
Access to CAS in Assessment		10

4.4.3 Task-based interview

For the task-based interview (Maher et al., 2014), the focus was on how the students interpreted and used their handheld CAS calculator in questions where errors are commonly made and/or where the CAS calculator produces ‘unexpected’ output, in that it is different to what they would obtain using the usual procedures for answering such a question that they have been taught in the unit.

In 2016, first-year students in the *Functions and Calculus* unit were invited to participate in a study investigating “students’ use of CAS in mathematics”. In 2017, second-year students in the *Advanced Calculus* unit were also invited similarly. The invitation to the study in each case was advertised during the second part of the semester on the university’s website for the unit, which is expected to be regularly accessed by all students during the semester. The announcement about the study informed students that “We are investigating how students approach solving Calculus problems using CAS calculators and/or computer software and how they reflect on their answers. We are looking for participants to take part in a case study analysis examining their approach to such problems”. It also included a link to the Participant Information Letter for the study, which outlined that participants would be asked to

- complete several mathematics problems using a CAS calculator and/or CAS software;
- answer related short questions about how you approached these problems and your interpretation of their answers;
- participate in verbal discussion with the researchers about your answers.

The main announcement informed students that the task was expected to take 90 minutes or less to complete and that they would be given a one-off payment of $50 for their participation. The consent form also asked students whether or not they gave permission for the interview about their answers to be digitally recorded, with the form specifying that this was not compulsory.

Three first-year *Functions and Calculus* students responded to the announcement in 2016 and one *Advanced Calculus* student responded to the announcement in 2017, with each of these students also consenting to have their interview digitally recorded. These students consequently participated in the study. They were expected to have completed at least some mathematics units in year 12, including Mathematical Methods or equivalent, with the second-year student also being expected to have completed the *Functions and Calculus* unit or equivalent. Within their learning at high school, students were also expected to have obtained some familiarity with using CAS calculators in class and in some assessments. It has not been possible for me to access more specific biographical information about the students who participated in the task-based interviews.

After consenting to participate, a problem solving/interview session was arranged with each participant. These sessions took place in a quiet room, where only the student, the Chief investigator[2] and student researcher were present. The students were asked to bring the CAS calculator they usually worked with to the session.

At the actual session, the students were first given a set of five mathematics problems to solve (see Appendix B). They were not given any time limit or time pressure and simply told to inform the researchers when they were finished. They were asked to use their CAS calculator to solve the problems, writing down all commands used on the CAS to do so. They were also asked to comment on whether the CAS output was what he/she expected. If it was not what was expected, they were asked how it could be reconciled with what was expected. They were also asked, where it was thought appropriate, to consider or demonstrate any methods by-hand or on CAS that could be used to check or interpret the answer.

I observed the students as they worked. Once each participant finished the set of problems, I interviewed the student, with the interview focussing on discussing each problem in turn, in relation to the questions just referred to. During the interviews, I made notes, but the students did not have any further opportunity to annotate their written responses. The students were also asked, for each problem whether they would usually prefer to solve such a problem using the CAS calculator or by-hand. The interview was audio recorded and transcribed by myself.

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[2] Former supervisor now retired.

4.5 Data Analysis

4.5.1 Analysis of questions on mid-year tests and final examination scripts

Test papers and examination scripts were available for a selection of first-year, second-year and third-year Calculus units. Task analysis was used to classify and investigate a sample of questions from these scripts, both in the absence of CAS and, where applicable, when CAS can be applied to answering a given question. Nine questions were purposively selected from 7 of the 11 undergraduate Calculus assessments (examinations and tests) collected. To analyse these test and examination questions to answer Research Question 1, Morgan and Sfard's (2016) commognitive analysis framework was used.

As with other types of discourse, Calculus discourse in the context of a CAS-active learning or assessment environment can be analysed using the four commognitive constructs (Sfard, 2008) that were described in Chapter 2: *word use, visual mediators, routines* and *endorsed narratives*. In the context of my study, *word use* focused on the specialised language of Calculus and associated topic areas such as Algebra and Functions, together with the word use of the CAS technology itself. As a result, this discourse includes not only number words, names of mathematical operations, commands and processes, but also words localised to the types of CAS considered, such as menu commands and other verbal instructions within CAS. All the types of symbolic and graphical *visual mediators* present were analysed, including any that were produced by CAS. The types of *routines* expected of students in answering examination questions and those carried out by the participants in the task-based interviews were analysed, with these being the types of action expected and demanded of the students in their roles as actors in the community of tertiary Calculus students in a CAS environment. The nature and presence of *endorsed narratives* were also analysed, both in the type of discourse present in examinations, including the extent of alienation of this discourse, and in the task-based interviews, where the extent to which the student participants produced endorsed narratives and the nature of these was investigated. These are all *mathematising* aspects of the discourse, which analyse the nature of the stories which are told about mathematical objects (Morgan & Sfard, 2016). *Subjectifying* aspects of the discourse, which give an account how an examination question speaks to students and the nature of any *autonomy* they have in answering it, including the *grain size* of each task, were also analysed.

Mathematising and *subjectifying* are aspects of discourses that are found in both examination papers and events such as interviews or problem solving sessions.

Morgan and Sfard's (2016) full commognitive analysis framework for analysing the discourse of examination papers is shown in Figure 4.4. However, given the nature of the university examination and test papers I wanted to examine in a CAS active/inactive assessment environment, like Alshwaikh (2016), I felt this framework would only go part way to meeting my needs. In particular, such examinations often contain graphical and symbolic (algebraic) visual mediators of different types and complexity levels. To first consider how Morgan and Sfard's framework could be extended in relation to analysing *graphical* visual mediators, I thus examined Alshwaikh's modification, published in Alshwaikh and Morgan (2013) (see Appendix C). The main modification that I took into account was his distinction between *narrative* and *conceptual* diagrams (Alshwaikh, 2016; Kress & Van Leeuwen, 2006), with the latter suggesting human construction or interaction with the diagram itself over time. This distinction between narrative and conceptual diagrams was important in my study, in investigating the influence of CAS on the types of visual mediators present in Calculus examinations and how such diagrams position students in relation to answering the associated examination questions.

Alshwaikh (2016) gives an example (see Figure 4.3) of the difference between *narrative diagrams* and *conceptual diagrams*, in showing part of a proof of the Exterior Angle Theorem for triangles in two ways. First, in diagram (a) there is a narrative diagram which includes two dotted lines to suggest *construction* of these lines (a line parallel to one side of the triangle and a second line extending one of the other sides), possibly by a human agent, at a time after the original triangle has been drawn. Second, in diagram (b) there is a conceptual diagram where all the lines are solid, so that the proof of the theorem is then shown as an object, while in the narrative diagram, the proof is shown as a process.

Figure 4.3

Distinction Between Narrative and Conceptual Diagrams (Alshwaikh, 2016, p. 170)

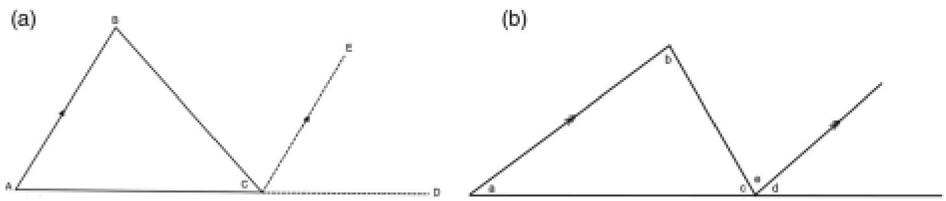


Figure 4.4a

Morgan and Sfard's (2016) Analytic Framework for Mathematising Aspects of Examination Discourse

Aspects of the discourse	**Questions guiding the analysis**	**Textual indicators**
Vocabulary and syntax (lexico-grammatical aspects)		
Specialisation	To what extent is specialised mathematical language used?	* Lexical items used in accordance with mathematical definitions, considered at the level of: ○ vocabulary ○ sentence ○ text unit * Extra-mathematical context ○ depth of engagement with context
Objectification of the discourse	To what extent does the discourse speak of *properties* of objects and *relations between them* rather than of *processes*?	* Nominalisation: use of a 'grammatical metaphor', converting a *process* (verb, e.g. rotate) into an *object* (noun, e.g. rotation). * the use of specialised mathematical nouns such as function, sequence which *encapsulate processes into an object* * complexity of compound nominal groups

Logical complexity	What kinds of logical relationships are present and how explicit are they?	The types and frequencies of conjunctions, disjunctions, implications, negations and quantifiers
Visual mediators		
The presence of multiple visual mediators	* To what extent does the discourse make use of specialised mathematical modes? * How are multiple visual mediators incorporated into the discourse?	* Presence of tables, diagrams, algebraic notation, etcetera * provided in the text or to be produced by the student * linguistic, visual and/or spatial relationships between modes
Transitions between visual mediators	* What transformations need to be made between different modes? * How are transformations indicated in the discourse?	* Presence of or demand for *two or more modes of communicating 'equivalent' information*, e.g. an equation formed from a word problem; a unit of text that involves table, graph and algebraic expressions corresponding to the same function * Provided in the text or to be produced by the student * Explicit linguistic or visual links between modes
Routines		
The types of action demanded of students	* What areas of mathematics are involved?	* Topics

	* What are the characteristics of the routine procedures?	* Algorithmic or heuristic? * Complexity * Explicitly hinted at
Endorsed narratives		
The origin of mathematical knowledge	* What is the degree of alienation of the discourse? * To what extent is mathematics construed as involving material action or as atemporal objects and their properties? * To what extent is mathematics presented as a human activity?	* Mathematical objects as agents in processes *Agency obscured by: ○ non-finite verb forms ○ passive voice * Mathematical objects involved in: ○ material processes ○ relational or existential processes * Human agents in mathematical processes ○ thinking ○ scribbling
The status of mathematical knowledge as absolute or contingent	To what extent does the text indicate that decisions or choices are possible during mathematical activity?	* Modifiers indicating degree of certainty (e. g. may, can, will …) * Conditional clauses (e.g., if … or when …) * Explicit decisions have been or need to be made

Source: Morgan & Sfard (2016, pp. 106-107)

Figure 4.4b

Morgan and Sfard's (2016) Analytic Scheme for Subjectifying Aspects of Examination Discourse

Aspects of the discourse	**Questions guiding the analysis**	**Textual indicators**
Student-author relationship	* What kind of relationship is constructed between the student and a mathematical community? * Is the student given instructions or invited to consider mathematical questions?	Use of personal pronouns ○ inclusive or exclusive we ○ other personal pronouns * Interrogative (questions) * Imperative (instructions)
Student autonomy	* In responding to an examination question, how many independent decisions is the student allowed/required to make in: * Designing the path to follow? * Interpreting the task? * Choosing the form of the 'answer'	* The grain size of the task * Complexity of utterances ○ lengths of a sentence ○ grammatical complexity: the depth of 'nesting' of subordinate clauses and phrases ○ logical complexity * The layout ○ the physical size of the answer ○ the space provided for the work to be done on the way toward solution

	*Choosing/constructing the mode of response?	○ format of the answer (units, precision, no. of solutions) ○ modality of the answer (graph? algebraic expression?) *Visual mediators: verbal, symbolic, or graphic: supplied or to be produced?

Source: Morgan & Sfard (2016, p. 108)

For my analysis of graphical visual mediators, I also introduced Ng's (2016) notion of *static* and *dynamic visual mediators* as an extension of the visual mediator aspects of the discourse, which has proved useful for coding of both the examination and test papers where CAS calculator or CAS-enabled computer software screens have been embedded in a static manner and the task-based interviews where students used their calculators dynamically in response to the problem-based interview questions.

In looking in more detail at the analysis of *algebraic* visual mediators, I in the *logical complexity* section of Morgan & Sfard's (2016) commognitive framework included a count of the number of, and types of, different mathematical operators and algebraic letters in the discourse. I also added a new subsection when analysing any visual mediators *offered* in a question to assist in answering it, where these can take the form of algebraic visual mediators (e.g., calculator or computer output provided with the question) and/or graphical visual mediators. This subsection I have called *level of visual complexity*, and this counts the number of lines of unnecessary CAS output (that do not assist in answering the question), the number of graphs and the number of different types of information presented in a graph or other types of diagrams.

Finally, in order to have a macro-level bird's-eye view of the analysis of these data, Commognitive analysis summary tables as used by Thoma (2018) were adapted. All the sections in Thoma's tables are from Sfard's commognitive framework, but analysed for two different aspects of tasks- in her case, showing the commognitive analysis of a particular task in one column and in the final column aligning each component of this commognitive analysis to the perspective of the lecturer in her study on assessment, as she had interview data from the lecturers who had set the examinations. As seen in Figure 4.5, the first column of my summary tables contains the four interrelated characteristic features of the mathematical discourse from the frameworks of Sfard and Alshwaikh (in bold italics), and based on the same structure as Thoma's tables, the second column the particular features evident in a task. The third column I have added collected commognitive analysis relating to technology use and a fourth column collected evidence about the positioning of the student based on how they had been taught in the course notes. This background corroborating evidence was actually necessary to interpret the data, for example, when deciding whether a routine was a ritual as the students had been taught a routine to interpret a slope field diagram, say, or whether it was a construction as they had not and had to construct their own routine to interpret it.

Figure 4.5

An Example Commognitive Summary Table for a Task

Q1, MATH314 Test A, 2017			
Commognitive analysis		**Relating to technology**	**Positioning of the student based on how they have been taught in the course notes**
Keywords and symbols			
Specialisation	Discourse of differential equations including the word DE and words used to classify them.		Students have been taught in the course notes how to classify DEs using the required specialised words of the type present in this question.
Objectification of the discourse	Discourse of differential equations is objectified: speaking of properties of DEs (as opposed to processes involving them). Longest compound nominal group length is 4.		Similar examples in the course notes, which also have objectified discourse.
Logical complexity	Relatively low: no conjunctions, disjunctions, implications, negations or quantifiers in the written discourse.		
Visual mediators			
Types of visual mediators and transitions between them	Symbolic visual mediator representing the DE being considered.		The same type of notation and presentation is used for the differential equations .
Logical complexity	Moderate: DE in question stem contains 2 variables: x and y.		DEs in the course notes are also often written in terms of x and y.
Routines			
*Type of routine expected (r*itual, recall, substantiation or construction)	Ritual.		Students taught how to identify the required aspects of the DE as a ritual procedure.
Endorsed Narratives			
Alienation of the discourse and how mathematics is construed	Alienated discourse. DE is presented as an atemporal object that needs to be classified in terms of some of its properties.		

Subjectification of the discourse			
Student-author relationship	Student is given an imperative: to choose the correct answer option.		
Student autonomy	The student can choose from a few different exam techniques in determining the correct answer. The form of the final answer is to write down the letter corresponding to the correct answer option.	No choice to use technology, as it would not help in answering this question.	

4.6 Conclusion

This chapter has detailed the research methods used in my study, which was an instrumental case study (Stake, 1995, 2005). This study was situated at an inner-city campus of a multi-campus Australian university, with the students involved studying Calculus in units in which they were taught in a CAS-active learning environment. The data collected included a purposive sample of questions from first-year to third-year tests and examinations for a selection of Calculus units, student responses for task-based interviews where they had solved problems using CAS and talked about their answers, and student responses to a questionnaire on their use of CAS and attitudes towards it. The theoretical framework applied in this study was the commognitive analysis framework of Morgan and Sfard (2016), together with additions from Alshwaikh and Morgan (2013) and Ng (2016), to capture qualitatively the discourse of the selected examination and test questions in these units and that of the student cohort who solved Calculus problems as part of their participation in task-based interviews. The results of the task-based interviews were then contextualised, through discussion of the percentage distribution of responses to each item on the questionnaire, for a larger, convenience sample of students attending classes who were surveyed about their attitudes to aspects of the use and utility of CAS, their opinion on whether or not students should have access to CAS in assessment, and their own preference for either using CAS or working by-hand when solving mathematics problems.

The results of the commognitive analysis of the test and examination papers will be reported in the next chapter. This will then be followed by Chapter 6, where the commognitive analysis of the task-based interviews will be presented, with the results to then be contextualised by the questionnaire data.

CHAPTER 5

RESULTS OF THE COMMOGNITIVE ANALYSIS OF WRITTEN TESTS AND EXAMINATION PAPERS

5.1 Introduction and Overview

As pointed out in Chapter 4, the mathematical discourse in written tests and examination papers has been examined to determine what is achieved by such discourse, particularly with respect to the impact of CAS technology on the institutionalised practice of preparing and sitting for such assessments. In particular, a commognitive framework (Morgan & Sfard, 2016; Sfard, 2008) has been applied to capture the complexity and difficulty level of written answer and multiple-choice examination questions asked in undergraduate Calculus units where use of CAS technology is available. In this chapter, a selection of the commognitive analyses of tasks from written mid-semester tests and examination papers is presented in Section 5.2. These tests and examinations were chosen from first-year, second-year and third-year Calculus units from two universities, with a focus on selecting for analysis examinations and mid-term tests where CAS calculators were permitted or where visual mediators containing CAS outputs were present. I had access to the course notes for each of the units considered and these were used to determine the types of routines students had been taught for solving the different types of test and examination questions to be analysed. The analysis presented in this chapter includes commognitive analysis summary tables, as indicated in Chapter 4, in order to have a macro-level view of the analysis of these data. The chapter ends with a conclusion where the major results coming from the analysis are detailed.

The following analyses have been selected to answer Research Question 1, which I remind the reader of now.

RQ1 How can a commognitive framework be applied to effectively capture the complexity and difficulty level of written answer and multiple-choice examination questions asked in undergraduate Calculus units where use of CAS technology is available?

In section 5.2, in keeping with the analytical framework (Morgan & Sfard, 2016), the nature of the mathematising and subjectifying of the discourse in the examination and test tasks are discussed. The nature of the examinations selected, reasons for this selection, and aspects of the features of the discourse that were looked at will now be presented.

5.2 Commognitive Analysis of Written Assessment Items

5.2.1 Overview

In section 5.2.1, I will provide an overview of how I applied a commognitive framework to analysing university-level test and examination questions on Calculus topics. In section 5.2.1.1 will describe the components of the framework and how they were applied, while section 5.2.1.2 will give an overview of the types of test and examination questions that were analysed, and the reason for selecting each of them. Sections 5.2.2 to 5.2.10 will provide commognitive analyses of each of the 9 test and examination questions considered, in turn.

5.2.1.1 Components of commognitive analysis framework to be applied

In considering the nature of the *mathematising* aspects of the discourse of each question, which concern the types of stories told about mathematical objects, consistent with the framework of Morgan and Sfard (2016) the components to be considered are *vocabulary and syntax (*which includes *word use), visual mediators, routines* and *endorsed narratives*. I now provide an overview of how each of these features has been examined in analysing the discourse of the questions.

Vocabulary and syntax. This included identifying the presence of any specialised mathematical terms, especially those which are associated with the discourse of Calculus and associated mathematical topics. The way in which a sentence is structured determines the extent of objectification of the discourse, so I was looking for the use of grammatical devices that contribute to *objectification*, such as *nominalisation*, where a mathematical process is converted into a noun. Mathematical processes are also sometimes encapsulated in mathematical objects when they are represented by symbolic visual mediators, so I also identified instances where this occurred. The *grammatical complexity* was also analysed, in terms of the length of the longest *compound nominal* group and whether there were any subordinate clauses present. Grammatical devices such as compound nominal groups and repetitive or recursive use of subordinate help determine “[h]ow much unpacking is needed to identify the mathematical information involved” (Morgan, 2016, p. 128). While the commognitive framework contained a separate section on visual mediators, which will be discussed next, these also can contribute to the grammatical complexity and objectification of written discourse, so that aspect was considered here in cases where symbolic visual mediators were included as part of a sentence and when they encapsulated mathematical processes into an object.

Visual mediators can be either symbolic or graphical, with CAS having the capability to produce visual mediators of both these types. Any graphical visual mediators were interpreted and classified as *narrative diagrams* or *conceptual diagrams* (Alshwaikh, 2016; Kress & Van Leeuwen, 2018). Narrative diagrams suggest human construction or interaction with the diagram itself over time. This is in contrast to *conceptual diagrams*, in which human actions are absent and “atemporal objects or relationships” are presented (Alshwaikh, 2016, p. 170). Any graphical visual mediators presented (offered to students) in written examination papers are *static visual* mediators (Ng, 2016), as the students cannot manipulate them directly. However, if any questions in which use of CAS was permitted presented the opportunity for students to produce their own graphical visual mediators using CAS, whether or not such diagrams could have been considered *dynamic visual* mediators was looked at. The *visual complexity* of any graphical visual mediators was also examined, in terms of how many graphs were present and the different types of information presented in them or in other types of diagrams.

Symbolic visual mediators are usually in the form of algebraic expressions, which can vary greatly in terms of complexity. The *logical complexity* of any symbolic visual mediators included in CAS outputs was examined, in terms of the number of distinct letters and subscripted letters they contained, whether these all referred to variables or whether some were constants, and the number of different types of mathematical operations included in the symbolic visual mediators (e.g. addition, multiplication, exponentiation). In cases where CAS output was presented in a question, the *visual complexity* of the output was also measured by determining whether any lines of this output were redundant, in not being necessary for the student to answer the question and, if so, how many such lines of output were present.

Transitions required between different visual mediators were considered by determining “what transformations need to be made between different modes”, if any, “in the presence of, or demand for, two modes of communicating ‘equivalent’ information” (Morgan & Sfard, 2016, p. 106). This could occur, for example, if a test or examination question contained both a symbolic and a graphical representation of the same mathematical object.

The type of *routines* expected of students were determined based on whether similar examples had been provided to them in the course notes and, if so, whether the solution process was been a *ritual* step-by-step process or more left to creative thinking by the student. Whether or not they were required to produce endorsable narratives about mathematics in a question was also an important aspect in determining when *exploration routines* were required of the students.

Endorsed narratives were analysed to examine the nature of the statements that were made about mathematical objects in the test and examination questions, which could be taken to be true. Such stories that are told about mathematical objects include those concerning agency of these objects, with respect to how they generate other mathematical objects or which describe their roles in mathematical processes. As a part of this, whether or not there was *alienation* of the discourse was recorded, as well as if any *material processes* were described or only *relational processes.*

Subjectifying aspects of the discourse, which give an account how an examination question speaks to students and the degree of autonomy they have in answering it, were also analysed. This included looking at the nature of the *student-author relationship*, by seeing if there were any personal pronouns such as *we* in the question, and whether the student was given an imperative or was invited to explore mathematical questions. In analysing the *student autonomy* aspect of the subjectifying of the discourse, possible solution paths the students could follow to answer a question were considered, leading to determining the *grain size* of each problem, as the minimal number of independent decisions a student needed to make to solve it. This took account of the effect of CAS on the solution process, in questions from tests and examinations where its use was permitted, as well as recording whether or not students had the choice to use CAS in answering the question. Whether or not the students had any autonomy in choosing the form of their answer was also examined, and for any questions where they were required to produce graphical visual mediators, this was recorded. The number of words and visual mediators in the longest sentence present was also recorded, as this is one measure of the relative complexity of information the students needed to process in answering test and examination questions.

5.2.1.2 Selection and characteristics of test and examination questions to be analysed

All the test and examination questions analysed in this chapter are from units in which the students were taught in a CAS-active learning environment, where they were expected to own a CAS calculator and where use of CAS and interpretation of its outputs was integrated into the teaching of the unit. However, student use of CAS was only permitted in some of the assessments considered, allowing an opportunity to analyse questions from both assessment environments.

Some of the analyses that follow are of questions which were selected due to their containing CAS outputs of different types of graphical and symbolic visual mediators produced by CAS. When looking in the tests and examinations for visual mediators produced by CAS, most of these were associated with the Calculus topic of differential equations, with slope field diagrams being the main type of *diagrammatic/graphical* visual mediator present; appearing in some of the second- and third-year tests and third-year examinations. These diagrams also sometimes had a solution curve superimposed on them and, in tests these were present in multiple-choice questions, while in the examinations they were a part of short answer questions. Therefore, in section 5.2.3, I have analysed a multiple-choice test question where the visual mediator shows local slopes only and the students were required to match the diagram with the equation of the DE and, in section 5.2.4, a written short answer examination question, which showed both local slopes and a particular solution curve where the students were required to both identify an initial solution and to produce endorsable narratives describing the local slopes in the diagram. Graphical visual mediators produced by other graphing software (that could be, but were not necessarily, CAS) were also considered, including in those assessments where use of a CAS calculator was not permitted. One question of this type, from a first-year mid-term test in which use of CAS calculators was not permitted, has been analysed in section 5.2.9. This question, on the Calculus topic of limits, provided a graphical visual mediator showing the graph of a piecewise continuous function, and the question required students to determine, from the graph, if the limit of the function existed at a particular point.

In the second- and third-year tests and examinations there were also screenshots from CAS calculators and software which displayed *symbolic* visual mediators, showing the solution of differential equations. These were in the form of output from Wolfram Alpha software and TI-NSpire calculator screens, so I have analysed one question of each of these types in sections 5.2.5 and 5.2.6, respectively.

I was also looking for Calculus questions where no CAS outputs were provided but where use of a CAS calculator was permitted and could assist in answering a question. There were some questions where CAS could be used directly, and in isolation, to answer a question, so one such question, where the students were required to determine a Maclaurin polynomial, was selected from a first-year examination paper and analysed in section 5.2.8. A few other questions required a more integrated approach between use of CAS and working by-hand, including requiring production of a graphical visual mediator as part of answering the question, so one question of this type from a first-year examination, which required students to determine the region bounded by two curves and its area, was analysed in section 5.2.10.

While both of the questions just described were short answer questions, I was also interested in the type of multiple-choice questions that could be asked when use of CAS was permitted. No questions of that type were present in the tests and examinations from the university which was the focus of this study, so I examined first-year mid-term test papers from a second university, where the questions were multiple-choice and where use of CAS was also permitted in the test. I selected a question which required using CAS to calculate a derivative, where some rearrangement of the resulting answer provided by the calculator was then required to determine the correct answer option. This question, which has been analysed in section 5.2.7, was also a context-separable application question, unlike the other questions selected for analysis.

Finally, I selected a question which could be considered CAS-free, in that it was from a third-year mid-term test where use of CAS was not permitted and it did not have any outputs produced by CAS (or other technology). This was a multiple-choice question on differential equations, for a third-year mathematics unit, and has been analysed in section 5.2.2.

5.2.2 Analysis of DE Classification Item in CAS-Inactive Mid-semester Test

This example in Figure 5.1 comes from a second-year Advanced Calculus mid-semester test for second-year students in 2017. CAS calculators were not permitted in the mid-semester test and would not have given any advantage in this particular question, even if permitted.

5.2.2.1 Vocabulary and syntax

Specialisation: Question 1 is a multiple-choice task, focusing on classifying a differential equation. The task involves the discourse of differential equations and contains the specialised mathematical language of Calculus and differential equations, both in the use of the word “DE” in the question stem and in the four words used to classify DEs in each of the answer options. The question stem includes a visual mediator in the form of an algebraic representation of the differential equation being considered. There are no CAS screens in this question, nor are any tables or graphs provided in the question.

Figure 5.1

DE Classification Item

1 The following DE is best classified as

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = y^3$$

A Second Order, Linear, Ordinary
B Second Order, Nonlinear, Ordinary
C First Order, Linear, Ordinary
D Second Order, Nonlinear, Partial
E Second Order, Linear, Partial.

[Q1, MATH203 Test A, 2017]

Objectification of the discourse: The use of the specialised mathematical term *DE* (meaning “differential equation”) encapsulates processes into the object “DE” as this term relates a function of one variable and its first- and second- order derivatives which represent rates of change. This appears to be nominalisation, as it converts the process of relating a function and its derivatives into the object DE. The longest compound nominal group (i.e., phrase with more than one word that has the same grammatical role as a noun) in the question stem has 3 words (i.e., “the following DE”). Each answer option contains 4 words for classifying three aspects of the DE. There are no diagrams or nested algebraic functions in this question. This question overall contains relatively low-level grammatical complexity compared with what is seen in certain other types of examination questions (examples of which will be seen later in this chapter).

Logical complexity: The logical complexity in this question is also relatively low compared to what will be seen in some subsequent examination questions. There are no conjunctions, disjunctions, implications, negations or quantifiers in the written discourse of this question. Logical complexity of visual mediators will be discussed in the coming sub-section.

5.2.2.2 Visual mediators

The actual algebraic expression $\frac{d^2y}{dx^2} + \frac{dy}{dx} = y^3$ in the question stem can be seen as either a single entity, or as a sentence, which makes it more complex. Looking at the equation as a sentence is the most appropriate here, where in turn this sentence then contains 5 entities: $\frac{d^2y}{dx^2}$, +, $\frac{dy}{dx}$, =, y^3. It is appropriate to think of it in this way because in order to answer the question asked, students need to consider these separate entities to correctly classify the differential equation. Tasara (2018) remarks that "**[s]ymbolic mediators** such as $\frac{dy}{dx}$ or $f'(x)$ have a dual role. On the one hand, they can be an **objectified narrative** for 'the derivative of ', and an **operational narrative** for 'the process of differentiation' on the other" (p. 7). In this case, $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ work as **objectified narratives**, as the differential equation in the context of this question serves as an **object** for the students to classify in terms of several of its characteristics. This aspect will therefore be elaborated in the alienation of the discourse sub-section below.

Transitions between visual mediators: In this case, there needs to be a transformation **between algebraic (symbolic) visual mediators and textual/verbal nominal groups** since the correct answer option is included that needs to be matched with the properties of the DE provided in the question stem.

Logical complexity of visual mediators: The visual mediator (DE) in the question stem contains two types of arithmetic operations: addition and exponentiation indicated by powers. Powers are also used in the notation for the second order derivative**.** Thus, powers are used in two different ways in the visual mediator that contains the equation under consideration in this question: as a mathematical operation and as a notation instruction, with, in the latter case, the entity $\frac{d^2y}{dx^2}$ representing *the second derivative of* y *with respect to* x. The question stem also contains two algebraic letters, x and y, which represent two variables. There are no algebraic letters in the answer options.

5.2.2.3 Routines

The students are asked to engage with a *ritual* routine, as they are required to identify three aspects of the differential equation in order to be able to determine the correct answer. The unit notes show students were taught how to identify each of these aspects of differential equations as a ritual procedure. I will elaborate more on this aspect in the sub-section on student autonomy in the Endorsed Narratives sub-section.

Instructions given regarding the procedure of the routine: Having access to a ritual routine does not preclude other approaches, so students are allowed to choose the procedure of the routine to apply. They must then write down the letter corresponding to the correct answer option on the first page of the test paper in a box provided.

5.2.2.4 Endorsed narratives

Alienation of the discourse/ The origin of mathematical knowledge: The object in this question is the DE, which is presented as being in an agentless relationship in which the agency is obscured by the use of passive voice "*DE is best classified as*" and a non-finite verb form: "*following,*", resulting in alienated discourse. In this case, the mathematics involved is construed as involving atemporal objects and their properties, as opposed to involving any material processes. When $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ work as objectified narratives they contribute to the alienation of the discourse through objectification of that discourse.

5.2.2.5 Subjectifying aspects of the discourse

Student-author relationship: There are no inclusive personal pronouns in the question or answer options. The student is given an imperative to choose the correct answer option. This suggests an impersonal student-examiner relationship, with no implied collaboration or mutual engagement in the process of doing mathematics.

Student autonomy: We look at this in terms of the students' autonomy in designing a solution path to follow, including determining the *grain size* of the task. This means we must determine the solution path/s which require the lowest number of independent decisions to be made by a student in obtaining the answer.

There are three approaches that students could have taken in determining the correct answer for this question. The first approach, summarised in Table 5.1, is the one that we would expect students to use if they are knowers of the ritual procedure of classifying differential equations. Firstly, they would read the answer options to determine the components they need to classify. Secondly, using the ritual rules they have learned in the unit, they would classify the order, the linearity and the nature (*ordinary/partial*) of the DE. Finally, they would check through the answer options and circle the one that matches the answer they have worked out. The student is therefore required to make four independent decisions (in classifying each of the three required aspects of the differential equation and then matching the combination of these three characteristics to the correct answer option).

The second approach the student could take would be to immediately check the DE against each answer option in turn until they find what they believe is the correct answer. Using this method would be positioning the student as a knower of multiple-choice exam technique more than as a knower of DEs. However, in this case, this method would not present an advantage over the first method described, nor would it reduce the total number of decisions required, as they would still need to correctly classify each component of the DE and, assuming they worked through the answer options in order, would need to evaluate the correctness of answer options A and B.

Table 5.1

Decisions for Solution of Q1, MATH203 Test A, 2017 Using Method 1

Result obtained	**Decision actions**
Second order	Determine the order of the differential equation
Non-linear	Determine if it is linear or non-linear
Ordinary	Determine if it is ordinary or partial
B. second order, non-linear, ordinary	Choose the correct answer option
	4 decisions

A third approach would be to look at the first required classification (‘order’ here) and cross off any answer options where this is incorrect (option C in this case as it is not first-order), then look at the next classification and cross of any incorrect options etcetera until the student is left with only the correct answer option. This would involve 6 decisions (see Appendix D, Table D1).

Therefore, the first approach, shown in Table 5.1, has the minimal number of decisions (as does the third approach), so the *grain size* of the task is 4.

In terms of CAS technology use, if the student is to successfully answer the question, we need to determine if they have any choice as to whether or not to use technology. In this question, students do not have this choice as use of technology will not help in classifying the DE to answer the question.

To interpret the task, the longest, and only sentence in the stem of the question, has length 8 (including 7 words and the differential equation object). Each answer option has a sentence of length 4 (classifying the required aspects of the DE). There is no nesting of subordinate clauses or phrases.

With respect to choosing the form of the 'answer', the student is required to write down the letter corresponding to the correct answer option (in a space provided on the first page of examination paper). Therefore, they have no autonomy in this respect.

In order to have a macro-level view of the analysis of these data, a Commognitive analysis summary table as adapted from those used by Thoma (2018) summaries the analysis (see Figure 5.3), to facilitate ease of looking across the Commognitive analyses of all tasks.

5.2.3 Analysis of DE Identification from Slope Field CAS Screenshot in CAS Inactive Test

This next example was present on the same second-year Advanced Calculus mid-term test paper as the previous question and on a third-year Differential Equations and Mechanics mid-semester test paper from 2017. In this both cases, CAS calculators were not permitted. The diagram shown in this question stem (see Figure 5.2) is a "direction fields" diagram, which was generated by a CAS calculator commonly used by students taking this unit).

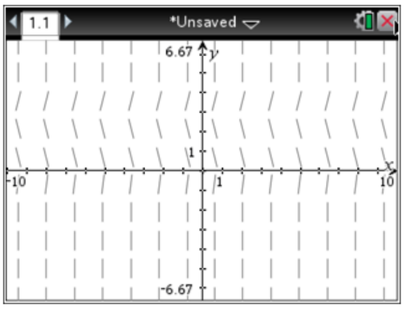
5.2.3.1 Vocabulary and syntax

Specialisation: Question 5 is a multiple-choice task which requires matching of direction field calculator screen output that is provided to the correct differential equation from the list of answer options. The task involves the discourse and specialised mathematical language of the Calculus topic of differential equations, including use of the word DE and the phrase "direction field" in the question stem and the word DE and the notation *y'* in the answer options.

Figure 5.2

DE Identification Given Slope Field CAS Screenshot Item

5 A DE has the direction field following in the region shown.



The DE could be

A $y' = y(y+3)$

B $y' = 2 + y$

C $y' = 2y(y - 3)$

D $y' = -2 - y$

E $y' = (3 - y)$

[Q5 MATH314 Test A and Q4 MATH203 Test A, 2017]

Figure 5.3

Commognitive Summary Table for Q1, MATH203 Test A

Q1, MATH203 Test A, 2017			
Commognitive analysis		**Relating to technology**	**Positioning of the student based on how they have been taught in the course notes**
Keywords and symbols			
Specialisation	Discourse of differential equations including the word DE and words used to classify them.		Students have been taught in the course notes how to classify DEs using the required specialised words of the type present in this question.
Objectification of the discourse	Discourse of differential equations is objectified: speaking of properties of DEs (as opposed to processes involving them). Longest compound nominal group has length 4.		Similar examples in the course notes, which also have objectified discourse.

Logical complexity	Relatively low: no conjunctions, disjunctions, implications, negations or quantifiers in the discourse		
Visual mediators			
Types of visual mediators and transitions between them	Symbolic visual mediator representing the DE being considered.		The same type of notation and presentation is used for the differential equations.
Logical complexity	Moderate: DE in question stem contains 2 variables: x and y.		DEs in the course notes are also often written in terms of x and y.
Routines			
*Type of routine expected (r*itual, recall, substantiation or construction)	Ritual.		Students taught how to identify the required aspects of the DE as a ritual procedure.
Endorsed Narratives			
Alienation of the discourse and how mathematics is construed	Alienated discourse. DE is presented as an atemporal object that needs to be classified in terms of some of its properties.		
Subjectification of the discourse			
Student-author relationship	Student is given an imperative: to choose the correct answer option.		
Student autonomy	The student can choose from a few different exam techniques in determining the correct answer. The form of the final answer is to write down the letter corresponding to the correct answer option.	No choice to use technology, as it would not help in answering this question.	

Objectification of the discourse: The mathematical term *"direction field* is encapsulating processes into the object "direction field," as this phrase signifies a diagram which contains slopes representing the rate of change of y with respect to x at different points. The longest compound nominal group in the question stem has 8 words "*the direction field following in the region shown,*" while each answer option contains a visual mediator in the form of an algebraic representation of a DE, meaning none of the answer options contain compound nominal groups. There are no nested algebraic functions in this question.

Logical complexity: There are no conjunctions, disjunctions, implications, negations or quantifiers in the written discourse of this question.

5.2.3.2 Visual mediators

The question stem includes a visual mediator in the form of output from a CAS calculator screen which shows a direction field plot associated with a first order DE. The five answer options each contain an algebraic/symbolic visual mediator representing a first-order DE.

Transitions between visual mediators: Transformations are needed between **graphical and algebraic (symbolic)** modes in both directions, since the correct algebraic answer option is included that needs to be matched with the behaviour of the local slopes shown in the direction field plot and checked.

Visual complexity: The slope field diagram relates two variables, x and y. The slopes shown at given points provide additional information, as they approximate the value of y' at those points. This introduces additional complexity to the diagram in a similar way to which the presence of a third variable or an additional dimension to the graph would, and this aspect of the diagram therefore increases its visual complexity. In evaluating the complexity of the CAS screenshot, it can also be seen the slopes change depending on the value of y but not across different values of x (given a fixed y value). While there are no visual distractors in the CAS screen output (i.e., no additional graphs not relevant to answering the question), students are positioned to be a knower of which parts or aspects of the output are most relevant (namely, where the slopes are positive and where they are negative and where they become steeper).

Logical complexity of visual mediators: In the visual mediators (DE) in the answer options there are three types of operations: addition, subtraction and multiplication. There is also an equals sign since each visual mediator is an equation. The answer options contain two algebraic letters: x and y, which represent two variables. There are no algebraic letters in the question stem.

5.2.3.3 Routines

The students are asked to engage with a *construction routine* in this task, as they are required to interpret and link the relevant aspects of the direction field diagram to the algebraic structure of a first order DE in order to be able to determine the correct answer from the multiple-choice answer options provided. In the unit, the students were taught in the course notes that a direction field diagram shows local slopes of a DE at different points, but they were not taught any step-by-step ritual routines for identifying specific characteristics to allow them to match such a diagram to the algebraic form of a DE.

Instructions given regarding the procedure of the routine: Students are allowed to choose the procedure of the routine. They must then write down the letter corresponding to the correct answer option.

The direction fields diagram is a *narrative diagram*, as it invites human action in the form of exploration. This is how it is positioning the student but is a student likely to take on that positioning? In other words, in what ways might students interact with this output and act within the practice of the mathematics they have been taught? The students had not been taught any rituals (i.e., step by step procedures) for answering a question of this specific type. Instead, they were assumed to be a knower of the conceptual interpretation of the line segments shown in the diagram as values of y' for different combinations of x and y. This output could invite some students to draw on, or annotate, the diagram to help them capture aspects of it that would assist them in determining the correct answer option. For example, they could identify (e.g., by circling or highlighting) areas of the diagram where the slopes are positive as separate from the areas of the diagram where the slopes are negative. Estimating actual approximate values of the slopes would be more difficult from the diagram provided, but some can clearly be identified as positive (if sloping up to the right) or negative (if sloping down to the right).

5.2.3.4 Endorsed narratives

Alienation of the discourse/ The origin of mathematical knowledge: In this question, agency is obscured by use of passive voice: "*DE has the direction field following*". There are also no material processes involving mathematical objects described in the discourse of the question. However, the slope field diagram is a process in this case, as it is showing rates of change of y with respect to x at different points. As the diagram is classified as a narrative diagram inviting human action, this mitigates the alienation of the discourse for a knower of these technological outputs as the author of the task is positioning them as such.

5.2.3.5 Subjectifying aspects of the discourse

Student-author relationship: As in the previous question, there are no inclusive personal pronouns in the question or answer options and the student is given an imperative to choose the correct answer option. This again suggests an impersonal relationship between the student and the examiner, with the examiner as the authority.

Student autonomy: In responding to an examination question, how many independent decisions the student is allowed/required to make will depend on how the student approaches answering the question and on what is interpreted as a "decision". For example, in the approach shown in Table 5.2, the student could be seen to be required to make 6 independent decisions, in interpreting the slopes shown in the direction field diagram and then relating these to the algebraic expressions shown in the answer options.

Designing the path to follow: An example of a path a student could choose to follow to answer the question after noticing the slopes take a constant value for each given *y* (as *y'* only function of *y*) is as follows, with the corresponding decisions also shown in Table 5.2.

1. Identifying (from diagram) approximate values of *y* where the slopes change from positive to negative;

2. Finding only possible answer option where this is true (C);

3. Verifying where the slopes are positive or negative is consistent for the graph and answer C.

All the decisions, except the last, optional verification step, which are summarised in Table 5.2 are necessary to determine the correct answer. Therefore, the grain size of this task is 6.

In terms of CAS technology use, in this question, students need to interpret the direction field diagram shown in the CAS calculator screen output provided in order to know the nature of the DE which is represented in the question, and to be able to therefore select the correct answer option. This is a test in which actual *use* of CAS was prohibited, so the students need to be a knower of CAS outputs of this type.

To interpret the task, the longest (and only) sentence in the stem of the question has length 11 words. Each answer option is an algebraic equation (symbolic visual mediator) with no additional written text (words).

With respect to choosing the form of the 'answer', the student is required to write down the letter corresponding to the correct answer option (in a space provided on the first page of the examination paper). There is thus no autonomy in this respect.

Table 5.2

Decisions for Solution of Q5 MATH314 Test A and Q4 MATH203 Test A, 2017

Choice taken	**Decision action**
For values of y less than 0 or greater than 3	Determine from the graph where the slopes are positive
For values of y between 0 and 3	Determine from the graph where the slopes are negative
At $y = 0$ and $y = 3$	Identify where the slopes change direction (sign)
y	Determine the type of factor that would change sign at $y = 0$
$y - 3$	Determine the type of factor that would change sign at $y = 3$
Only C. $2y(y\text{-}3)$, suggesting it is correct answer	Identify all answer options that have factors of y and $y - 3$
True	(optional) verify by substitution of 'test values' that it is negative between $y = 0$ and $y = 3$ and positive elsewhere
	6 decisions

Figure 5.4 is a Commognitive analysis summary table that summaries the analysis of this task.

5.2.4 Analysis of DE Output Interpretation of Slope Field CAS Screenshot Item in CAS-Active Examination

The next example comes from a third-year Differential and Difference Equations final examination paper from 2015. This was an examination for students where CAS calculators were permitted. The question in Figure 5.5 is a stand-alone question (i.e., it is independent of the questions and any work done by the students in answering parts (a) – (c) of the question). The diagram shown in this question is once again a "direction fields" diagram, which was generated by a CAS calculator, but, unlike the question discussed in section 5.2.3, this direction fields diagram also has a curve superimposed on as plotted points, representing one particular solution to the differential equation provided.

Figure 5.4

Commognitive Summary Table for Q5, MATH314 Test A and Q4 MATH203 Test A, 2017

Q5, MATH314 Test A and Q4, MATH203 Test A, 2017			
Commognitive analysis		**Relating to technology**	**Positioning of the student based on how they have been taught in the course notes**
Keywords and symbols			
Specialisation	Discourse of differential equations including the word DE.	Discourse of direction fields.	The students have been introduced to direction fields and their interpretation in the context of first order DEs
Objectification of the discourse	Discourse of differential equations is objectified: speaking of properties of DEs (as opposed to processes involving them) Longest compound nominal group length is 11.		Similar examples in the course notes which also have objectified discourse.
Logical complexity	Relatively low: no conjunctions, disjunctions, implications, negations or quantifiers in discourse.		
Visual mediators			
Types of visual mediators and transitions between them	Symbolic visual mediators in the answer options, representing equations of different DEs. Direction fields plot in question stem. Transformations are needed between graphical and symbolic modes in both directions.	Direction fields plot. This is a narrative diagram as it indicates visually the rate at which y is changing with respect to x at different points.	Examples of direction fields diagrams and their corresponding algebraic DEs are provided in the course notes, together with CAS calculator instructions on how to produce a direction field plot for a first-order DE.
Logical complexity	Moderate for DE in question stem, which contains 2 variables: x and y.	High: CAS screenshot showing direction fields plot relates x and y, but for local slopes at different points. That the gradients of the line segments shown provide additional information: indicating the slope y' at different points (x, y), increasing complexity of the diagram.	The general interpretation of direction fields plots is given in the course notes.

Routines			
*Type of routine expected (r*itual, recall, substantiation or construction)	Construction.		The course notes do not provide any ritual step-by-step procedures for interpreting direction fields diagrams or linking them to the equations of particular DEs.
Endorsed Narratives			
Alienation of the discourse and how mathematics is construed	Alienated discourse. DEs in the answer options are presented as atemporal objects, to be linked to the direction fields diagram.	Direction fields diagram can be seen to contain processes within it, as it is showing the rate at which y *changes* with respect to x at different points.	
Subjectification of the discourse			
Student-author relationship	Student is given an imperative to choose the correct answer option.		
Student autonomy	The student can choose from a few different exam techniques in determining the correct answer. The form of the final answer is to write down the letter corresponding to the correct answer option.	To successfully answer the question, the student must use the direction fields diagram (they do not have the choice to ignore this output generated by a CAS calculator).	

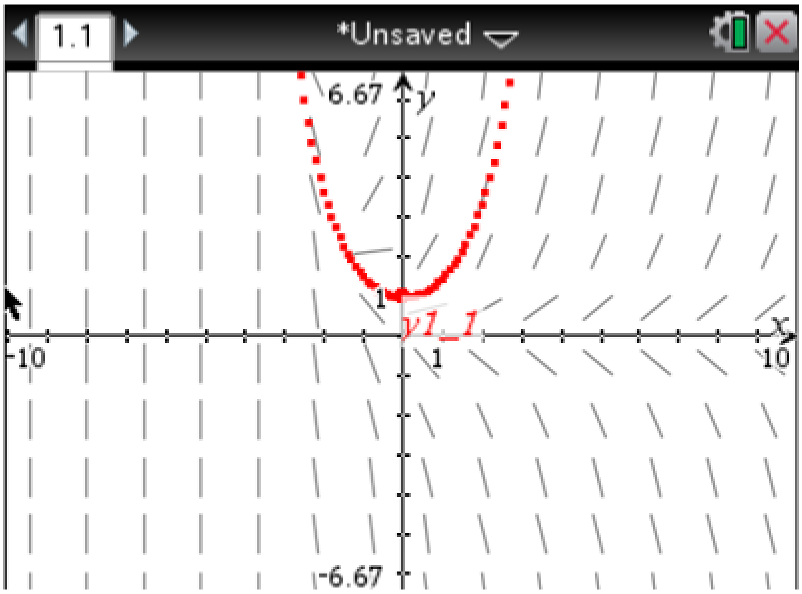
5.2.4.1 Vocabulary and syntax

Specialisation: Question 1(d) is a short-answer question, which firstly requires the students to identify the values of x and y that correspond to the initial condition which satisfies the dotted curve on the diagram. It then requires the students to produce a written interpretation of what they see in the diagram, by asking them to "explain the local slopes shown" in relation to the differential equation provided. This task involves the discourse and specialised mathematical language of differential equations, with the question including use of the word DE and the symbolic visual mediator/notation y'. The mathematical phrases *initial condition*, *particular solution curve*, *slope field* and *local slopes* are also from the discourse of differential equations, while the individual words contained within these phrases also have different meanings in other mathematical contexts.

Figure 5.5

DE Initial Condition Identification and Slopes Interpretation Given CAS Slope Field Item

(d) The slope field for the DE $y' = y - e^{-x}$ is shown below. Axes are shown



Identify the initial condition which generates the particular solution curve shown and explain the local slopes shown in terms of the DE.

[Q 1(d), MATH310 Exam, 2015]

Objectification of the discourse: The discourse is mostly objectified, with the mathematical term *slope field* present in the first sentence encapsulating processes into the object “slope field,” because, as in the question considered in section 5.2.3, this phrase signifies a diagram which contains slopes representing the rate of change of *y* with respect to *x* at different points. However, in the second sentence, the wording, *“the initial condition which generates the particular solution curve”* contains a descriptive adjectival clause, with the verb “generates” referring to the initial condition being involved in the production of the solution curve. The longest compound nominal group in the question is "*the slope field for the DE* $y' = y - e^{-x}$*",* which has 6 words and one symbolic visual mediator. The composite (nested) function e^{-x} appears in the question stem.

Logical complexity: The conjunctions *and* and *which* each appear once in the question, increasing the complexity relative to that of the questions in sections 5.2.2 and 5.2.3. There are no disjunctions, implications, negations or quantifiers in the written discourse of this question.

5.2.4.2 Visual mediators

The question stem includes a visual mediator in the form of output from a CAS calculator screen, which shows a direction fields diagram associated with a first order DE, with a particular solution curve also plotted.

Transitions between visual mediators: Transformations are needed between **graphical** and **textual** modes, as the first part of the question instructs the students to "identify the initial condition which generates the particular solution curve shown." This can be done by looking at the slope fields diagram and reading off the value of y on the curve for which $x = 0$. This would be the most likely way for students to approach this part of the question. However, in answering the second part of the question, the students are required to make a transition between the graphical visual mediator and the algebraic visual mediator giving the associated differential equation, when explaining how the local slopes shown in the diagram relate to the differential equation.

Visual complexity: The plot relates two variables, x and y. The slopes shown at given points provide additional information, as they approximate the value of y' at those points, meaning that, as in the previous example, this is like having a third variable or additional dimension to the graph present, thus again increasing its visual complexity. The inclusion of the dotted solution curve also adds further visual complexity, in providing additional information which is required to answer the question. The solution curve being shown with dotted lines also suggests the graphical visual mediator provided as a whole could be considered to be a *narrative diagram* (Alshwaikh, 2016, 2018; Kress & Van Leeuwen, 2006), as the dotted nature of the curve suggests a representation of a mathematical activity which takes place over time, in first determining the slope field (local slopes) and then following them to superimpose onto the diagram the solution curve corresponding to the initial condition provided. While there are no visual distractors in the CAS screen output, as in the previous question, students are positioned to be knowers of which parts or aspects of the output are most relevant. These are the y value of the solution curve when $x = 0$ and, in answering the second part of the question, being able to describe how the local slopes in the diagram behave, near the solution curve. The latter should include describing where such slopes are positive, where they are negative and where they become steeper. CAS screenshot complexity could also be considered to be greater than in the previous question, as the slopes are affected by both different values of y and different values of x, whereas in the previous example the slopes were only affected by different values of y. In addition, the solution curve is an extra piece of graphical information which was not included in the CAS screenshot in the question from Example 5.2.3.

Logical complexity of visual mediators: In the symbolic visual mediator (DE) in the question there are two types of arithmetic operations: addition and exponentiation. There is also an equals sign since it is an equation. The question contains two algebraic letters: x and y, which represent two variables. This could be considered a moderate amount of logical complexity.

5.2.4.3 Routines

The first requirement of the task, determining the initial condition, could potentially be considered as a *ritual routine*, as the students have had experience in reading values from a graph and are being positioned as knowers that an *initial* condition usually (although not always) occurs when the independent variable, in this case *x*, is equal to 0. With the absence of any other specifications for the initial condition in the question, choosing the initial condition in this way (i.e., at $x = 0$) would be an acceptable solution and the most likely one for students to use, as the examples of initial conditions provided in the course notes all occur when the value of the independent variable is 0. However, the students are asked to engage with a *construction routine* in the second requirement of this task, as to correctly address it they are required to produce endorsable narratives which describe the nature of the local slopes shown in the diagram and how these relate to the corresponding differential equation in the question. Analysis of the course notes indicated that the students were not taught any step-by-step ritual routines for identifying specific characteristics to describe such a slope field or how it relates to the algebraic form of a differential equation. This means they need to construct their own endorsable narratives without having previous knowledge of a *template* style of answer to work from.

Instructions given regarding the procedure of the routine: Students are allowed to choose the procedure of the routine. They must write down the initial condition that satisfies the curve shown and also provide a written interpretation of how the slopes shown in the graphical visual mediator relate to the differential equation. As with Figure 5.2 for the previous question, the plot invites human action in the form of exploration, where, for example, the students again could draw on the diagram to identify areas where the slopes are positive as separate from the areas where the slopes are negative. In this case, they could also potentially draw on the diagram by tracing the path taken by the local slopes near the solution curve shown and/or add a drawing of a horizontal local slope at the turning point of the solution curve at (0,1), to help in describing their interpretation of the local slopes in terms of the differential equation. Here, students were again assumed to be a knower of the conceptual interpretation of the line segments shown in the diagram as values of y' for different combinations of x and y. Estimating actual approximate values of the slopes would again be difficult from the diagram provided but many can clearly be identified as positive or negative, including those near the particular solution curve shown.

5.2.4.4 Endorsed narratives

Alienation of the discourse/ The origin of mathematical knowledge: In the text of the question, the discourse is alienated from human actions, with "*the initial condition which generates the particular solution curve"* suggesting a material process with non-human agency, where it is the initial condition carrying out or contributing to the action of generating the curve of the particular solution. As in the previous question described in section 5.2.3, the slope field diagram for this question represents a process, in showing the rates of change of y with respect to x at different points. It is also again a *narrative diagram,* inviting human interaction with it. As a result, the alienation of the discourse is reduced for any students who are knowers of this type of narrative diagram.

5.2.4.5 Subjectifying aspects of the discourse

Student-author relationship: There are no inclusive personal pronouns in the question or answer options, again suggesting an impersonal relationship between the student and the examiner. The student is given an imperative to identify the initial condition corresponding to the solution curve. They are also asked to *explain* the local slopes shown in terms of the differential equation. This second instruction, unlike the other questions considered so far, is construing the student as an actor in the role of *thinking* (Morgan, 2016) about how to *construct* narratives about mathematics in producing such an explanation.

Student autonomy: In responding to the first requirement of the question, the minimal number of decisions would be 2, which would involve firstly recognising that the initial condition will occur when $x = 0$ and secondly reading the value of y from the graph when $x = 0$. In addressing the second requirement of the question, there are different ways in which the local slopes could be explained in terms of the DE. A description of how the local slopes behave near the solution curve on each side of it and at its turning point at $x = 0$, together with an explanation that relates the slopes in each of these sections of the graph back to the algebraic form of the differential equation, would capture this.

Designing the path to follow: In order to bring out all these features with the minimal number of decisions, an example of a solution path a student could choose to answer the question would be:

1. Identifying that they will need to determine the value of y when $x = 0$ on the solution curve in order to find the initial condition.

2. Reading the value $y = 1$ when $x = 0$ from the graph of the solution curve, thus giving the initial condition $y(0) = 1$.

3. Identifying that the local slopes near the left side of the solution curve are negative and getting steeper as y increases.

4. Relating this back to the equation by explaining that $y' = y - e^{-x}$ will become an increasingly large negative number for increasingly large negative values of x combined with increasingly large positive values of y.

5. Identifying that the local slopes near the right side of the solution curve are positive and getting steeper as y increases.

6. Relating this back to the equation by explaining that $y' = y - e^{-x}$ will become an increasingly large positive number for increasingly large positive values of x combined with increasingly large positive values of y.

7. Identifying that at (0,1) the local slope would be horizontal, as this is at a stationary point (a local minimum for the solution curve).

8. Relating this to the equation in that when $x = 0$ and $y = 1$, $y' = 1 - e^{-0} = 0$.

This solution path involves 8 decisions. This is the minimal number of decisions to solve this problem, as two decisions are required to find the initial condition, followed by positive, negative and horizontal slopes near the solution curve each needing to both be described and then related to the differential equation as in the solution path just described. Therefore, the grain size of the task is 8.

The corresponding table of decisions is shown in Table 5.3.

In terms of CAS technology use, students need to interpret the direction fields diagram and associated solution curve plot shown in the CAS calculator screen output provided, in order to determine the initial solution to the differential equation and, using the diagram, to describe the behaviour of the local slopes near the solution curve in relation to the associated differential equation provided in the question. They are thus positioned as knowers of this output.

To interpret the task, the longest sentence in the question has length 22 words, which is a greater individual sentence length than that of most the other questions considered in this chapter. “Initial condition which generates the particular solution curve shown” is an adjectival clause describing “condition,” but there is no nesting of phrases in subordinate clauses.

With respect to choosing the form of the ‘answer,’ there is little autonomy in answering the first requirement of the question, which is to write down the initial condition associated with the curve, beyond possibly whether they present the answer in symbolic form as $y(0) = 1$ or in a form such as “when $x = 0$, $y = 1$”. However, in addressing the second requirement, in explaining the local slopes shown in terms of the differential equation, the students have some autonomy in the nature of the written description they produce, both in the wording they use and in the specific details they select from the diagram to focus on in writing their answer, as well as the nature of their explanation in relating these slopes back to the differential equation itself.

Figure 5.6 is a Commognitive analysis table that summarises the analysis of this task.

Table 5.3

Decisions for Solution of Q 1(d), MATH310 Examination, 2015

Choice taken	**Decision action**
$x = 0$	Recognise which value of x an initial condition occurs at.
$y = 1$, so $y(0) = 1$	Read off from the solution curve the value of y at $x = 0$ to determine the initial condition.
On the left side of the solution curve, the slopes are negative and getting steeper as y increases.	Determine and describe the nature of the slopes on the left side of the solution curve.
$y' = y - e^{-x}$ will become an increasingly large negative number for increasingly large negative values of x combined with increasingly large positive values of y.	Relate the description from the previous step to the differential equation.
On the right side of the solution curve, the slopes are positive and getting steeper as y increases.	Determine and describe the nature of the slopes on the right side of the solution curve.
$y' = y - e^{-x}$ will become an increasingly large positive number for increasingly large positive values of x combined with increasingly large positive values of y.	Relate the description from the previous step to the differential equation.
Horizontal local slope at (0,1) as this is at a stationary point on the curve.	Determine and describe the nature of the local slope at the turning point (0,1) of the solution curve.
When $x = 0$ and $y = 1$, the differential equation gives $y' = 1 - e^{-0} = 0$	Relate the description from the previous step to the differential equation.
	8 decisions

Figure 5.6

Commognitive Summary Table for Q1(d), MATH310 Examination, 2015

Q1(d), MATH310 Final Exam, 2015			
Commognitive analysis		**Relating to technology**	**Positioning of the student based on how they have been taught in the course notes**
Keywords and symbols			
Specialisation	Discourse of differential equations including the word DE and the phrases *initial condition, particular solution* and *slope field*.	Discourse of direction fields	Students have been introduced to solution curves, direction fields and their interpretation in the context of first order DEs.
Objectification of the discourse	Discourse of differential equations is objectified: mostly speaking of properties of DEs, although reference to “the initial condition which *generates* the particular solution curve” also suggests reference to the initial condition being involved in the production of the solution curve. Longest compound nominal group has length 7.		Similar examples in the course notes, which also have objectified discourse
Logical complexity	Relatively low: the conjunction "and” appears once and there are no disjunctions, implications, negations or quantifiers in the written discourse.		
Visual mediators			
Types of visual mediators and transitions between them	Symbolic visual mediator representing the DE being considered. Direction fields plot in the question. Transformations are needed between graphical and symbolic visual mediators in both directions, in answering the second requirement of the question.	Direction fields plot containing also solution curve. This is a narrative diagram as it indicates visually the rate at which y is changing with respect to x at different points and the dotted nature of the curve also suggests a process occurring over time, with it being superimposed on the existing slope field.	Examples of direction fields diagrams with solution curves and their corresponding algebraic DEs are provided in the course notes, together with CAS calculator instructions on how to produce a direction fields plot for a first-order DE.

Logical complexity	Moderate: The DE in the question stem contains 2 variables: x and y	Very high: CAS screenshot showing direction fields plot relates x and y, showing the slope y' at different points (x,y) which increases the complexity of the diagram. These slopes are also influenced by both the corresponding x and y values, further increasing the visual complexity, together with the inclusion of the solution curve for a particular initial condition being also present on the diagram.	The general interpretation of direction field plots is given in the course notes.
Routines			
*Type of routine expected (r*itual, recall, substantiation or construction)	The first part of the question invites a ritual routine in reading the initial solution off the graph. The second part requires a construction routine in producing endorsable narratives about the local slopes and how they relate to the equation of the DE.		The course notes do not provide any ritual step-by-step procedures for interpreting direction fields diagrams or linking them to the equations of particular DEs.
Endorsed Narratives			
Alienation of the discourse and how mathematics is construed	Alienated discourse. The DE, the local slopes and the initial condition are presented in the written discourse as atemporal objects. Agency is attributed to the initial condition in generating the particular solution curve.		
Subjectification of the discourse			
Student-author relationship	Student is given an imperative: to identify the initial condition that generates the solution curve shown and to explain the local slopes shown in terms of the DE.		

Student autonomy	In the first part of the question, a student could either read the value of y when $x = 0$ from the graph or substitute $x = 0$ into the DE to determine the value of y at that point. In the second part of the question, there are different ways and parts of the diagram the students could refer to, in describing the local slopes shown and how they relate to the equation of the DE. The form of the final answer is a written answer, specifying the initial condition and a descriptive part of the answer, relating the local slopes shown to the equation of the DE.	To successfully answer the question, the student must use the direction fields diagram (they do not have the choice to ignore this output generated by a CAS calculator). Use of an actual CAS calculator is permitted, but its Calculus functionality would not help in answering this question.	

5.2.5 Analysis of Identification of the Complementary Function of a DE Item in a CAS-Inactive Test

This example (see Figure 5.7) is from the same third-year Differential Equations and Mechanics mid-semester test paper from 2017 as the question in section 5.2.3. This question was on a test where CAS calculators were not permitted. The output shown in the question stem is output produced when solving the second-order DE $y'' + 3y' + 2y = e^{-x}$ using the computer program Wolfram Alpha. The image in Figure 5.7 captures output that appears when solving the DE using Wolfram Alpha on a computer or other internet-connected device.

5.2.5.1 Vocabulary and syntax

Specialisation: The question is a multiple-choice task, focusing on identifying the part of the general solution of a differential equation that corresponds to its *complementary function*. The task involves the discourse of differential equations and contains the specialised mathematical language of Calculus and differential equations, including the words DE and *complementary function* in the question stem. The question stem and answer options also include the specialised mathematical notation y' and y'' to represent derivatives of y and include the specialised mathematical function e^{-x}, which is an exponential function. The specialised mathematical functions $cosh(x)$ and $sinh(x)$ also appear as part of the Wolfram Alpha output that is present but actually are not required to answer the question.

Objectification of the discourse: The use of the specialised mathematical term *DE* (meaning "differential equation") encapsulates processes into the object "DE" as this term relates a function of one variable and its (first- and second-order) derivatives which, in turn, represent rates of change. The longest compound nominal group in the question stem has 3 words ("the complementary function"). Each answer option contains only a symbolic visual mediator, meaning that none of the answer options contain compound nominal groups. This is a relatively low-level of grammatical complexity. The composite (nested) function e^{-x} appears in both the question stem and in three of the answer options.

Logical complexity: The logical complexity in this question is relatively low compared with that of some subsequent examination questions to be analysed. There is one conjunction, 'and', in the question stem, but no disjunctions, implications, negations or quantifiers in the written discourse of the question.

5.2.5.2 Visual mediators

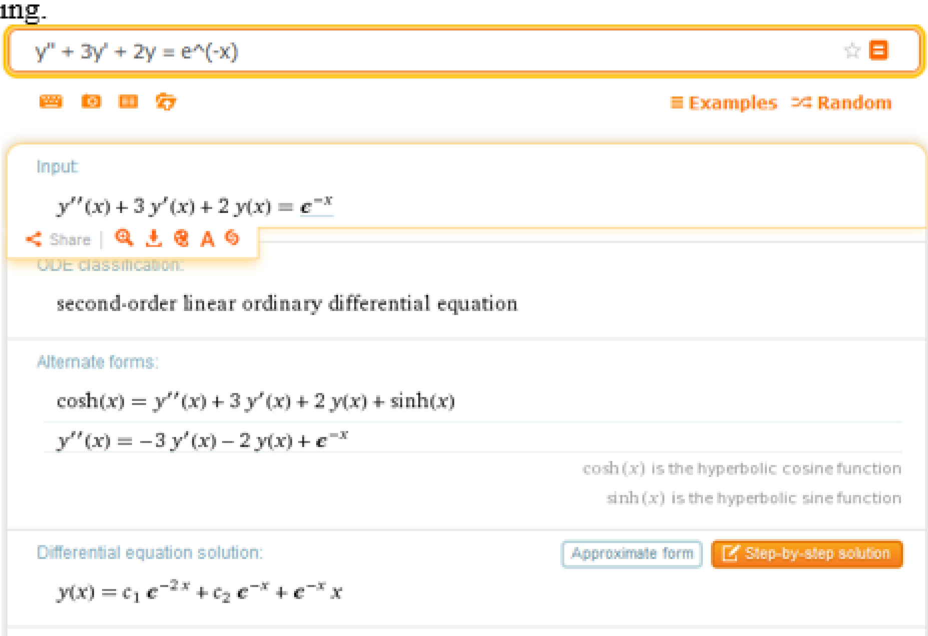
The question stem includes two visual mediators: (1) a symbolic visual mediator showing the differential equation itself and (2) a computer screenshot showing the algebraic representation of the differential equation being considered and its solution in Wolfram Alpha, as well as several intermediate lines of output also produced by the Wolfram Alpha software when it is instructed to solve the DE. There are no tables or graphs provided in the question. Every answer option is a symbolic visual mediator, in the form of algebraic expressions to be considered as the required complementary function of the DE. The actual algebraic expressions provided can again be viewed as either single entities or as sentences.

In addition, the individual lines of output provided in the Wolfram Alpha screenshot each provide mathematical information; but in this case only the first and the last lines of the computer output are actually required to help determine the correct answer. The first line indicates the DE which is being solved and the last line indicates the general solution of the DE, from which the required *complementary function* can be extracted.

Figure 5.7

Identification of the Complementary Function of a DE Item

9 The DE $y'' + 3y' + 2y = e^{-x}$ is placed in Wolfram Alpha and generates the output following.



From this we can see that the *complementary function* can be written

A Ae^{-x}
B $Ae^{-2x} + Be^{-x}$
C xe^{-x}
D $k^2 + 3k + 2 = 0$
E $y'' + 3y' + 2y = 0$

[Q9, MATH314 Test A]

Considering the original differential equation, its general solution and the multiple-choice answer options as sentences is appropriate here, as in turn these sentences are each comprised of symbolic entities. It is appropriate to think of them in this way as students need to consider these separate entities in order to correctly identify which part of the general solution in the Mathematica output is the required complementary function and to then recognise an equivalent form in the answer options provided.

Transitions between visual mediators: In this case there needs to be a transformation between **algebraic (symbolic) visual mediators and algebraic (symbolic) visual mediators** since the correct answer option is included that needs to be related to the

Wolfram Alpha output provided in the question stem that includes the DE being considered and its general solution.

Visual complexity: How visually complex students find the Mathematica output will depend on whether or not they are a knower of such outputs, as there are three intermediate lines of output that are not required to answer the question. Students studying this unit were being positioned as a *knower* of such outputs, including of which parts are relevant. Given the usual time pressure of tests and examinations, the expectation of the examiner is that the students would immediately proceed to the last line of the output, however the visual complexity becomes much greater for any students who are not a knower of such outputs, and who thus try to analyse the additional lines of output as well.

Logical complexity of visual mediators: The visual mediator, DE, in the question stem contains three types of operations: addition, multiplication and exponentiation. The question stem contains two algebraic letters, *x* and *y*, which represent the two variables being related in the DE. The visual mediator in the stem also contains two additional subscripted letters: c_1 and c_2, which represent arbitrary constants in the differential equation solution generated by Wolfram Alpha. Three of the answer options contain the letter *x* and one contains the letter *y*, where these again represent the variables associated with the DE. Another answer option includes the letter *k* which is sometimes used in the auxiliary equation associated with a DE (although in the course notes for this unit, the letter *r*, not *k*, was used in all the auxiliary equation examples). The letters *A* and *B* also appear in the correct answer option, where these represent arbitrary constants, consistent with how these letters have been shown in the section of the course notes relevant to solving second-order DEs. The logical complexity of this question is increased by the number of distinct letters which need to be considered, along with requiring an understanding of whether they represent variables (*x*, *y* here), arbitrary constants (c_1, c_2, A, B) or relate to equations other than the DE or its solution (*k* here).

5.2.5.3 Routines

The students are expected to engage with a *ritual routine*, as they are required to identify the complementary function of the DE, which is taught as a ritual routine in this unit's course notes. This is because the students have been taught both how a DE of this type can be solved by-hand and how the required information can be extracted from Wolfram Alpha output, in examples that are similar to this question. I will elaborate more on this aspect in the sub-section on student autonomy.

Instructions given regarding the procedure of the routine: As with other questions in this test paper, having access to a ritual routine does not preclude other approaches, so students are allowed to choose the procedure of the routine to apply. They must then write down the letter corresponding to the correct answer option on the first page of the test paper, in a box provided.

5.2.5.4 Endorsed narratives

Alienation of the discourse/ The origin of mathematical knowledge: The object in this question is the DE, which is initially presented as being in a relationship in which the agency is obscured, by the use of passive voice "*DE* $y'' + 3y' + 3y' = e^{-x}$ *is placed* in Wolfram Alpha" but which then is described as being an agent in a material process as it "*generates* the output following". In other words, in this case the mathematical discourse appears to be alienated, in construing the DE as generating the Wolfram Alpha output, as opposed to it being generated as a result of human activity. However, the wording also could suggest that, although not explicitly stated, the output was originally generated by a human agent *placing* the DE in Wolfram Alpha.

5.2.5.5 Subjectifying aspects of the discourse

Student-author relationship: Unlike the previous three questions discussed, the inclusive personal pronoun *we* is included in the final sentence of the question stem: *"we* can see that the complementary function can be written", suggesting a more personal student-author (examiner) relationship in this case, although the student is again given an imperative to choose the correct answer option.

Student autonomy

Designing the path to follow: There are a few different approaches that students could have taken in determining the correct answer for this question. The first approach, summarised in Table 5.4, is the one that we would expect students to use if they are knowers of how to interpret Wolfram Alpha output of the type provided in the visual mediator in the question stem. First, identify the general solution of the DE from the final line of Wolfram Alpha output. Second, extract the complementary function from that output. Finally, check through the answer options to recognise that answer option B is consistent with the complementary function shown in the Wolfram Alpha output, despite the arbitrary constants *A* and *B* used in that answer option being named differently from the c_1 and c_2 arbitrary constants generated by Wolfram Alpha. It can therefore be considered that the student is required to make three independent decisions, as shown in Table 5.4.

The second approach is one we might expect students to take if they feel more confident as a knower of multiple-choice exam technique and of the general expected form of a complementary function of a second order linear DE, rather than of Wolfram Alpha screens like the one provided. In such a case, the student could look at the five answer options provided and recognise that, as the DE is second-order, its complementary function should have two arbitrary constants and be in terms of the independent variable *x*. They would then correctly choose answer option B, as the only option that meets these requirements. This decision process is shown in Table D2 (see Appendix D).

Table 5.4

Decisions for Solution of Q9, MATH314 Test A, 2017

Choice taken	Decision action
$y(x) = c_1e^{-2x} + c_2e^{-x} + e^{-x}x$	Locate the general solution of the DE (in the last line of the Wolfram Alpha output).
$c_1e^{-2x} + c_2e^{-x}$	Identify the part of the above solution that is the complementary function
$Ae^{-2x} + Be^{-x}$	Match to the correct answer option, noticing the constants are named differently in the answer
	3 decisions

Finally, a third approach students could take if they are not knowers of the Wolfram Alpha output would be to work out the answer by-hand, as CAS calculators were not permitted in this test. To do this, they could first determine the auxiliary equation $r^2 + 3r + 2 = 0$ associated with the DE, then solve it for *r* (using factorisation or the quadratic formula). Based on the nature of the resulting solutions (roots of the auxiliary equation), they could then determine the correct form of the complementary function by using the associated formula taught in the unit. As shown in Table D3 (see Appendix D), this solution process would require 4 decisions, which is greater than the number required when using the

Wolfram Alpha output effectively. Therefore, the solution processes shown in Tables 5.5 and D2 have the minimal number of independent decisions required and the grain size of the task is 3.

In terms of CAS technology use, in this question students have a choice to not use output produced by the Wolfram Alpha, either by using multiple-choice examination techniques as illustrated in Table D2, or by using a by-hand routine they had been taught in class, as shown in Table 5.4. However, the latter would require an additional decision to be made, in determining the correct answer.

To interpret the task, the longest sentence in the stem of the question has length 13 (including 12 words and the differential equation object). Each answer option is an algebraic equation (symbolic visual mediator) with no additional written text (words).

With respect to choosing the form of the 'answer', the student is required to write down the letter corresponding to the correct answer option (in a space provided on the first page of examination paper). There is thus no autonomy in this respect.

Figure 5.8 is a Commognitive analysis summary table that summaries the analysis of this task.

5.2.6 Analysis of Interpretation of a CAS Output DE Screenshot Item in CAS-Inactive Mid-semester Test

The example in Figure 5.9 comes from the same third-year Differential Equations and Mechanics mid-semester test paper from 2017 as two of the previous four questions we have just considered, where CAS calculators are not permitted in the test. The image shown in the question stem captures what is shown on the screen when solving the second-order DE $y'' - 4y = e^{2x}$ using a TI-Nspire CX CAS calculator.

5.2.6.1 Vocabulary and syntax

Specialisation: The above question is a multiple-choice task, focusing on finding the correct interpretation for part of the output shown on a CAS calculator screen. The task involves the discourse of differential equations and contains the specialised mathematical language of Calculus and differential equations, including the word *DE* and the specialised mathematical notation y' and y'' in the question stem. The answer options include additional specialised mathematical language associated with DEs: including *complementary function, particular integral* and *general solution*.

Objectification of the discourse: The use of the specialised mathematical term *DE* (meaning "differential equation") encapsulates processes into the object "DE" as this term relates a function of one variable and its (first- and second- order) derivatives. As in the previous question discussed, the longest compound nominal group in the question stem has 3 words ("the complementary function"). Each answer option contains both a symbolic visual mediator and written text. The longest compound nominal group in any of the answer options is *the complementary function for this problem*, which has length 6. Overall, this indicates a slight increase in grammatical complexity of the answer options when compared to the previous question, as the result of the inclusion of both written text for interpretation and visual mediators which need to be related back to the CAS screen output provided. At least one of the composite (nested) exponential functions e^{2x}*and* e^{-2x} appears in each of the answer options.

Figure 5.8

Commognitive Summary Table for Q9, MATH314 Test A, 2017

Q9, MATH314 Test A, 2017			
Commognitive analysis		**Relating to technology**	**Positioning of the student based on how they have been taught in the course notes**
Keywords and symbols			
Specialisation	Discourse of differential equations, including the specialised words DE, complementary function.	Discourse of classifying DEs.	Students have been taught about the complementary function in the course notes, and how it relates to the general solution of a DE.
Objectification of the discourse	Discourse of differential equations is objectified: speaking of properties of DEs (as opposed to processes involving them). Longest compound nominal group has length 3.		Similar examples in the course notes which also have objectified discourse.
Logical complexity	Relatively low: no conjunctions, disjunctions, implications, negations or quantifiers in the written discourse.		

Visual mediators			
Types of visual mediators and transitions between them	Symbolic visual mediators showing equation of DE, its general solution and other forms of the DE.	Wolfram Alpha screen output showing the equation of the DE, some of its written properties, other forms of the DE and its general solution.	The same type of notation and presentation is used for the differential equations
Logical complexity	DE in question stem and the Wolfram Alpha output contain 2 variables: x and y. The Wolfram Alpha output also contains 2 constants c_1 and c_2, while the answer options include 2 differently named constants A and B. The student needs to recognise that c_1 and c_2 are equivalent to A and B, as arbitrary constants.	Level of complexity depends on whether students are a knower of Wolfram Alpha outputs- as if they are then they can ignore most of the lines of output, while otherwise they may 'waste' time trying to interpret most or all of the output.	DEs in the course notes are also often written in terms of x and y, with constants A and B also used in their solution. Students are shown examples of similar Wolfram Alpha output with constants c_1 and c_2.
Routines			
*Type of routine expected (r*itual, recall, substantiation or construction)	Ritual		Students are taught in the course notes how to relate Wolfram Alpha output of this type to the components of the solution of a second order DE.
Endorsed Narratives			
Alienation of the discourse and how mathematics is construed	The DE is described as a (non-human) agent that "*generates* the output following."		
Subjectification of the discourse			
Student-author relationship	Student is given an imperative: to choose the correct answer option. The inclusive personal pronoun "we" is included in the final sentence of the question stem.		
Student autonomy	The student can choose from a few different exam techniques in determining the correct answer. The form of the final answer is to write down the letter corresponding to the correct answer option.	The student can choose whether or not to use the Wolfram Alpha output provided to determine correct answer- but working the answer out by-hand without using this CAS output would be more time-consuming.	

Logical complexity: The logical complexity in this question is still relatively low compared with what will be seen in some subsequent examination questions. There are no conjunctions, disjunctions, implications, negations or quantifiers in the written discourse of this question.

5.2.6.2 Visual mediators

The question stem includes two visual mediators: (1) a symbolic visual mediator showing the differential equation itself and (2) a CAS calculator screenshot showing on the first line the calculator instruction that was entered to solve the differential equation being considered and on the next line the resulting general solution of the DE. There are no tables or graphs provided in the question. The answer options each include symbolic visual mediators as noted in the previous section, in the form of algebraic expressions related to part of the original DE, together with a possible written interpretation of the results. The actual algebraic expressions provided are either single entities or expressed as sentences.

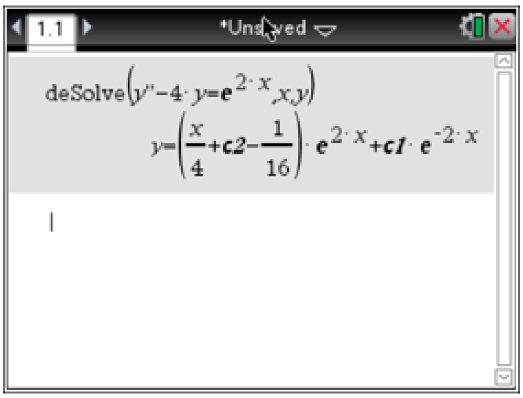
Figure 5.9

Interpretation of CAS DE Output

12 The DE

$$y'' - 4y = e^{2x}$$

was entered into a CAS calculator which gave the following output.



From this, we can see that (renaming constants),

A $Ae^{2x} + Be^{-2x}$ could be a general solution to the DE
B Ce^{2x} could be used to find a particular integral
C $\frac{x}{4}e^{2x}$ could be a particular integral
D The complementary function for this problem is $Axe^{2x} + Be^{-2x}$
E $Axe^{2x} + Be^{x}$ could be a general solution to the DE.

[Q12, MATH314 Test A, 2017]

Considering the original differential equation, its general solution and the multiple-choice answer options as sentences is appropriate here, where in turn these sentences are each comprised of symbolic entities. We can think of them in this way as students need to consider these separate entities in order to correctly identify the component parts of the differential equation (*complementary function and particular integral*) and to hence determine which of the answer options provides an equivalent form of that output, together with its correct interpretation.

Transitions between visual mediators: In this case there needs to be a transformation between **algebraic (symbolic) visual mediator** and **algebraic (symbolic) visual mediator,** since the correct answer option is included that needs to be related to the CAS calculator output provided in the question stem that includes the DE being considered and its general solution.

Visual complexity: How visually complex students find the CAS calculator output will depend on whether or not they are a knower of such outputs. Students studying in this unit are being positioned as a *knower* of such outputs, including of which parts are relevant. In one way, this question could be judged as having less complexity than the output for Q9, MATH314 Test A, 2017, as there are fewer lines of CAS output. As in Q9, MATH314 Test A, 2017, recognition that the output shows the complementary function plus a particular integral is important if using this output to determine the correct answer. But it is not immediately clear which part of the output will correspond to a correct answer option, as there are answer options involving the particular integral, or the complementary function, or the general solution, so potentially a deeper understanding and more effective use of the output is required than for Q9, MATH314 Test A, 2017 which was analysed in section 5.2.4. Furthermore, the term $(\frac{x}{4} + c2 - \frac{1}{16})e^{2x}$ has one part of the bracketed expression which is part of the complementary function, one part that is a particular integral for the DE and one part (the constant c_2) that could be included in either the complementary function or as part of the expression for a particular integral for the DE.

Logical complexity of visual mediators: The visual mediator (DE) in the question stem contains three types of arithmetic operations: subtraction, multiplication and exponentiation. The main statement in the question stem contains two algebraic letters, x and y, which represent the two variables being related in the DE. The visual mediator in the stem also contains two additional numbered letters: $c1$ and $c2$, which represent arbitrary constants. All of the answer options contain the letter *x*, which represents one of the variables associated with the DE. One of the answer options also includes the letter *C*, while three other answer options include the letters *A* and *B*, where these all represent arbitrary constants. This is consistent with how these letters have been shown in the section of the course notes relevant to solving second-order DEs. As in Q9, MATH314 Test A, 2017 analysed in section 5.4, the logical complexity of this question is increased by the number of distinct letters which need to be considered, along with requiring an understanding of whether they represent variables (*x*, *y* here) or arbitrary constants ($c1$, $c2$, *A*, *B* or *C* here).

5.2.6.3 Routines

The students are asked to engage with a ritual routine, as they are effectively required to identify the complementary function and particular integral from the general solution of the DE provided by the CAS calculator, in order to determine which of the answer options is correct. This was taught as a ritual routine in this unit's course notes, with students given examples of how this information can be extracted from CAS calculator output of this type. I will elaborate on this aspect in the subsection on student autonomy in the Subjectifying aspects of the discourse sub-section.

Instructions given regarding the procedure of the routine: As with the other questions on the MATH314 Test A, 2017 paper, having access to a ritual routine does not preclude other approaches, so students are allowed to choose the procedure of the routine to apply. They must then write down the letter corresponding to the correct answer option on the first page of the test paper in a box provided.

5.2.6.4 Endorsed narratives

Alienation of the discourse/ The origin of mathematical knowledge: The object in this question is the DE, which is initially presented as being entered into the CAS calculator by an unknown agent, in that the agency is obscured by the use of passive voice "*the DE was entered into a CAS calculator*". However, in practice it would be assumed the agent entering the DE is human. In turn, the CAS calculator is portrayed as a non-human agent in the material process of solving the DE as it *gave* the output that followed. In other words, in this case the mathematics involved in solving the DE is construed, through the use of alienated discourse, as being the result of the CAS calculator generating the solution to the DE, as opposed to the solution resulting from human actions.

5.2.6.5 Subjectifying aspects of the discourse

Student-author relationship: As in Q9, MATH314 Test A, 2017, the inclusive personal pronoun *we* is included in the final sentence of the question stem: *"we can see that* ", suggesting a more personal student-author (examiner) relationship in this case. The student is again given an imperative to choose the correct answer option.

Student autonomy

Designing the path to follow: There are a few different approaches that students could have taken in determining the correct answer for Q9, MATH314 Test A, 2017. The first approach, shown in Table 5.5, is the one that we would expect students to use if they are knowers of how to interpret CAS calculator output of the type provided in the visual mediator in the question stem. First, they would identify the general solution of the DE from the second line of the calculator screen output. Then, upon recognising that $c_2 - \frac{1}{16}$ is another arbitrary constant, they would rewrite it as, say, *A*. Next, the brackets in the expression $(\frac{x}{4} + c_2 - \frac{1}{16})e^{2x}$ should be expanded. The complementary function (which will include arbitrary constants) and the particular integral (which will be any remaining terms) can then easily be identified. Finally, they would check the answer options and recognise that answer option D is consistent with the complementary function shown in the calculator output, even though the arbitrary constants are named differently from the c_1 and c_2 arbitrary constants generated by the calculator. If using this solution method, students are required to make 6 independent decisions, as shown in Table 5.5.

If the student feels confident as a knower of multiple-choice examination technique they could, as in the example in section 5.2.5, consider the general form of the complementary function, recognising that, as the DE is second-order, it must contain two constants. However, in this case, they would also need to be familiar with the general form of a particular integral, including how it relates to the roots of the auxiliary equation, and with the form of the general solution of a DE, as there are also answer options corresponding to those possibilities. This would complicate the process of finding the correct answer option using this method, requiring a greater number of decisions. An implication is that this question effectively counteracts knowers of multiple-choice question techniques minimising decision making, through using a discourse of multiple-choice examinations that does not require knowledge of some of the mathematical discourse taught in the unit.

Finally, a third approach students could take if they are not knowers of the CAS calculator output would be to work out the answer by-hand, by first determining the auxiliary equation $r^2 - 4 = 0$ associated with the DE, then solving it for *r* (using factorisation or the quadratic formula), then determining the correct form of the complementary function by using ritual routines taught in the unit. They would then need to find a particular integral (which would also involve several decisions), and then match it with the correct answer option (C). This would require 10 decisions (see Table D4 in Appendix D), which is also more than in the solution path in Table 5.5 for solving the problem by using the CAS calculator output; therefore, the grain size of the task is 6.

Table 5.5

Decisions for Solution of Q12, MATH314 Test A, 2017

Choice taken	**Decision action**
$y = \left(\frac{x}{4} + c_2 - \frac{1}{16}\right) e^{2x} + c_1 e^{-2x}$	Identify the general solution of the DE from the calculator output
$y = \left(\frac{x}{4} + A\right) e^{2x} + c_1 e^{-2x}$	Rename $c_2 - \frac{1}{16}$ as a single constant (A would be the best choice)
$y = \frac{x}{4} e^{2x} + Ae^{2x} + c_1 e^{-2x}$	Expand the brackets
$y = \frac{x}{4} e^{2x} + Ae^{2x} + Be^{-2x}$	Rename the second constant as B
Complementary function is $Ae^{2x} + Be^{-2x}$, particular integral is $\frac{x}{4} e^{2x}$	Split into the complementary function and particular integral
Option C: particular integral is $\frac{x}{4} e^{2x}$	Match up with the correct answer option
	6 decisions

In terms of CAS technology use, in this question, students have a choice to not use output produced by the CAS calculator, but as described above, not using it would make the process of finding the correct answer slower, as use of CAS calculators was not permitted in this test.

To interpret the task, the longest sentence in the stem of the question has length 14 (including 13 words and the differential equation object). None of the answer options have sentences as long but they are all several words in length.

With respect to choosing the form of the 'answer', the student is required to write down the letter corresponding to the correct answer option in a space provided on the first page of examination paper. Therefore, there is no autonomy regarding this.

Figure 5.10 is a Commognitive analysis table that summaries the analysis of this task.

5.2.7 Analysis of a Context-separable Task on a CAS-Active Mid-semester Test

The example in Figure 5.11 is from a first-year Engineering Mathematics (Product Design) mid-semester test paper from 2010, where CAS calculators were permitted in the mid-semester test.

Figure 5.10

Commognitive Summary Table for Q12, MATH314 Test A, 2017

Q12, MATH314 Test A, 2017			
Commognitive analysis		**Relating to technology**	**Positioning of the student based on how they have been taught in the course notes**
Keywords and symbols			
Specialisation	Discourse of differential equations, including the specialised words DE, complementary function and particular integral.	Technical command *deSolve* for solving DEs on the CAS calculator.	Students have been taught about the complementary function and particular integrals in the course notes, and how they relate to the general solution of a DE
Objectification of the discourse	Discourse of differential equations is objectified: speaking of properties of DEs (as opposed to processes involving them). Longest compound nominal group length is 6.		Similar examples in the course notes which also have objectified discourse.
Logical complexity	Relatively low: no conjunctions, disjunctions, implications, negations or quantifiers in discourse.		

Visual mediators			
Types of visual mediators and transitions between them	Symbolic visual mediators in question stem showing equation of the DE , CAS calculator screenshot. Answer options include symbolic visual mediators.	Wolfram Alpha screen output showing the equation of the DE, some of its written properties, other forms of the DE and its general solution.	The same type of notation and presentation is used for the differential equations
Logical complexity	Slightly higher than in previous question. DE in question stem and the CAS output contains 2 variables: x and y. The calculator output also contains 2 constants c_1 and c_2, while the answer options include 3 differently named constants: A, B and C. Students need to recognise that $c_1$and $c_2 - \frac{1}{16}$are equivalent to A and B as arbitrary constants.	Level of complexity depends on whether students are a knower of CAS calculator outputs for solving DEs- as if they are then they should recognise the connection the constants shown in the CAS calculator output and those in the correct answer option.	DEs in the course notes are also often written in terms of x and y, with constants A and B also used in their solution. Students are shown examples of similar CAS calculator output with constants labelled and written in a similar form.
Routines			
*Type of routine expected (*ritual, recall, substantiation or construction)	Ritual.		Students are taught in the course notes how to relate CAS calculator output of this type to the components of the solution of a second order DE.
Endorsed Narratives			
Alienation of the discourse and how mathematics is construed	The CAS calculator is portrayed as a non-human agent in the process of solving the DE.		
Subjectification of the discourse			
Student-author relationship	Student is given an imperative: to choose the correct answer option. The inclusive personal pronoun “we” is included in the final sentence of the question stem.		
Student autonomy	The student can choose from different exam techniques in determining the correct answer. The form of the final answer is to write down the letter corresponding to the correct answer option.	The student can choose whether or not to use CAS calculator output provided to determine the correct answer- but working the answer out by-hand without using this CAS output would be more time-consuming.	

This question has both greater grammatical complexity and greater algebraic complexity than the previous examples shown. Because it is assumed students will use their CAS calculator to differentiate the appropriate function/s, this has allowed the examiner to set a question which is in other ways more complex, as we will see in the analysis that follows.

5.2.7.1 Vocabulary and syntax

Specialisation: This question is a multiple-choice task, focusing on interpreting a written scenario in the form of a worded problem, which requires differentiation of the provided function and then using the result to determine the correct answer option. The task involves the discourse of Calculus and the question stem contains the specialised mathematical language of science. Specialised phrases such as “rate of the temperature variation”, “altitude variation” and specialised words such as “elevation” are used. Because this is an applied mathematical problem, the question stem does not explicitly instruct students to use differentiation, even though the required answer involves combining and differentiating the functions provided, to find the derivative of air temperature T with respect to time t. However, the requirement to find this derivative is clearly telegraphed in the answer options. The question contains extra-mathematical content, with a real world object (hot air balloon) and a person (a balloon passenger) named. However, the depth of engagement with the context is very low, making this question what Galbraith and Stillman (2001) would classify as a context-separable problem, with the hot air balloon and its passenger playing no real role in the solution and able to just be stripped away from the mathematical question.

Four of the answer options are symbolic visual mediators, representing possible algebraic expressions for the required derivative and these options also include the specialised mathematical notation $\frac{dT}{dt}$ to represent the derivative of temperature T with respect to time t and the specialised mathematical function e^{-kz}, which is an exponential function, as well as sine and cosine functions. The remaining answer option is “none of the above”, which could be seen as inviting the student to attempt the question rather than just looking at the answer options, in case the correct algebraic answer is not given.

Figure 5.11

A Context-separable Mathematising Task

7. The air temperature T depends on the time of day t and the elevation above the sea level z as $T = T_0(1 + \cos(\pi t/_{12}))e^{-kz}$ (T_0 and k are constant). A hot air balloon rises vertically up so that its altitude changes as $z = \sin(vt)$ (v is constant). Therefore the rate of the temperature variation observed by a balloon passenger is

(A) $\frac{dT}{dt} = -\frac{\pi}{12} T_0 e^{-kz} sin\frac{\pi t}{12}$

(B) $\frac{dT}{dt} = -T_0 e^{-kz}(\frac{\pi}{12} sin\frac{\pi t}{12} + k(1 + cos\frac{\pi t}{12}))$

(C) $\frac{dT}{dt} = -T_0 e^{-kz}(\frac{\pi}{12} sin\frac{\pi t}{12} + kv\left(1 + cos\frac{\pi t}{12}\right)\cos vt)$

(D) $\frac{dT}{dt} = -T_0 e^{-kz}(\frac{\pi}{12} sin\frac{\pi t}{12} + kv\left(1 + cos\frac{\pi t}{12}\right)\sin vt)$

(E) none of the above

[Q7, HMS112P Test B, 2010]

Objectification of the discourse: The use of the specialised mathematical term, $\frac{dT}{dt}$, which is a symbolic visual mediator, encapsulates processes into the object, as in the context of this question, this term represents the rate of change of temperature *T* with respect to time *t*. The nature of the mathematisation of the discourse in this question itself differs from that in the others considered so far in this chapter, in that the question stem contains the description of a **process** taking place over time ("*a hot air balloon rises vertically up so that its altitude changes"*), rather than only containing **objectified** discourse. The grammatical complexity of the question stem is far greater than that of the answer options, with the longest compound nominal group in the question stem containing 10 words ("the rate of temperature variation observed by a balloon passenger"), while all answer options but one contain only a symbolic visual mediator. The total volume of text, the applied context of the problem and the number of variables and constants involved (which will be discussed further in the 'logical complexity' sub-section below) together indicate that a greater amount of "unpacking" is required to extract the mathematical information that is necessary to answer this question, compared to the other questions analysed so far in this chapter.

Logical complexity: The logical complexity in this question is greater than that seen in the previous questions we have considered. There is one conjunction 'and' and one implication 'therefore' in the question stem, but no disjunctions, negations or quantifiers in the written discourse of this question. The composite (nested) function $T = T_0\left(1 + \cos\left(\frac{\pi t}{12}\right)\right)e^{-kz}$ appears in the question stem, as does the composite function $z = \sin(vt)$. These two functions must be combined either by substituting $\sin(vt)$ for z in the first function or by using a suitable chain rule for differentiation when answering the question, which relates both of the above functions and their derivatives. However, the question does not explicitly telegraph to students this logical connection between these the two functions.

Four of the five answer options are also composite (nested) functions. To determine which one corresponds to the correct answer again requires an understanding of the connection between the functions $T = T_0\left(1 + \cos\left(\frac{\pi t}{12}\right)\right)e^{-kz}$ and $z = \sin(vt)$. These composite functions will be discussed further in the next sub-section.

5.2.7.2 Visual mediators

The question stem includes two symbolic visual mediators: the equations for the functions T and z described in the previous sub-section on logical complexity. Four of the five answer options are symbolic visual mediators, in the form of algebraic expressions to consider, to determine which one (if any) is the required derivative, $\frac{dT}{dt}$. The actual algebraic expressions provided can again be considered as either single entities or as sentences.

Having access to a CAS calculator could lead some students to treat the two algebraic expressions presented in the question stem as single entities, to be entered directly into the calculator for differentiation or, at most, to consider the expression for z to be a single entity that can be substituted into the expression for T. To successfully answer the question with assistance from a CAS calculator, a student needs to recognise that in the discourse of functions, T, t and z are variables, while v, T_0 and k are constants. This can be determined from the discourse in the question stem as it tells us that "[t]he air temperature T **depends on** the time of day t and the elevation above sea level z." A knower of the discourse of differentiation should realise it is communicating that T is a variable that is **dependent on** and **changing value based on** the values of the variables z and t. In the question stem, it is also explicitly stated that v, T_0 and k are constants.

Transitions between visual mediators: In this case there needs to be a transformation between **algebraic (symbolic) visual mediator** to **algebraic (symbolic) visual mediator** since the correct answer option is included that needs to be related to the symbolic visual mediators provided in the question stem, together with the context of the scenario being considered.

Logical complexity of visual mediators: The visual mediators in the question stem and answer options contain up to five types of operations: addition, subtraction, multiplication, division and exponentiation. The question stem and the answer options contain up to three algebraic letters which represent variables: T, t and z, where the student is required to find the rate of change of T with respect to t. The visual mediators in the stem also contain two additional letters: v and k, which represent constants, as well as a subscripted letter, T_0, which is a constant. The logical complexity of this question is increased by the number of distinct letters to be considered, along with requiring an understanding of whether they represent variables (T, t and z), or constants (T_0, v and z). If the equations in answer options A to D are viewed as sentences containing separate entities, we can also see that these symbolic visual mediators have greater logical complexity than we have seen in the other examples discussed in this chapter. There are nested brackets with two levels of nesting in three of these answer options, up to three types of specialised mathematical functions included (exponential, sine and cosine) in individual answer options, and a greater total number of symbols to consider in relating the answer options back to the answer obtained when working out the derivative using a CAS calculator (or by-hand).

5.2.7.3 Routines

The students are asked to engage with a *ritual routine*, as they are required to find the derivative of a function (and to possibly use a chain rule, depending on how they choose to answer the question). Differentiation is taught as a ritual routine in this unit’s course notes, in teaching students how this can be done by-hand, and in classes, how this can be done on a CAS calculator.

Instructions given regarding the procedure of the routine: Having access to a ritual routine does not preclude other approaches, and students sitting this examination were allowed to choose the procedure of the routine to apply. They were then required to write down the letter corresponding to the correct answer option on the first page of the test paper in a box provided.

5.2.7.4 Endorsed narratives

Alienation of the discourse/ The origin of mathematical knowledge: Unlike the other examination questions considered so far in this chapter, of which the only one in section 5.2.5 suggested human agency, and not explicitly, here human presence is evident in a non-mathematical process when the final sentence in the question stem asks for "the rate of temperature variation observed by a balloon passenger," That is, the discourse in this question is not fully alienated, although the previous sentence about the hot air balloon rising includes alienated discourse, as it does not attribute the cause of the balloon rising to any human agent. The question also describes relational processes, when relating the air temperature T to the other variables.

5.2.7.5 Subjectifying aspects of the discourse

Student-author relationship: There are no inclusive personal pronouns in the question or answer options. The examiner is presented as an impersonal authority, with the given an imperative to choose the correct answer option.

Student autonomy: There are at least two approaches that students could have taken in determining the correct answer for this question, using their CAS calculator to help them obtain the answer, both of which relate to what the students had been taught in the unit.

Designing the path to follow: The first approach (see Table 5.6) is the one that students would be expected to use if they are knowers of differentiation of a function of one variable and of relating two associated functions to each other. First, substitute $z = \sin(vt)$ into $T = T_0\left(1+\cos\left(\frac{\pi t}{12}\right)\right)e^{-kz}$ to obtain $T = T_0\left(1+\cos\left(\frac{\pi t}{12}\right)\right)e^{-ksin(vt)}$. This then gives T as a function of t only, so that this expression could be entered into the CAS calculator and immediately differentiated with respect to t. For example, this can be done using the TI-NSpire CAS calculator's **derivative** command. Note that the constant T_0 could be renamed with another letter when doing this on the calculator. The TI-NSpire then gives the result for $\frac{dT}{dt}$ (if written in terms of T_0 again) as:

$$\left(-k.T_0.\left(\cos\left(\frac{\pi.t}{12}\right)+1\right).\cos(t.v).v-\frac{T_0.\pi.\sin\left(\frac{\pi.t}{12}\right)}{12}\right).e^{-ksin(t.v)}$$

While this is a correct answer for $\frac{dT}{dt}$, it appears to be different from any of the answer options provided. The actual equivalent answer option is C, but to obtain it several further steps need to be taken. Firstly, $e^{-ksin(t.v)}$ needs to be recognised as being equal to e^{-kz} again. Secondly, $-T_0e^{-kz}$ needs to be taken out as a common factor. Thirdly, $t.v$ must be recognised as vt, and the students also need to recognise that the terms that are multiplied by $-T_0e^{-kz}$ are the same as in C, but are shown in a different order.

The second approach is one might expect students to take if they feel more confident as a knower of chain rules for functions of two variables. In this approach, as a first step, they would need to identify that in the original equation $T = T_0\left(1 + \cos\left(\frac{\pi t}{12}\right)\right)e^{-kz}$, *T* is a function of *t* and *z*, while in the equation $z = \sin\,(vt)$, z is a function of t.

Next, they must identify that in this situation the chain rule $\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial z}.\frac{dz}{dt}$ can be used. The students could then work out each of these derivatives, substitute them into the above formula and simplify to match the correct answer option. This solution process requires more decisions than the process in Table 5.6, with 1 decision required to identify the required chain rule, 3 decisions to obtain the required derivatives $\frac{\partial T}{\partial t}$, $\frac{\partial T}{\partial z}$ and $\frac{dz}{dt}$, 1 decision to combine the results to get $\frac{dT}{dt}$, at least 3 decisions to take out the required common factors and rewrite the expression in the required form, followed by a final decision in matching the result with the correct answer option. That is, at least 9 decisions, which is more than the 8 required if using the solution path in Table 5.6; therefore, the grain size of this task is 8.

In terms of CAS technology use, in this question, students have a choice whether or not to use their CAS calculator for the required differentiation. However, not using the calculator would require a greater number of decisions to obtain the required answer, as a result of the complexity of the function/s to be differentiated, requiring use of the chain rule for differentiation and also, depending on the method used, the product rule for differentiation.

To interpret the task, the longest sentence in the stem of the question has length 27 (including 23 words, three letters and the equation object representing T). The four answer options are algebraic equations (symbolic visual mediators) with no additional written text (words), while the fifth answer option contains 4 words.

With respect to choosing the form of the 'answer', the student is required to write down the letter corresponding to the correct answer option in a space provided on the first page of examination paper. There is thus no autonomy in this respect.

Figure 5.13 is a Commognitive analysis summary table that summaries the analysis of this task.

5.2.8 Analysis of a Maclaurin Polynomial on a CAS-Active Examination Paper

The example in Figure 5.12 was on a first-year Functions and Calculus final examination paper from 2015, where CAS calculators were permitted. The researcher also has the class notes from the MATH104 unit.

Table 5.6

Decisions for Solution of Q7, HMS112P Test B, 2010

Result obtained	**Decision action**
$T = T_0\left(1+\cos\left(\frac{\pi t}{12}\right)\right)e^{-ksin(vt)}$	Substitute $z = \sin(vt)$ into $T = T_0\left(1+\cos\left(\frac{\pi t}{12}\right)\right)e^{-kz}$ to get T as a function of *t* only.
$\frac{d}{dt}(T_0.\left(1+\cos\left(\frac{\pi.t}{12}\right)\right)e^{-ksin(t.v)}))$	Identify that require derivative of T with respect to t, which will require use of the **derivative** command next
$\left(-k.T_0.\left(\cos\left(\frac{\pi.t}{12}\right)+1\right).\cos(t.v).v - \frac{T_0.\pi.\sin\left(\frac{\pi.t}{12}\right)}{12}\right).e^{-ksin(t.v)}$	Use the command $\frac{d}{dt}(T_0.\left(1+\cos\left(\frac{\pi.t}{12}\right)\right)e^{-ksin(t.v)}))$ to find $\frac{dT}{dt}$ using the CAS calculator.
$-T_0\left(k\left(\cos\left(\frac{\pi.t}{12}\right)+1\right).\cos(t.v).v + \frac{\pi.\sin\left(\frac{\pi.t}{12}\right)}{12}\right).e^{-ksin(t.v)}$	Take out $-T_0$ as a common factor.
$-T_0\left(k\left(\cos\left(\frac{\pi.t}{12}\right)+1\right).\cos(t.v).v + \frac{\pi.\sin\left(\frac{\pi.t}{12}\right)}{12}\right)e^{-kz}$	Rewrite $e^{-ksin(t.v)}$ as e^{-kz}
$-T_0e^{-kz}\left(k\left(\cos\left(\frac{\pi.t}{12}\right)+1\right).\cos(t.v).v + \frac{\pi.\sin\left(\frac{\pi.t}{12}\right)}{12}\right)$	Move e^{-kz} to the left side of the brackets so that $-T_0e^{-kz}$ is a common factor to the left side of the brackets
$-T_0e^{-kz}\left(kv\left(\cos\left(\frac{\pi.t}{12}\right)+1\right).\cos(t.v) + \frac{\pi.\sin\left(\frac{\pi.t}{12}\right)}{12}\right)$	Move v to the right of the bracketed expression $k\left(\cos\left(\frac{\pi.t}{12}\right)+1\right).\cos(t.v)$
Answer option C.	Check answer options, match to
	8 decisions

Figure 5.12

Maclaurin Polynomial on a CAS-active Examination Paper for a CAS Teaching/learning Environment

1 Write down the *Maclaurin polynomial* of order 4 which approximates the function

$$f(x) = x \sin x$$

[Q1, MATH104 Final examination, 2015]

Figure 5.13

Commognitive Summary Table for Q7, HMS112P Test B, 2010

Q7, HMS112P Test B, 2010			
Commognitive analysis		**Relating to technology**	**Positioning of the student based on how they have been taught in the course notes**
Keywords and symbols			
Specialisation	Discourse of Calculus and motion, including specialised phrases such as “rate of the temperature variation”, “altitude variation” and specialised words such as “elevation.”		Students have been taught how to differentiate functions involving constants also labelled by letters, but examples in the course notes are typically less complex than this example.
Objectification of the discourse	Discourse contains description of a **process** taking place over time (“*a hot air balloon raises vertically up so that its altitude changes”*), rather than only containing **objectified** discourse. Longest compound nominal group has length 10.		
Logical complexity	There is one conjunction: ‘and’ in the question stem but no disjunctions, implications, negations or quantifiers.		

Visual mediators			
Types of visual mediators and transitions between them	The question stem includes symbolic visual mediators: the equations for the functions T and z. Most answer options are also symbolic visual mediators. Transformations are needed between the two symbolic mediators in the main question (they need to be combined appropriately) and also between the required derivative and the answer options.	Use of a CAS calculator (which is permitted) can produce the required derivative, which in itself is also a symbolic visual mediator.	Students have been taught in classes how to differentiate functions of this type on the calculator, but usually functions that do not involve as many constants with different letter names.
Logical complexity	Very high. There are three variables: T, t and z, with the students being required to find the rate of change of T with respect to t. There are also three constants: v, k, and T_0. Students need to not only understand which letters represent variables or constants but also that differentiation of T with respect to t is required.	Entering the required expressions into the calculator for differentiation is relatively complex as students need to enter or rename T_0 and to recognise that with the calculator used in this unit, a multiplication sign must be included between any two letters, so it recognises them as distinct variables or constants.	
Routines			
*Type of routine expected (r*itual, recall, substantiation or construction)	Ritual.		Students taught in the course notes how to differentiate functions of this type.
Endorsed Narratives			
Alienation of the discourse and how mathematics is construed	Here human presence is evident in a non-mathematical process when the final sentence in the question stem asks for “the rate of temperature variation observed by a balloon passenger.”		
Subjectification of the discourse			
Student-author relationship	The student is given an imperative to choose the correct answer option.		

	There are no inclusive personal pronouns in the question or answer options.		
Student autonomy	The student can choose from a few different methods techniques in determining the correct answer. The form of the final answer is to write down the letter corresponding to the correct answer option.	The student can choose whether or not to use the CAS calculator output provided to determine the correct answer- but working the answer out by-hand without using this CAS output would be much more time-consuming.	

5.2.8.1 Vocabulary and syntax

Specialisation: The question is in short answer format, and requires students to produce a Maclaurin polynomial of order 4. The question involves the discourse of Calculus and contains the specialised mathematical language of Calculus, including the specialised phrases, *Maclaurin polynomial* and *of order 4*, and specialised general mathematical language including the words *approximates* and *function*. There is also a symbolic visual mediator $f(x) = xsin(x)$ which includes the specialised mathematical function sin(x).

Objectification of the discourse

The discourse is objectified, with the mathematical objects referred to including the Maclaurin polynomial and the function $f(x) = xsin(x)$. Although the question includes only one sentence, it contains a relatively long compound nominal group "the Maclaurin polynomial of order 4 which approximates the function $f(x) = xsin(x)$" (of length 11), which increases the grammatical complexity overall.

Logical complexity: There are no conjunctions, disjunctions, implications, negations or quantifiers in the written discourse of this question. Logical complexity of the visual mediators will now be described in the next section.

5.2.8.2 Visual mediators

The question contains one symbolic visual mediator: the equation $f(x) = xsin(x)$. This equation can be viewed as either a single entity or a sentence. The logical complexity of $f(x) = xsin(x)$ for a student depends on their solution approach. If they approach the question by-hand, they need to recognise the right-hand side of the function is a product of two functions of *x* to correctly apply the product rule for differentiation to help in answering the question. Alternatively, if students use their CAS calculators to answer this question, they could treat the right-hand side of the function as a single entity when entering it into the calculator (entering it either to immediately find the Maclaurin polynomial, or to find the required derivatives). A detailed description of both these methods using a CAS calculator is provided in the student autonomy sub-section.

Transitions between visual mediators: In this case there is only one (symbolic) visual mediator present: the function $f(x) = xsin(x)$ so there is no demand for multiple modes of communicating equivalent information.

Logical complexity of visual mediators: The visual mediator $f(x) = xsin(x)$ contains only one operation: multiplication. The question contains only one algebraic letter, *x*, which represents a variable and the letter *f* is used to represent the function being considered.

5.2.8.3 Routines

The students are asked to engage with a *ritual routine,* as in the unit they were taught how to find Maclaurin polynomials of any order as a ritual routine procedure, both with and without the use of a CAS calculator. I will elaborate more on this aspect in student autonomy in the Subjectifying aspects of the discourse sub-section.

Instructions given regarding the procedure of the routine: As with the other questions in chapter, having access to a ritual routine does not preclude other approaches, as students are allowed to choose the procedure of the routine to apply. They are required to write down their final answer and any intermediate working in the space provided below the question. Students were advised in this and other short-answer questions to show their working but also were informed in classes that they could still obtain full marks if they merely wrote down the final answer for a question of this type, provided the answer was fully correct.

5.2.8.4 Endorsed narratives

Alienation of the discourse/ the origin of mathematical knowledge: The object in this question is the function $f(x) = xsin(x)$. Students are then required to find "*the Maclaurin polynomial of order 4 which approximates the function*", suggesting the required

Maclaurin polynomial is an agent in a material process in which it approximates the original function, with the discourse being alienated from any human actions.

5.2.8.5 Subjectifying aspects of the discourse

Student-author relationship: There are no inclusive personal pronouns in the question or answer options, suggesting an impersonal relationship between the student and the examiner. The student is given an imperative: to *write down* the required Maclaurin polynomial, with the instruction “write down” suggesting the student is being “instructed to engage in ‘scribbling’ activities” (Morgan, 2016), without being expected to engage in any mathematical reflection or “thinking.” This is in contrast to the short-answer question analysed in section 5.2.4, where the instruction *explain* suggested the students were invited to engage in *thinking* activities (Morgan, 2016) to answer the question.

Student autonomy

Designing the path to follow: In responding to this examination question, there are at least three different approaches that students could have taken in determining the correct answer for question 1 using their CAS calculator to help them obtain the answer, based on what the students had been taught in the unit. These are:

1. CAS could be used to obtain the required Maclaurin polynomial of order 4 in one step, using the calculator’s inbuilt **Taylor** command (see Table 5.7)

2. CAS could be used to directly calculate the derivatives at $x = 0$ (without the need to determine them for general x first). The students would then substitute the results into the general formula for a Maclaurin polynomial of order 4 (see Table D5 in Appendix D).

3. CAS could be used to calculate the first, second, third and fourth order derivatives of $f(x)$, for general x. The students would then evaluate the function and its derivatives at $x = 0$ and substitute the results into the general formula for a Maclaurin polynomial of order 4 (see Table D6 in Appendix D).

The minimum number of decisions is 2, for the solution shown in Table 5.7; therefore, the grain size of the task is 2. Note also that the number of decisions required for the task would increase if the students were not permitted to use a CAS calculator, as working out the derivatives would then require repeated use of the *product rule* for differentiation, together with simplifying the results. In this question, students could choose not to use their CAS calculator for the required differentiation, but as stated above, not using the calculator would require a greater number of decisions to be made in the solution process, as the result of the complexity of the functions to be differentiated, requiring use of the product rule for differentiation and some simplification of the resulting derivatives.

Interpreting the task: The longest sentence in the question has length 13 (including 11 words, a numeral and the visual mediator representing the function being considered).

Choosing the form of the 'answer': The student is required to write their answer, including any intermediate working, in space provided below the question on the examination paper.

Table 5.7

Decisions for Solution of Q1, MATH104 final examination, 2015 if Use CAS Calculator (Method 3)

Result obtained	**Decision actions**
Awareness that will need to find a Taylor polynomial about $= 0$	Recognise that a Maclaurin polynomial is a Taylor polynomial expanded about $x = 0$
$p_4(x) = x^2 - \frac{x^4}{6}$	Use the taylor($xsin(x)$, x, 4, 0) command on the CAS calculator to find the required Maclaurin polynomial
	2 decisions

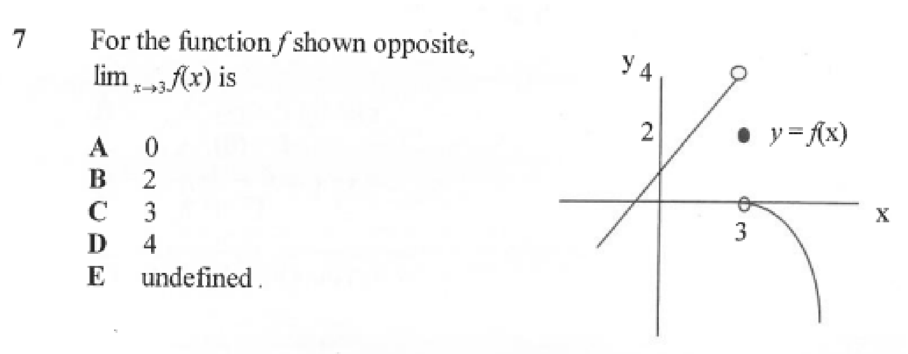
Figure 5.15 is a Commognitive analysis table that summaries the analysis of this task.

5.2.9 Analysis of a Limits Question with a Graphical Visual Mediator on a CAS-Inactive Mid-semester Test

This example comes from a first-year Functions and Calculus mid-semester test paper from 2017, where CAS calculators were not permitted. The diagram shown in this question stem (see Figure 5.14) shows a graph of a piecewise continuous function, where this image was generated by computer software, but not a CAS calculator or the Wolfram Alpha software used in the unit, so analysis of this question emphasises interpreting this *type of graph*, rather than specifically interpreting CAS output.

Figure 5.14

Limit Identification Function Question



[Q7, MATH104 Test B, 2017]

Figure 5.15

Commognitive Summary Table for Q1, MATH104 Final Examination, 2015

Q1, MATH104 exam, 2015			
Commognitive analysis		**Relating to technology**	**Positioning of the student based on how they have been taught in the course notes**
Keywords and symbols			
Specialisation	Discourse of Calculus, including the specialised the phrase *Maclaurin polynomial* and specialised general mathematical language including the words *approximates* and *function*.		Students are familiar with these terms from the course notes.
Objectification of the discourse	Discourse of Maclaurin functions and polynomials is objectified: speaking of properties of DEs (as opposed to processes involving them). Longest compound nominal group has length 11.		Similar examples in the course notes which also have objectified discourse.
Logical complexity	There are no conjunctions, disjunctions, implications, negations or quantifiers in the written discourse.		

Visual mediators			
Types of visual mediators and transitions between them	Symbolic visual mediator representing the function being considered.	The required Maclaurin polynomial and/or derivatives can be produced using the types of CAS calculators the students had access to.	The course notes show how to work out Taylor (and hence Maclaurin) polynomials on a CAS calculator and how to work out derivatives of functions.
Logical complexity	The question contains only one algebraic letter, x, which represent a variable and the letter f is used to represent the function being considered.		Functions in the course notes are frequently also of the form $f(x)$.
Routines			
*Type of routine expected (r*itual, recall, substantiation or construction)	Ritual.		In the course notes students are shown the ritual routine for producing Maclaurin polynomials by-hand and on the calculator.
Endorsed Narratives			
Alienation of the discourse and how mathematics is construed	The required Maclaurin polynomial is assigned the role of a non-human agent doing the approximation of the original function.		
Subjectification of the discourse			
Student-author relationship	Students are required to write down their final answer and any intermediate working in space provided below the question. There are no inclusive personal pronouns.		
Student autonomy	The student can choose from a few different techniques in determining the correct answer. The final answer and any intermediate working is required to be written in space provided below the question.	The student can choose whether or not to use the CAS calculator to help determine the correct answer- but working the answer out by-hand without using the calculator would be more time-consuming.	

5.2.9.1 Vocabulary and syntax

Specialisation: The question is a multiple-choice task, which requires students to select the correct answer regarding the limit as x approaches 3, of the function $f(x)$ shown, based on looking at the graph of $f(x)$ to determine if the limit of the function shown exists at $x = 3$ and, if so, what the value of the limit is. The task involves the discourse and specialised mathematical language of Calculus and Functions, including use of the word "function" and the symbolic visual mediator $\lim_{x \to 3} f(x)$ which includes the specialised mathematical abbreviated notation *lim* to represent the limit (as x approaches 3 in this case).

Objectification of the discourse: The discourse is objectified, and speaks of the specialised mathematical object $\lim_{x \to 3} f(x)$ and "the function *f*." The notation $\lim_{x \to 3} f(x)$ could be interpreted as an *objectified* narrative for the *value of* "the limit of" $f(x)$ as x approaches 3 or as an *operational* narrative for the *process of* taking a limit of $f(x)$ as x approaches 3. Which of these applies would depend on the context in which it is being used. Since the answer options for this question are possible *values* of the limit or that (a value of) it does not exist (which is actually the correct answer), in this case $\lim_{x \to 3} f(x)$ would be an *objectified* narrative for the *value* of this limit. The stem of the question only includes one written phrase, which is of relatively low grammatical complexity, with "the function *f*" being the longest and only compound nominal group, of length 3. The answer options are single numbers except for the final answer option, which is the single word "undefined."

Logical complexity: There are no conjunctions, disjunctions, implications, negations or quantifiers in the written discourse of this question.

5.2.9.2 Visual mediators

The question stem includes a visual mediator in the form of output from graphing software (but not from a CAS calculator or Wolfram Alpha, which students used in this unit). This graphical visual mediator is the graph of a piecewise continuous function, with two open circles signifying that the corresponding points on the graph are approached by the function as x approaches 3 from the left and from the right, but are not included as part of the function. This is a common way to represent such values in the graphical discourse of functions and limits. The five answer options do not contain any symbolic or graphical visual mediators, with four of the answer options a single number and the fifth one being the word "undefined."

Transitions between visual mediators: Transformations are needed between **graphical and algebraic (symbolic)** modes, since the correct answer option needs to be determined by associating the symbolic visual mediator $\lim_{x \to 3} f(x)$ with the information provided in the graphical visual mediator about the behaviour of the function f(x) near $x = 3$.

Logical complexity of visual mediators: In the symbolic visual mediator $\lim_{x \to 3} f(x)$ in the question stem, there are no arithmetic operations. There are two letters present, with f signifying the name of the function and x the independent variable. The limit notation increases the logical complexity of the symbolic visual mediator.

Visual complexity: The graph relates two variables, x and y, and shows the graph of a function $y = f(x)$. The discontinuities in the graph of $f(x)$, combined with the associated open circle present at $(3, 0)$ and the filled in circle representing a point at (3, 2) increase the visual complexity, in terms of the amount and variety of information that needs to be processed. However, this diagram has less visual complexity than the slope fields diagrams previously discussed (from Figures 5.3 and 5.5). The diagram in this question positions students to be a knower of discerning which part of the graph is most relevant to consider, in this case the values $f(x)$ approaches from the left and from the right at $x = 3$, with the point at (3, 2) a potential distractor as the actual value of $f(x)$ at $x = 3$ does not influence consideration of the existence or the value of the limit. This curve being shown with solid lines also suggests the graphical visual mediator provided as a whole could be considered to be a *conceptual diagram* (Alshwaikh, 2016, 2018; Kress & Van Leeuwen, 2006), as it does not represent any mathematical activity taking place over time and also does not record rates of change at any points (contrary to the slope fields diagrams in Figures 5.3 and 5.5). The diagram also does not potentially invite students to interact with it by drawing on it in the way the slope fields diagrams did, as here the correct answer can be determined by simply seeing that the function approaches different values from the left and the right of x = 3 and concluding therefore that $\lim_{x \to 3} f(x)$ is undefined (does not exist).

5.2.9.3 Routines

The students are asked to engage with a *ritual routine* in this task, as in this unit they were taught how to read piecewise continuous functions of this type, including being taught to check the left-hand limit as x approaches the value from below and the right-hand limit as x approaches the value from above, and, that if these are not equal, then the limit is undefined.

Instructions given regarding the procedure of the routine: Students are allowed to choose the procedure of the routine, but given they only have the graph provided to work with, there is only one clear way to approach solving the problem (reading off and comparing the left-hand and right-hand limits as x approaches 3). They must then write down the letter corresponding to the correct answer option.

5.2.9.4 Endorsed narratives

Alienation of the discourse/ The origin of mathematical knowledge: The mathematical object in the question text is the expression $\lim_{x \to 3} f(x)$. As outlined in section 5.2.9.1, in the context of this question, this expression is an objectified narrative for the value of the limit, meaning this expression does not suggest any relational or material processes taking place. Likewise, since the conceptual diagram for the graphical visual mediator for this question does not suggest any mathematical activity place through human actions, the discourse for this question is fully alienated and does not include any material actions by, or on, mathematical objects.

5.2.9.5 Subjectifying aspects of the discourse

Student-author (examiner) relationship: There are no inclusive personal pronouns in the question or answer options. The student is given an imperative to choose the correct answer option.

Student autonomy:

Designing the path to follow: In answering this question, given the information provided there is only one possible solution path for answering this question, with the corresponding decisions shown in Table 5.8.

1. Inspect the graph to see that when $x \to 3$ from below, $f(x) \to 4$.

2. Inspect the graph to see than when $x \to 3$ from above, $f(x) \to 0$.

3. Conclude that because these two values are not equal, the limit as x approaches 3 of $f(x)$ is undefined.

4. Select answer option E, which is that the limit is undefined.

This being the only solution path, we can conclude that the minimal number of steps required to solve the problem is 4 and therefore the grain size of the task is 4.

Table 5.8

Decisions for Solution of Q7, MATH104 Test B, 2017

Choice taken	**Decision action**
4	Determine the value $f(x)$ approaches as x approaches 3 from below
0	Determine the value $f(x)$ approaches as x approaches 3 from above
Undefined (as left and right limits not equal)	Draw a conclusion about whether or not the limit as x approaches 3 of $f(x)$ is defined
E (Undefined)	Select the correct answer option
	4 decisions

In terms of CAS technology use, use of a CAS calculator is not applicable to this question as answering it correctly relies solely on using the graphical visual mediator provided. This visual mediator has been produced by graphing software (other than a CAS calculator or Wolfram Alpha as used in the relevant unit).

To interpret the task, the longest (and only) sentence in the stem of the question has length 6 words and also includes the symbolic visual mediators f and $\lim_{x \to 3} f(x)$. Four of the five answer options are single numbers and the fifth answer option is the single word "undefined."

With respect to choosing the form of the 'answer', the student is required to write down the letter corresponding to the correct answer option (in a space provided on the first page of examination paper). There is thus no autonomy in this respect.

Figure 5.17 is a Commognitive analysis summary table that summaries the analysis of this task.

5.2.10 Analysis of Area between two Curves Item on a CAS-Active Examination Paper

The example in Figure 5.16 comes from the same first-year Functions and Calculus final examination paper from 2015, as the question in Figure 5.13, in which CAS calculators were permitted. In this case, unlike the other examples shown in this chapter, the students are required to draw the region bounded by the graphs of two curves (and to then find the area of this bounded region).

Figure 5.16

Area Between Two Curves Question

13 (a) Find the area of the region which is enclosed by the curve $y = x^3 - 2x^2 + 5$ and the line $y = x + 3$. (Draw this and identify intersections first)

[Q13 (a), MATH104 Final examination, semester 2 2015]

5.2.10.1 Vocabulary and syntax

Question 13(a), MATH104 Final examination, semester 2, 2015, is a written answer question, which requires drawing the region bounded by the two curves and finding the area of this region. The task involves the discourse and specialised mathematical language of Functions, including use of the words *area, region, enclosed, curve, line* and *intersections* as specialised terms, given the mathematical context of the question.

Logical complexity: In the text of the question there are two occurrences of the conjunction "*and*": one that associates the two curves being considered, while the other occurrence is part of the instruction to the students regarding two aspects of the information they need to produce as part of their answer.

5.2.10.2 Visual mediators

The question includes two symbolic visual mediators, in the form of the equations of the two curves being considered. In answering the question, students are required to produce a graph showing two intersecting curves as part of their answer.

Transitions between visual mediators: Transformations between **graphical** and **algebraic (symbolic)** modes are needed, since the students are to produce a graphical visual mediator based on the algebraic equations provided for the two curves being considered.

Visual complexity: The students are required to produce a graphical visual mediator in the form of a 2-dimensional sketch (relating *x* and *y*) which requires them to show (on the same diagram) the region enclosed by the curves $y = x^3 - 2x^2 + 5$ and $y = x + 3$, which requires drawing, at minimum, the part of each of these two curves that bounds the region between them, and an identification (e.g., labelling the points) of where these curves intersect.

Figure 5.17

Commognitive Summary Table for Q7, MATH104 Test B, 2017

Q7, MATH104 Test B, semester 2 2017			
Commognitive analysis		**Relating to technology**	**Positioning of the student based on how they have been taught in the course notes**
Keywords and symbols			
Specialisation	Discourse of Calculus and functions including the word *function* and, as part of a symbolic mediator, the word *lim* signifying a limit.		Students have been given examples of questions of this type with the same type of discourse, where they need to check the left-hand and right-hand limits to see if a limit exists
Objectification of the discourse	Discourse is objectified, treating the function and limit objects as nouns. Longest compound nominal group has length 3.		Similar examples in the course notes, which also have objectified discourse
Logical complexity	Low: there are no conjunctions, disjunctions, implications, negations or quantifiers in the written discourse.		
Visual mediators			
Types of visual mediators and transitions between them	Symbolic visual mediator representing the limit of the function. Graphical visual mediator showing the function.		Examples of similar piecewise continuous functions are also provided in the course notes, in the context of determining the value or existence of limits at particular points.
Logical complexity	Moderate: The limit in the question stem contains 2 letters: f signifying the name of the function and x signifying the independent variable.	Moderate: The graph relates two variables, x and y. The discontinuities in the graph, the open circles at $x = 3$ and the individual point at (3, 2) add to the amount of information to be processed.	The general interpretation of piecewise continuous functions of this type and how these relate to finding limits is included in the course notes.

Routines			
*Type of routine expected (r*itual, recall, substantiation or construction)	This question involves a ritual routine, as students have been given examples of answering this type of question in the course notes.		The course notes provide examples of checking the left-hand and right-hand limits at a point in problems of this type, to see if the limit exists.
Endorsed Narratives			
Alienation of the discourse and how mathematics is construed.	Alienated discourse, with no actions by or on mathematical objects.		
Subjectification of the discourse			
Student-author relationship	The student is given an imperative: to choose the correct answer option.		
Student autonomy	Given the information provided, there is only one way to answer the question: by checking the left-hand and right-hand limits at $x = 3$ on the graph to determine if they are equal and hence whether or not the limit exists.	To successfully answer the question, the student must use the graphical visual mediator to determine the values of the left-hand and right-hand limits at $x = 3$	

The visual complexity of the resulting diagram is increased by having *two* curves present. This diagram to be produced by the students on paper is a *static visual mediator* (Antonini et al., 2020; Ng, 2016). However, they could potentially treat the image showing the graphs on their calculator screen as a *dynamic visual mediator* (Antonini et al., 2020; Ng, 2016) when producing and examining the required curves if, for example, they zoom in on them or otherwise adjust them on their calculator screen, which some might do in order to see the required bounded areas between the curves more clearly or to determine their intersection points.

Logical complexity of visual mediators: In the visual mediators (functions) provided in the question there are four types of operations: addition, subtraction, multiplication and exponentiation. There is also an equals sign present, for defining each function.

5.2.10.3 Routines

The students are asked to engage with a ritual routine. This is because they are given instructions in the actual question text as to the initial steps to take in answering the question and because the course notes show that the procedure for finding the area between two curves has been taught in the unit as a ritual procedure. This is also a type of question that they would have been taught to solve during their study of Calculus topics in secondary school, meaning that manifestations of the *discursive footprint* of Calculus and functions from their secondary school studies could also influence the way they approach solving this type of problem, when accessing their *precedent search space*.

Instructions given regarding the procedure of the routine: Students are instructed to first draw the region and identify the intersections of the curves enclosing it. They are otherwise allowed to choose the procedure of the routine for finding the area of the region enclosed by the two curves. This means that they could utilise solution methods taught in this unit, in senior secondary school, or, since they are still at an early stage of their university studies and transitioning from the discourse of secondary school Calculus, they could potentially combine aspects of what they have been taught in both these learning environments in solving the problem.

5.2.10.4 Endorsed narratives

Alienation of the discourse/ the origin of mathematical knowledge: The objects in this question are the equations of the two curves, which are presented as being in a relationship with the area bounded by them, with the description "*the region which is enclosed by the curve[s]"* suggesting alienated discourse with non-human agency, with the curves carrying out the material process of enclosing the region.

5.2.10.5 Subjectifying aspects of the discourse

Student-author (examiner) relationship: There are no inclusive personal pronouns in the question or answer options. The student is given imperatives as to how to approach answering the question: to find the area that encloses the two curves, after first drawing the curves and identifying their intersections, suggesting an impersonal student-examiner relationship with the examiner in a role of authority, commanding the student to perform these two parts of the task in the order specified.

Student autonomy: In this question, the students are required to draw the curves and the area enclosed by them, followed by identifying the intersection points, before proceeding to find the area enclosed by the curves. This level of detail in the instructions provided, together with how the students have been taught to find such areas in the course, and the fact that this is a well-rehearsed routine in their precedent search space from upper secondary school, means that the path described below would be the most likely way students would approach answering this question. This solution path assumes use of CAS calculator technology, although the later steps of finding the intersection points and areas could also be done by-hand, which would have required a greater number of decisions to be made in the solution process.

Designing the path to follow: An example of a path a student could choose to follow to answer the question is shown below. The corresponding individual decisions which are required are shown in Table D7 (see Appendix D), with 8 decisions in total.

1. Drawing the two curves and shading the region enclosed by them.

2. Identifying their intersection points.

3. Determining and evaluating the integrals that correspond to the area in each enclosed region (these could also be done in the opposite order to that shown in the table).

4. Adding up these areas to find the total area enclosed by the two curves.

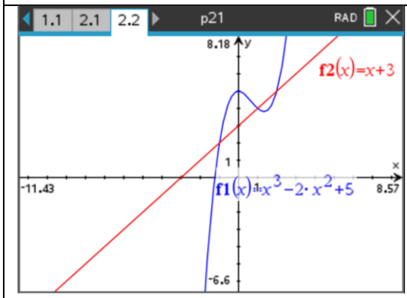
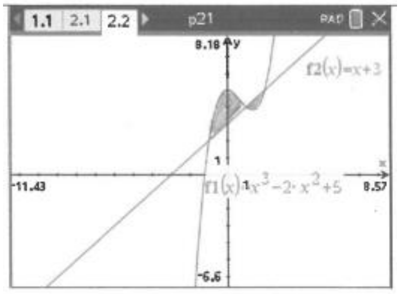
This problem could also be solved by interacting directly with the CAS calculator. Such a solution approach, using the TI-NSpire CAS calculator, is shown in Table 5.9. This method requires only 6 decisions to be made, which is a lower number of decisions than for other solution methods; therefore, the *grain size* of the task is 6.

Interpreting the task: The longest sentence in the question has length 17 (including 15 words, and two symbolic visual mediators which represent the equations of the curves being considered).

Choosing the form of the 'answer': The student is required to produce and draw a graphical visual mediator showing the region bounded by the two curves involved in the question as well as indicating the intersection points.

Table 5.9

Decisions for Solution of Q13(a), MATH104 final examination, 2015

Choice taken	**Decision actions**
	Sketch the two curves, making sure to include the region which is enclosed by them in the sketch.
	Shade (or otherwise identify) the required area bounded by the curves.
$(-1,2)$, $(1,4)$, $(2,5)$	Find the x and y values where the curves intersect, by using a graphing intersection tool on the calculator and write down the resulting points.
	Go to **Menu – Analyze Graph- Bounded Area.** Move the cursor until it is on the minimum x-value (at the intersection) point of the first bounded area ($x = -4$ here) and click it.
Area bounded by the curves between $(-1,2)$ and $(1,4)$ is $\frac{8}{3}$ Area bounded by the curves between $(1,4)$ and $(2,5)$ is $\frac{8}{3}$	Drag the cursor across until reaching the maximum x-value (at the intersection) point of the second bounded area ($x = -2$ here) and click it, so that the required bounded areas between the two curves are shaded and calculated.
$\frac{8}{3} + \frac{5}{12} = \frac{37}{12}$	Add up the two areas to get the total area of the region bounded by the curves.
	6 decisions

They are then required to find and write down the answer for the area enclosed between the curves. This question was worth 4 marks and in this examination paper the students could obtain full marks without needing to show any other intermediate steps in their working. However, showing their intermediate solution steps would increase the chances of their obtaining more marks through consequential marking, if there were any small errors in their solution process.

Figure 5.18 is a Commognitive analysis summary table that summaries the analysis of this task.

5.3 Discussion of Common Findings Across all the Test and Examination Questions

In this section, I draw across the commognitive analysis summary tables to see what can be concluded about the different aspects of the test and examination questions which were analysed in section 5.2. The use of a commognitive analysis framework in this chapter to analyse the selected written response and multiple-choice questions from these assessment items, demonstrates its application in capturing the complexity and difficulty level of these questions. This commognitive analysis has shown that in terms of the complexity and difficulty level of assessment questions where CAS technology is available, although the number of procedural steps which would be present if working by-hand have been reduced in some cases, questions that are in other ways logically and grammatically complex (with more variables and relatively complex discourse) are sometimes asked in test and examination assessments. While most of the questions considered in this study were found to have relatively low grammatical complexity, the question considered which was a context-separable application question (see section 5.2.7) had high grammatical complexity, which would be expected to increase how difficult the students find it to solve. It also contained constants represented by letters, which previous research has indicated increases how difficult students find such problems to solve (Leigh-Lancaster & Stacey, 2022).

The presence of visual mediators that need to be considered increases when CAS technology is used, which can also increase the complexity of a question. For example, when use of CAS technology is available, questions can be asked which contain symbolic visual mediators with a relatively high number of different letters, signifying both variables and constants (see section 5.2.7).

Figure 5.18

Commognitive Summary Table for Q13(a), MATH104 Final Examination, 2015

Q13(a), MATH104 exam, 2015			
Commognitive analysis		**Relating to technology**	**Positioning of the student based on how they have been taught in the course notes**
Keywords and symbols			
Specialisation	Discourse and specialised mathematical language of functions, including use of the words *curve*, *line* and *intersections* in the question		Students have been taught in the course notes how to plot curves, find their intersections and also the areas bounded by such curves
Objectification of the discourse	The discourse speaks of properties of the two curve objects and the area between them rather than speaking of processes. Longest compound nominal group has length 16 (including 2 symbolic visual mediators).		Similar examples in the course notes which also have objectified discourse
Logical complexity	Two occurrences of the conjunction "*and*", but no disjunctions, implications, negations or quantifiers in the written discourse of this question.		
Visual mediators			
Types of visual mediators and transitions between them	Symbolic visual mediators for the equations of the two curves being considered	Required to produce the two curves, one of which can only accurately be produced by a calculator with graphing capability. Could also use the calculator to find intersections of the curves and/or to integrate appropriate expressions to find the required areas enclosed by the curves.	Students shown in the unit how to graph curves.
Logical complexity	Two functions are considered, both which relate two variables: x and y		Functions in the course notes are also often written in terms of x and y

Routines			
*Type of routine expected (r*itual, recall, substantiation or construction)	Ritual		Students taught in the course notes as ritual routines how to draw curves using the CAS calculator and how to find intersections and areas between curves
Endorsed Narratives			
Alienation of the discourse and how mathematics is construed	The discourse suggests non-human agency with the curves carrying out the action of enclosing the region		
Subjectification of the discourse			
Student-author relationship	There are no inclusive personal pronouns in the question or answer options. The student is given imperatives as to how to approach answering the question: to find the area that encloses the two curves after first drawing the curves and identifying their intersections.		
Student autonomy	The student can choose from a few different techniques in determining the correct answer, but the structure of the question and the steps it demands limit the number of different ways students would be encouraged to approach this question.	One of the functions can only accurately graphed using a calculator (such as their CAS calculator) with graphing capability. They have a choice as to whether to calculate the intersection of the curves and areas enclosed by them by-hand or with the calculator, but would take longer by-hand.	

As was seen in Chapter 3, students typically find such questions involving parameters and more general forms of equations relatively difficult, compared to those which only have numerical coefficients (Leigh-Lancaster & Stacey, 2022). In the case of multiple-choice questions involving either direct use of CAS technology or using CAS outputs, algebraic manipulation or renaming and simplifying constants is also sometimes required to determine the correct answer option (see sections 5.2.5 and 5.2.6). CAS technology is able to generate graphical visual mediators such as slope field diagrams and graphs of some functions which would be very time-consuming to require students to produce by-hand in an examination. In turn, some of these graphical visual mediators, including slope field diagrams, have a high level of visual complexity. These graphical visual mediators can be valuable for testing students' ability to make connections between different representations associated with the same mathematical function or process, such as linking the algebraic solution of a DE directly with its corresponding slope field diagram (see section 5.2.3) or requiring the students to describe how the local slopes in such a diagram behave and how this is connected with the equation of the DE (see section 5.2.4). Such questions, which require students to produce their own endorsable narratives, as in section 5.2.4, or to mediate between different mathematical representations, as in both sections 5.2.3 and 5.2.4, are among those students typically find relatively difficult, when compared to questions which test the ability to carry out standard ritual routines, by replicating specific step-by-step processes that they have been taught. Therefore, CAS can increase the level of difficulty of examination questions, through the presence of symbolic and graphical mediators of the types described above.

However, how difficult or complex students might find questions which involve CAS outputs also, in some cases, partly depends on the extent to which they are a knower of such outputs. Examinations with such outputs are positioning students as a knower of technology, but if the student is not confident or familiar with such outputs, the logical complexity of interpreting the task can be much greater for them than for students able to use alternate approaches as knowers of the CAS outputs. For example, it was seen that the symbolic output in the question considered in section 5.2.5, contained lines of Wolfram Alpha CAS output that would not actually need to be considered to answer the question. This means a student who was a knower of such outputs would quickly be able to dismiss those lines of output as not relevant. The CAS calculator symbolic output in section 5.2.6 would also be quicker and easier to interpret for students familiar with output of that type, in the way it presents constants in solutions of DEs of that general type. Similarly, CAS outputs in the form of graphical visual mediators will also appear more complex to students who are not knowers of such outputs, with it being important for students to be aware of what features to focus on in interpreting outputs such as slope field diagrams (see sections 5.2.3 and 5.2.4) and graphs of functions, such as in section 5.2.9, based on what the question requires.

However, in some types of *multiple-choice* questions, such as the question analysed in section 5.2.5 involving extracting the correct information from Wolfram Alpha output, being a knower of multiple-choice question answering strategies can allow students to avoid needing to know how to use and interpret the CAS output provided (and also how to work out the answer to the question by-hand). This can occur if they know some particular aspects that correct answer options would need to have, which only are contained in one of the answer options (in the case of the section 5.2.5 question, that the complementary function in the general solution of a DE must contain two arbitrary constants, with only one of the answer options having this form). This demonstrates the importance of designing answer options to such multiple-choice questions in a way to avoid this, which is done in the section 5.2.6. multiple-choice question, which also involves interpreting CAS output but which has answer options that contain not only different parts of the DE output, but also properties connected with different parts of the DE solution (complementary function, but also particular integral and the general solution). This is an important finding for how to design multiple-choice questions in a CAS environment.

The written discourse of examination questions can also vary in the way in which mathematics is talked about, which was investigated. The examination questions analysed contained discourse that was mostly objectified and which used specialised mathematical words from the discourse of Calculus, Functions and Differential Equations. This discourse was mostly alienated, with an absence of human presence in the mathematical actions being described in the questions, which focused on relational statements and material processes in which mathematical objects were the agents. The exceptions were the question in section 5.2.7 which referred to a human participant in the scenario described in the question, but not in the generation of the actual mathematics involved, and the question in section 5.2.5, which did not explicitly mention a human agent but where the description that the DE was placed into Wolfram Alpha would be assumed to be by a human agent.

There are also some questions asked when CAS technology is available, where the discourse in the question text and any visual mediators appear to have similar grammatical and logical complexity to CAS-free questions on the same topics, but where the *grain size* of the task would be much greater if CAS is not used or allowed. These include when graphing functions of degree 3 or higher in many cases and when finding Taylor (or Maclaurin) polynomials of functions involving products or composite functions that are time consuming to repeatedly differentiate. In some such cases (e.g., accurately graphing polynomials of high degree that have no rational roots), it also might not be feasible for students to fully answer these questions without use of CAS technology. However, the emphasis in such a question is also important, as, for example, in the question of this type in section 5.2.10, an approximate sketch of the cubic function involved would have sufficed, as the question required an accurate sketch only of the region *bounded* by the curves, not of their x-intercepts.

The majority of the questions analysed would be expected to be solved by the use of *ritual routines*, in that they were testing skills that the students had been taught step-by-step processes for in the course notes. The exceptions were the two questions which involved interpreting a slope field diagram (see sections 5.2.3 and 5.2.4), in requiring *construction routines*. This is because, while students had been taught how to interpret slope field diagrams, they had not been given any previous examples which used the types of reasoning needed to answer these questions, with the question in section 5.2.4 also requiring the students to construct endorsable narratives in their answer, in describing how the local slopes in the diagram behave and are related to the associated differential equation provided in the question. The question in section 5.2.10 where the students were required to find the area bounded by two curves could potentially also invite explorative behaviour by the students, in interacting with a graphical visual mediator showing the two curves on their calculator screen and treating it as a dynamic visual mediator. However, the actual expected solution process for answering that question was a ritual routine, familiar to the students not only from the course notes, but also from experience with solving similar types of problems in high school.

5.4 Conclusion

In addressing Research Question 1, these overall findings indicate that the presence of CAS does have an effect on the difficulty and complexity of test and examination questions, with such questions usually able to be solved with less procedural steps, but with some CAS outputs containing relatively high levels of complexity. In answering some questions, students are required to make connections between graphical and symbolic visual mediators, including in finding the correct answer option to some multiple-choice questions. The number of distinct letters (including subscripted letters), which represent different variables and constants is one aspect which affects how difficult students typically find questions, and, in cases where CAS outputs are provided, and in applications questions where use of CAS is permitted, the amount of such letters present is often relatively high. How difficult and complex students find questions involving CAS is also affected by how individual students are positioned in relation to such questions and the extent to which they are knowers of the outputs produced by CAS.

Chapter 6 now follows, which will explore how undergraduate students used their CAS calculators in practice, when answering Calculus questions, and how they talked about their answers to such problems in task-based interviews. The findings will then be contextualised by analysing the results of a questionnaire given to a similar cohort of students during the same time period. Chapter 7 will be a discussion of the findings of my thesis in light of the associated literature and the research questions, while Chapter 8 will conclude the thesis by summarising the key findings of this study and considering implications and possibilities for future research in this field.

CHAPTER 6

RESULTS OF ANALYSIS OF TASK-BASED INTERVIEWS AND STUDENT ATTITUDE QUESTIONNAIRE

6.1 Overview

In Chapter 5, the effect that Computer Algebra Systems (CAS) technology can have on the discourse of Calculus examination questions, as set by the examiner, has been seen. However, the effect of CAS technology on how students respond to, and talk about, their answers to Calculus questions of this type is also of interest, including in cases where the nature of the technology has the potential to generate commognitive conflicts (Nachlieli & Heyd-Metzuyanim, 2022) for its users. In this chapter, the findings of the analyses of task-based interviews (Maher et al., 2014) with four volunteer university students will be presented. Section 6.2 describes the tasks the participants were required to complete and expected approaches they could take, together with the grain sizes of the different approaches. In section 6.3, further commognitive analysis (Sfard, 2008) to answer the research questions is applied to the students' written attempts at the tasks and to their responses in interviews. The focus is on looking for evidence of *ritual* and/or *explorative routines* (Lavie et al., 2019; Sfard, 2008; Viirman & Nardi, 2018) in their work and at how they managed potential *intrapersonal commognitive conflicts* (Kontorovich, 2021) arising from their interactions with their CAS calculators, in attempting to solve the problems provided, as well as looking at the nature of their discourse and how this was influenced by the presence of CAS technology and by their *precedent search spaces* (Lavie et al., 2019; Viirman & Nardi, 2021). To contextualise the results from sections 6.2 and 6.3, the results of a qualitative analysis of questionnaire data on undergraduate students' use of, and attitudes towards, CAS technology at the time the study took place are presented in section 6.4. The chapter ends with a conclusion (Section 6.5), where the major results coming from the analysis are detailed, as well as what will follow in coming chapters to complete the thesis.

The analyses of the questions that follow in section 6.2 supplement the work from the previous chapter in addressing Research Question 1, through commognitive analysis of the discourse of the mathematics problems the students were instructed to solve. In addressing Research Question 2(a) in section 6.3, I am interested in identifying the extent to which the students used *ritual routines* and any evidence of them moving from rituals to explorations, such as *substantiation routines* or *construction routines*, as this relates to how effectively they were using and interpreting the CAS output obtained in solving the set tasks. These included some where the CAS output was in a different format to what they would have obtained if solving the problems by-hand using the methods they had been taught in class. The parts of the commognitive analysis framework (Morgan & Sfard, 2016) that relate to different types of routines will be applied to help investigate these matters.

Another aspect in addressing Research Question 2(a), in examining how effectively the students were using their CAS calculator in solving the problems and in interpreting the resulting output, is looking for any sources of error in their work. From a commognitive analysis perspective, this includes looking to see if, and, how any *commognitive conflicts* occurred. As was noted in Chapter 2, commognitive conflicts are situations where “the encounters between interlocutors use the same mathematical signifiers (words or written symbols) in different ways or perform the same mathematical tasks according to differing rules” (Sfard, 2008, p. 161).

In this current study, I investigated possible sources of commognitive conflicts that could be generated by certain types of Calculus examination questions when answered using CAS calculators, and examined students’ mathematical discourse when these potential conflicts occurred. That is, I considered situations where students’ errors might be the result of commognitive conflicts associated with the use of the Computer Algebra Systems calculators when completing mathematical tasks. Some of these, I argue, can be the result of the nature of the commands and expected expressions to be entered into the calculator, where the format can be different to what would still be part of a correct solution process in classroom mathematical discourse, if doing the same tasks ‘by-hand’. How the students managed these commognitive conflicts is also of interest.

Research Question 2(b) will also be addressed in section 6.3, when applying the commognitive analysis framework to analyse the nature of the participants’ mathematical discourse associated with their solving of the series of mathematics problems, and in analysing their responses to the interviews conducted immediately after they finished the tasks. Research Question 3 will be addressed in section 6.4, when the analysis of the questionnaire data from undergraduate students reporting on their use of, and attitudes towards, CAS technology at the time of the study will be presented.

6.2 The Participants, Tasks and Expected Solution Approaches

This section introduces the participants in the task-based interviews part of the study, and the tasks they were given to complete, and provides commognitive analysis of each of the questions they were given in the task. Subsection 6.2.1 begins with a description of the participants and the types of calculators they used when completing the tasks. A description of the tasks themselves then follows, including the instructions, the mathematics problems each participant was given to solve and the conditions under which the participants completed these tasks. Subsections 6.2.2 and 6.2.3 relate to addressing Research Question 1, which I remind the reader of below.

RQ1 How can a commognitive framework be applied to effectively capture the complexity and difficulty level of written answer and multiple-choice examination questions asked in undergraduate Calculus units where use of CAS technology is available?

In particular, in subsection 6.2.2, commognitive analysis of the discourse of the five questions that the three first-year students were given is presented, followed by the commognitive analysis of the additional three alternative questions that the second-year student was also presented with, omitting one of the questions for the first-year students. Subsection 6.2.3 describes the rationale for choosing each question for the task and then applies part of Sfard’s commognitive analysis framework (Morgan & Sfard, 2016) to analyse the expected solution paths for the students to take for each question, both with a CAS calculator and by-hand, based on the subject learning materials, together with indicating the *grain size* of the task in each case.

6.2.1 Participants, tasks and testing conditions

Task-based interview sessions were conducted individually with four volunteer students. These comprised three first-year university students (participants A, B, and C) and one second-year student (participant D), studying a Calculus subject as part of a degree where they were preparing to become secondary mathematics teachers in schools. One of the first-year students (participant B) used a Casio Classpad 330 calculator whereas the other three students (participants A, C and D) all had TI-Nspire CX CAS calculators.

They were given a worksheet with five (for the first-year students) or seven (for the second-year student) Calculus problems to solve (see Figure 6.1), with space for their solutions to be recorded. A session started with a student being given the set of mathematics problems to solve.

Figure 6.1

Instructions and Problems for the Task-based Interviews

Use your CAS calculator to solve the following problems. Write down all commands you use on CAS to do so. In each case, also comment on whether the CAS output is what you expected. If it is not what you expected in some cases, how do you think it could be reconciled with what you expected? Where you think it appropriate, also consider/demonstrate any methods by-hand or on CAS you could use to check or interpret your answer.

Question 1 for Participants A, B, C and D

1. Find $\frac{dy}{dx}$ given $y = (x^3 + x)^{\frac{2}{3}}$.

Question 2 for Participants A, B, C and D

2. Find $\frac{dy}{dx}$ given $x^2y + e^{2y} = 3xy$.

Question 3 for Participants A, B, C and Question 7 for participant D

3. Find the area bounded by the two curves $y = x^3 + 9x^2 + 22x + 12$ and $y = -4x - 12$.

Question 4 for Participants A, B, C

4. A particular quantity is known to have value $P(t) = P_0(1 + t^{2.5})e^{-kt}$ at time t, where P_o and k are constants. Find the rate of change of the quantity P with respect to time t.

Question 5 for Participants A, B, C and Question 6 for participant D

5. Determine $\int(\sin x + \cos x)\, dx$ and $\int_0^{\pi}(\sin x + \cos x)\, dx$

Question 3 for participant D

3. Solve $\frac{dy}{dx} = \frac{1}{2}(1 + y^2)$ such that y(0) = -1. Write your answer in the form $y = f(x)$.

Question 4 for participant D

4. Solve $3x^3y' = y^4$ such that $y(2) = 1$.

Question 5 for participant D

5. Find the general solution of $\frac{d^2y}{dx^2} - 4y = e^{2x}$. Hence write down complementary function and a particular integral.

There was no time limit, or time pressure, and each student was simply told to inform the researchers (i.e., the former doctoral supervisor as a silent observer and myself as interviewer) when finished. Figure 6.1 shows the instructions given to each participant. Section 6.2.2 now follows, with a commognitive analysis of these problems.

6.2.2 Commognitive analysis of task-based interview problems

In these analyses, in keeping with the analytical framework, the extent of specialisation in word use, grammatical and logical complexity, types of visual mediators, routines, endorsed narratives and the degree of student autonomy, were first analysed and detailed in order to allow comparison to the examination task analyses from Chapter 5 and to gain an overview of major results from the data on the questions the student participants were asked.

Specialisation in word use. The tasks all contain the discourse of Calculus, but mainly in the form of visual mediators rather than specialised mathematical words, although the words *bounded*, *area* and *curves* that occur in question 3 and the words *constants* and *rate of change* in question 4 are specialised in that they are being used in the context of Calculus discourse. The specialised mathematical $sinx$ and $cosx$ functions appear in question 5 and the *exponential* mathematical function appears in three of the questions as e^{2x}, e^{2y} and e^{-kt}.

Grammatical Complexity. The majority of these questions have very low grammatical complexity, as they mostly comprise visual mediators, connected by only one or two words. The only questions with compound nominal groups are questions 3 and 4 (given to participants A, B and C) and question 3 version 2 (given to participant D). The longest compound nominal groups in these questions are of lengths 10 (including 2 visual mediators), 8 and 3 (including 1 visual mediator), respectively. Questions 3 (version 1) and 5 (version 1) contain the conjunction *and* once. There are no nested subordinate clauses in any of the questions.

Logical Complexity. The main source of logical complexity of the questions lies in the symbolic visual mediators provided, especially in question 4 (version 1) which is the question with the greatest number of different letters present in the symbolic visual mediator (3), where in that case, two of these signify variables, while the other one signifies a constant and there is also the symbol P_0, which signifies another constant. Questions 1, 4 (version 1) and 3 (version 2) include nesting of a function inside brackets.

Visual Mediators. The only visual mediators present in the questions are the actual symbolic visual mediators, such as equations of functions, and Calculus commands such as $\frac{dy}{dx}$. Question 3 (version 1) is the only one where the participants would be expected to produce a graphical visual mediator in answering the question, to identify the parts of the curves that are bounded by each other and that hence need to be used in determining the area between the curves.

Routines. Students could participate in *rituals* to answer these questions if they used just the calculator to obtain the answers without investigating further. However, the questions invite *substantiation* and even potentially *construction explorative routines*, in asking for students to reconcile their answers with what they would have obtained by-hand and to think about whether the answers obtained are what they expected, and if not, why that was the case. This is especially the case for question 1, where the answer is very different to that obtained by-hand, and for question 3 (version 1), where producing and interacting with a graphical visual mediator showing the curves and their intersection, would be beneficial in helping answer the question.

Endorsed Narratives. The discourse is alienated and objectified throughout all the questions, with the majority of them describing agentless relationships between mathematical objects. The exception is question 3 (version 1), which suggests non-human agency, with the curves themselves carrying out the action of bounding the area between them.

Student Autonomy. The students were instructed to use their CAS calculators to help answer each question but given some autonomy as to whether or not they also worked out the answer by another method to compare the results. There are also no imperatives about using any specific methods if checking their work by-hand, with questions 2 and 3 (version 1), in particular, having more than one 'by-hand calculation' approach that could be taken to obtain the correct answer.

6.2.3 Rationale for choosing each question, possible solution paths and task grain size

The reasons for choosing each question given to the participants are outlined below, followed by the ways in which the questions could be solved, based on the information the students could be expected to be knowers of, from their study of the Calculus subject involved. Tables are used to record solution steps and corresponding decision actions. The number of decisions involved in each solution method is then shown at the bottom of each table with the *grain size* of each task then determined as the *minimal* number of decisions required.

Figure 6.2

First Task-based Interview Question Given to All Four Participants

Question 1

1. Find $\frac{dy}{dx}$ given $y = (x^3 + x)^{\frac{2}{3}}$

As shown in Table 6.1, the different calculators generate different forms of output for Question 1 (Figure 6.2). If using a TI-Nspire CX CAS calculator, the answer provided by the calculator is very different from the answer obtained by-hand using the chain rule for differentiation that the students were taught in class and which is most commonly used to solve problems of this type by-hand. As shown in Table 6.2, this would yield:

$$\frac{dy}{dx} = \frac{2}{3}(x^3 + x)^{-\frac{1}{3}}(3x^2 + 1)$$

This question was therefore designed to see if students with the TI CAS calculator would engage in exploration routines, such as substantiation, by also working out the answer by-hand and directly comparing it with the answer obtained on the CAS calculator and/or construction routines by, upon seeing the answers were different, further exploring algebraically, or otherwise, why this might be the case.

Table 6.1

Decisions to Determine Grain Size of CAS Calculator Solution to Question 1

Result obtained by calculator	**Decision actions**
$\frac{2(x^2+1)^{\frac{2}{3}}}{3x^{\frac{1}{3}}} + \frac{4x^{\frac{5}{3}}}{3(x^2+1)^{\frac{1}{3}}}$ (TI-Nspire) or $\frac{2.(3.x^2+1)}{3.(x^3+x)^{\frac{1}{3}}}$ (Casio Classpad)	Use the command $\boldsymbol{d}((x^3 + x)^{\frac{2}{3}}, x)$ to find $\frac{dy}{dx}$ using the CAS calculator
	1 decision

Table 6.2

Decisions to Determine Grain Size of By-hand Solution to Question 1

Result obtained by-hand	**Decision actions**
Let $u = x^3 + x$	Identify that will need to use the chain rule to obtain the answer, with "inner function" $u = x^3 + x$
$y = u^{\frac{2}{3}}$	Rewrite y in terms of u
$\frac{dy}{du} = \frac{2}{3}u^{-\frac{1}{3}}$	Find the derivative of y with respect to u
$\frac{du}{dx} = 3x^2 + 1$	Find the derivative of u with respect to x
$\frac{dy}{dx} = \frac{2}{3}u^{-\frac{1}{3}}(3x^2 + 1)$	Work out $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
$\frac{dy}{dx} = \frac{2}{3}(x^3 + x)^{-\frac{1}{3}}(3x^2 + 1)$	Write the final answer in terms of x
	6 decisions

The solution paths shown in Table 6.1 and 6.2 have the minimal number of decisions required for solving the problem using CAS and by-hand, respectively, based on the methods the students had been taught for solving these problems; therefore, the grain size of the task is 1 if CAS is available and 6 if it is not.

Figure 6.3

Second Task-based Interview Question Given to All Four Participants

<u>**Question 2**</u>

2. Find $\frac{dy}{dx}$ given $x^2y + e^{2y} = 3xy$

The second question (Figure 6.3) for the task-based interview was designed to see if the students would encounter commognitive conflicts, due to the calculator signifying *partial* differentiation in its use of the *d* symbol and due to the TI-Nspire CX calculator only producing the correct answer if the product xy is written with a * multiplication sign included, as $x * y$. As shown in Tables 6.3 and 6.4, there are two methods which can be used to solve this problem, using a CAS calculator. The **impDif** command, which is present on both models of calculators used, can be used to find the answer in one step, as shown in Table 6.3, or it can be used to implicitly differentiate each term in turn, followed by using the **Solve** command to rearrange the result to find $\frac{dy}{dx}$, as shown in Table 6.4. Table 6.5 shows the decisions required to solve the problem by-hand.

Table 6.3

Decisions to Determine Grain Size of CAS Calculator Solution to Question 2 (Method 1)

Result obtained using CAS calculator	**Decision actions**
	Recognise that will need to use the **impDif** command on the equation.
$\dfrac{3y-2xy}{x^2+2e^{2y}-3x}$	Use the command **impDif**$(x^2y+e^{2y}=3x*y,x,y)$ on the CAS calculator
	2 decisions

Table 6.4

Decisions To Determine Grain Size of CAS Calculator Solution to Question 2 (Method 2)

Result obtained	**Decision actions**
$\frac{d}{dx}(x^2y)+\frac{d}{dx}(e^{2y})=\frac{d}{dx}(3xy)$	Recognise must differentiate both sides (and hence each term) with respect to x
$x^2y'+2xy$	Use **impDif**(x^2y) command to find $\frac{d}{dx}(x^2y)$
$2.y'.e^{2y}$	Use **impDif**(e^{2y}) command to find $\frac{d}{dx}(e^{2y})$
$-3.x.y-3.y$	Use **impDif**$(3x*y)$ command to find $\frac{d}{dx}(3xy)$
$2xy+x^2\frac{dy}{dx}+2e^{2y}\frac{dy}{dx}=3y+3x\frac{dy}{dx}$	Substitute these derivatives into the relation found at the first step
$2xy+x^2z+2e^{2y}z=3y+3xz$	Substitute a variable (e.g., z) for $\frac{dy}{dx}$
$z=\frac{dy}{dx}=\dfrac{-y(2x-3)}{2e^{2y}+x^2-3x}$	**Solve**$(2xy+x^2z+2e^{2y}z=3y+3xz,z)$ will solve for z (and hence for $\frac{dy}{dx}$).
	7 decisions

The first method requires the lowest number of decisions to be made; therefore the grain size of the problem when CAS is available is 2, while by-hand it is 13.

Table 6.5

Decisions to Determine Grain Size of Solution By-hand Solutions to Question 2

Result obtained by-hand	**Decision actions**
$\frac{d}{dx}(x^2y) + \frac{d}{dx}(e^{2y}) = \frac{d}{dx}(3xy)$	Recognise must differentiate both sides (and hence each term) with respect to x
$\frac{d}{dx}(x^2y) = \frac{d}{dx}(x^2)y + x^2\frac{dy}{dx}$	Apply product rule to differentiate x^2y
$\frac{d}{dx}(x^2) = 2x$	Differentiate x^2 with respect to x
$\frac{d}{dx}(x^2y) = 2xy + x^2\frac{dy}{dx}$	Combine by applying the product rule
$\frac{d}{dx}(3xy) = \frac{d}{dx}(3x)y + 3x\frac{dy}{dx}$	Apply product rule to differentiate $3xy$.
$\frac{d}{dx}(3x) = 3$	Differentiate $3x$ with respect to x
$\frac{d}{dx}(3xy) = 3y + 3x\frac{dy}{dx}$	Combine by applying the product rule
$\frac{d}{dx}(e^{2y}) = \frac{d}{dy}(e^{2y})\frac{dy}{dx}$	Identify that will need to apply the chain rule to differentiate e^{2y} with respect to x
$\frac{d}{dx}(e^{2y}) = 2e^{2y}\frac{dy}{dx}$	Differentiate e^{2y} with respect to x
$2xy + x^2\frac{dy}{dx} + 2e^{2y}\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$	Substitute the above results into the equation from step 2
$x^2\frac{dy}{dx} + 2e^{2y}\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 2xy$	Rearrange so all terms involving $\frac{dy}{dx}$ are on one (left) side of the equation and all other terms are on the other (right) side
$\frac{dy}{dx}(x^2 + 2e^{2y} - 3x) = 3y - 2xy$	Take $\frac{dy}{dx}$ out as a common factor
$\frac{dy}{dx} = \frac{3y - 2xy}{x^2 + 2e^{2y} - 3x}$	Divide both sides by $x^2 - 3x$ to make $\frac{dy}{dx}$ the subject
	13 decisions

The third question version 1 (Figure 6.4) for the task-based interview is similar to the MATH104 examination question from Figure 5.16 which was discussed in Chapter 5, both in the type of discourse used in the question and the required solution process (although the question from Chapter 5 also explicitly instructed the students to draw the two curves and the area between them). This question was designed to see if students would plot the curves on the CAS calculator to assist in determining the area between them, and if they would then treat the resulting output as a *dynamic visual mediator* (Antonini et al., 2020; Ng, 2016), by zooming in on the curves, which would be encouraged by the required area between the curves in this case not being clearly visible with the calculator's default 'window' settings. From a research perspective, I was also interested to see if they showed any other evidence of tracing around or clicking over parts of the graph to find the values of specific points on the curves, such as axis intercepts, points where the two curves intersect, or the actual area between the curves, or if they would just treat the output as a static diagram by only looking at the diagram initially produced by the calculator without interacting with it in any way on the calculator screen.

Figure 6.4

Third Task-based Interview Question Given to All Four Participants

Question 3

3. Find the area bounded by the two curves $y = x^3 + 9x^2 + 22x + 12$ and $y = -4x - 12$

Table 6.6 shows the steps and expected decisions for working out the intersection points and the required definite integrals using either model of CAS calculator, while Tables 6.7 and 6.8 show the expected decisions a student would make if using the TI-Nspire CX CAS calculator and the Casio Classpad 330 calculator respectively as much as possible to solve this problem, including finding the area between the curves within the calculator's graphing window in each case. The solution path shown in Table 6.9, shows the most likely, efficient method to be used if solving this problem fully by-hand.

Table 6.6

Decisions to Determine Grain Size of CAS Calculator Solution to Question 3 (Version 1) Method 1

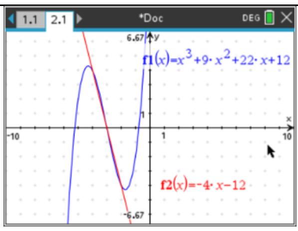
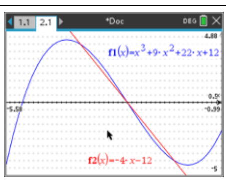
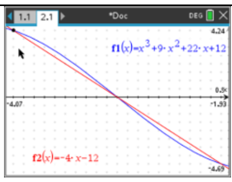
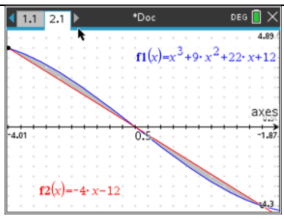
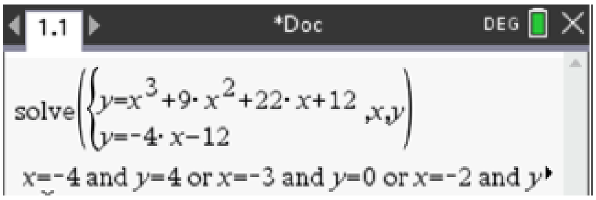
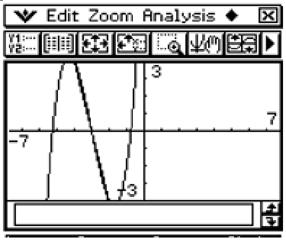
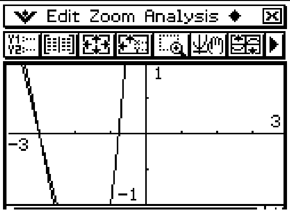
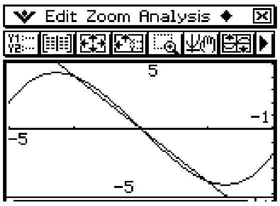
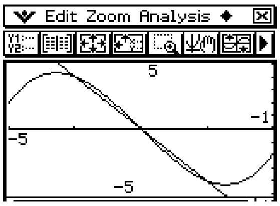
Result obtained	**Decision actions**
	Plot the two curves (using the calculator)
 	Adjust the **window** settings or zoom in on the curves to clearly see the required areas
	Shade (or otherwise identify) the area bounded by the curves.
$x = -4 \; or \; x = -3 \; or \; x = -2$ 	Find the x values where the curves intersect, using **Solve**$(x^3 + 9x^2 + 22x + 12 = -4x - 12, x)$ on the CAS calculator
$\int_{-4}^{-3} x^3 + 9x^2 + 22x + 12 - (-4x - 12\,)dx$	Write down the integral for the area of the leftmost bounded region
$\int_{-4}^{-3} x^3 + 9x^2 + 22x + 12 - (-4x - 12\,)dx = \frac{1}{4}$	Evaluate the resulting integral using the CAS calculator
$\int_{-3}^{-2} -4x - 12 - (x^3 + 9x^2 + 22x + 12\,)dx$	Write down the integral for the area of the second bounded region
$\int_{-3}^{-2} -4x - 12 - (x^3 + 9x^2 + 22x + 12\,)dx = \frac{1}{4}$	Evaluate the resulting integral using the CAS calculator
$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$	Add up the two areas to get the total area enclosed by the curves
	9 decisions

Table 6.7

Decisions to Determine Grain Size of CAS Solution to Question 3 (Version 1) Using Casio Classpad 330- Method 2 (finding area within graphing window).

Result obtained	**Decision actions**
	Plot the two curves (using the calculator)
 	Adjust the **window** settings or zoom in on the curves to clearly see the required areas
	Shade (or otherwise identify) the area bounded by the curves.
	Remain in the graphing window and go to **Analysis- GSolve- Integral-** $\int dx$ **Integration** option.
	The left-most intersection point should be selected, if not then scroll across to near it (that is, to near -4). Press **Execute.**
$\frac{1}{2}$	Scroll across to near the right-most intersection (that is, to near -2). Press **Execute.** The area between the curves should then be given.
	6 decisions

Working in the calculator's graphing window, as shown in Tables 6.7 and 6.8 requires the fewest decisions to be made; therefore, the grain size of the task is 6 or 7, depending on whether using a TI-NSpire or a Casio Classpad calculator, respectively.

Table 6.8

Decisions to Determine Grain Size of Solutions to Question 3 (Version 1) Using a TI-Nspire CX Calculator- Method 2 (finding area within graphing window).

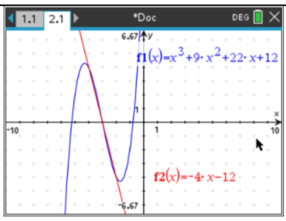
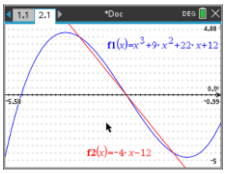
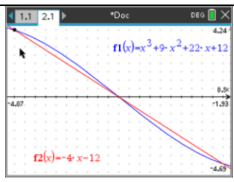
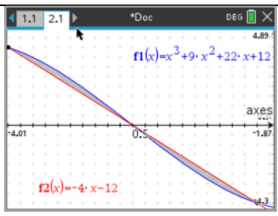
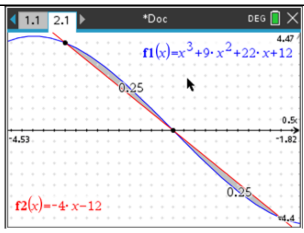
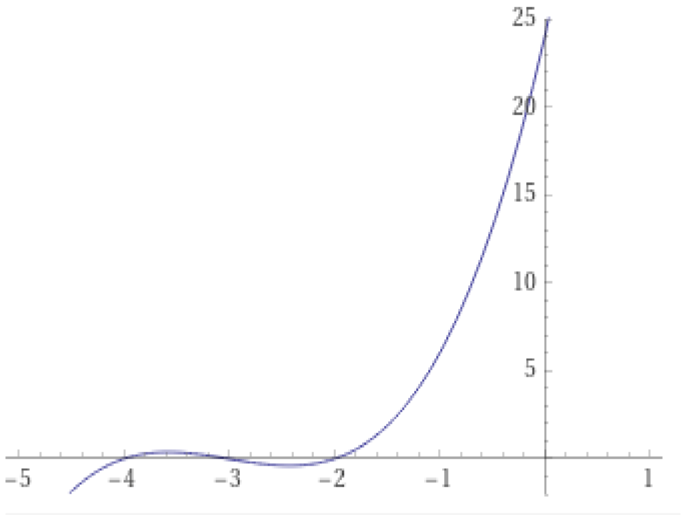
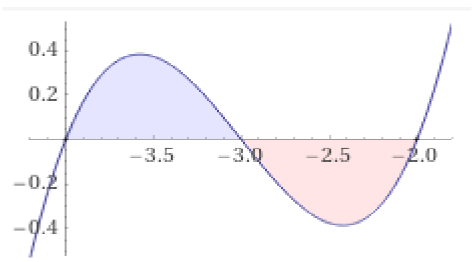
Result obtained	**Decision actions**
	Plot the two curves (using the calculator)
 	Adjust the **window** settings or zoom in on the curves to clearly see the required areas
	Shade (or otherwise identify) the area bounded by the curves.
	In graphing window, go to **Menu – Geometry – Points & Lines – Intersection Point(s)**. Select the two curves in turn, to display intersections.
	Go to **Menu – Analyze Graph- Bounded Area.** Move the cursor until it is on the minimum x-value (at the intersection) point of the first bounded area ($x = -4$ here) and click it
 Each of the two areas = 0.25	Drag the cursor across to maximum x-value (at the intersection) point of the second bounded area and click it, so that the required bounded areas between the two curves are shaded and calculated
$Area = 0.25 + 0.25 = 0.5$	Add up the two areas to get total area
	7 decisions

Table 6.9

Decisions to Determine Grain Size of Solutions to Task-based Interview Question 3 (Version 1) Working By-hand

Result obtained	**Decision actions**
$x^3 + 9x^2 + 26x + 24 = 0$	Determine that to find the x values where the curves intersect, will need to solve $x^3 + 9x^2 + 22x + 12 = -4x - 12$, which should be rearranged to obtain 0 on the right-hand side
$x = -2$	Substitute in values of x that could be a solution, which are those which are factors of the constant term, 24. Do this until obtain a value that satisfies the equation (will use the example of $x = -2$ here, but could also use $x = -3$ or $x = -4$)
$(x + 2)$	Determine the associated factor that satisfies the above solution
$\frac{x^3 + 9x^2 + 26x + 24}{x - 2} = x^2 + 7x + 12$	Perform polynomial long division to obtain the required quadratic function that is also a factor of $x^3 + 9x^2 + 26x + 24$
$x^2 + 7x + 12 = (x + 3)(x + 4)$ Therefore, $x = -3$ or -4	Factorise or use the quadratic formula to solve for x
When $x = 0$, $y = 24$	Determine the y-intercept of $y = x^3 + 9x^2 + 26x + 24$, by substituting in $x = 0$

 	Use the information about the x and y intercepts to plot $y = x^3 + 9x^2 + 26x + 24$ by-hand
 	Shade (or otherwise identify) the area bounded by the curves.
$\int_{-4}^{-3} x^3 + 9x^2 + 26x + 24\, dx$	Write down the integral for the area of the leftmost bounded region (from -4 to -3)
$\left[\frac{x^4}{4} + 3x^3 + 13x^2 + 24x\right]$ from -4 to -3	Simplify and determine the antiderivative $\int_{-4}^{-3} x^3 + 9x^2 + 26x + 24dx =$ $\left[\frac{x^4}{4} + 3x^3 + 13x^2 + 24x\right]$ from -4 to -3
$\frac{(-3)^4}{4} + 3(-3)^3 + 13(-3)^2 + 24(-3) - (\frac{(-4)^4}{4} + 3(-4)^3 + 13(-4)^2 + 24(-4) = \frac{1}{4}$	Apply the Fundamental Theorem of Calculus, to obtain $\frac{(-3)^4}{4} + 3(-3)^3 + 13(-3)^2 + 24(-3) - (\frac{(-4)^4}{4} + 3(-4)^3 + 13(-4)^2 + 24(-4) = \frac{1}{4}$

$-\int_{-3}^{-2} x^3 + 9x^2 + 26x + 24\,dx$	Write down the integral for the area of the second bounded region (from -3 to -2, multiplying $x^3 + 9x^2 + 26x + 24$ by -1, since this part of the curve is below the x-axis
$\left[-\frac{x^4}{4} - 3x^3 - 13x^2 - 24x\right]$ from -3 to -2	Simplify and determine the antiderivative $-\int_{-3}^{-2} x^3 + 9x^2 + 26x + 24dx =$ $\left[-\frac{x^4}{4} - 3x^3 - 13x^2 - 24x\right]$ from -3 to -2
$-\frac{(-2)^4}{4} - 3(-2)^3 - 13(-2)^2 - 24(-2) - (-\frac{(-3)^4}{4} - 3(-3)^3 - 13(-3)^2 - 24(-3) = \frac{1}{4}$	Apply the Fundamental Theorem of Calculus, to obtain $\frac{(-2)^4}{4} + 3(-2)^3 + 13(-2)^2 + 24(-2) - (\frac{(-3)^4}{4} + 3(-3)^3 + 13(-3)^2 + 24(-3) = \frac{1}{4}$
$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$	Add up the two areas to get the total area enclosed by the curves
	15 decisions

If access to a CAS calculator is not permitted, the grain size of the problem is 15 (see Table 6.9), which is much higher than when working by-hand.

The fourth question (Figure 6.5) was designed to see if the students would encounter commognitive conflicts, due to the calculator interpreting a letter followed by a bracketed expression as a function (as opposed to multiplication of the letter outside the brackets by the expression inside), and due to the TI-Nspire CX calculator only producing the correct answer if the product kt is written with a * multiplication sign included, as $k * t$.

Figure 6.5

Fourth Task-based Interview Question Given to First-year Participants

<u>Question 4</u>

4. A particular quantity is known to have value $P(t) = P_0(1 + t^{2.5})e^{-kt}$ at time t, where P_0 and k are constants. Find the rate of change of the quantity P with respect to time t.

As shown in Table 6.10, if entered correctly this question can be answered in one step using either model of CAS calculator. Table 6.11 shows it requires more decisions to solve by-hand, as it is necessary to apply both the product rule and the chain rule for differentiation if answering by-hand.

Table 6.10

Decisions to Determine Grain Size of CAS Calculator Solutions to Question 4 (Version 1)

Result obtained	**Decision actions**
	Identify that will need to use the **derivative** command to find the derivative of P with respect to t, treating all other letters as constants
$\dfrac{-(2k.t^{\frac{5}{2}}.P0 - 5.t^{\frac{3}{2}}.P0 + 2.k.P0)e^{-kt}}{2}$	Use the command $\boldsymbol{d}(P0 * (1 + t^{2.5}) * e^{-k*t}, t)$ to find $\frac{dT}{dt}$ using the CAS calculator
	2 decisions

In showing the minimal number of decisions required to solve the problem if using CAS or solving the problem by-hand, Tables 6.10 and 6.11 indicate grain sizes of 2 and 7, respectively, depending on whether CAS is available.

The first part of the fifth question (Figure 6.6) for the first version was designed to see if students would write down the answer in the form provided by the calculator, suggesting potentially ritual behaviour, or if they would add the required constant of integration, as shown in Table 6.12. The second part is relatively quick to also work out by-hand, as is shown in Table 6.13, so I was interested to see if this would invite more participants to carry out a substantiation routine by also checking the calculator answer by-hand.

Table 6.11

Decisions to Determine Grain Size of By-hand Solutions to Task-based Interview Question 4 (Version 1)

Result obtained by-hand	**Decision actions**
	Identify that will need to differentiate P with respect to t, treating all other letters as constants
$\frac{dP}{dt} = v\frac{du}{dt} + u\frac{dv}{dt}$	Identify will need to use the product rule. E.g., could let $u = e^{-kt}$ and $v = P_0 * (1 + t^{2.5})$
$\frac{du}{dt} = -ke^{-k*t}$	Differentiate e^{-k*t} with respect to t (in one step with formula sheet provided in this subject).
$\frac{dv}{dt} = P_0 * 2.5t^{1.5}$	Differentiate $P_0 * (1 + t^{2.5})$ with respect to t
$P_0 * (1 + t^{2.5}).(-ke^{-kt}) + e^{-kt}.P_0 * \ 2.5t^{1.5}$	Apply the product rule $\frac{dP}{dt} = v\frac{du}{dt} + u\frac{dv}{dt}$
$-P_0 * k(1 + t^{2.5}).e^{-kt} + 2.5P_0e^{-kt}.t^{1.5}$	Simplify to get the final answer
	7 decisions

Figure 6.6

Fifth Task-Based Interview Question Given to Participants

<u>**Question 5**</u>

5. Determine $\int(sinx + cosx)dx$ and $\int_0^{\pi}(sinx + cosx)dx$

Tables 6.12 and 6.13 show solution paths which require a minimal number of decisions to be made; therefore the grain size of the task is very similar, at 4 and 5, when using CAS or working by-hand, respectively.

Table 6.12

Decisions to Determine Grain Size of CAS Calculator solution to Question 5 (Version 1)

Result obtained	**Decision actions**
	Make sure the calculator is in **radian** mode
$-cosx + sinx$	Use the command $\int(sinx + cosx, x)$ to integrate the function with respect to x
$-cosx + sinx + C$	Add a constant of integration to obtain the answer
2	Use the command $\int(sinx + cosx, x)$ to integrate the function with respect to x from 0 to π
	4 decisions

Table 6.13

Decisions to Determine Grain Size of By-hand Solution to Question 5 (Version 1)

Result obtained	**Decision actions**
$\int sinxdx + \int cosxdx +$	Recognise, from the linearity rule for integration, that $\int(sinx + cosx, x)\, dx$ can be split up into two integrals
$-cosx + sinx$	Use standard antiderivatives to determine that $$\int sinxdx + \int cosxdx = -cos + sinx$$
$-cosx + sinx + C$	Add a constant of integration to obtain the answer
$(-\cos(\pi) + \sin(\pi) - (-\cos(0) + \sin(0))$	Using the Fundamental Theorem of Calculus, determine that $\int_0^\pi (sinx + cosx)dx = (-\cos(\pi) + \sin(\pi)) - (-\cos(0) + \sin(0))$
$(--1+0) - (-1+0) = 2$	Evaluate the values of $sinx$ and $\cos x$ at 0 and π and substitute into the expression from the previous step to obtain final answer
	5 decisions

The next three questions discussed were only given to Participant D. This is because these questions covered differential equations topics from a core second-year mathematics unit, meaning that the first-year participants had not learnt about these topics. In addition, these differential equations questions were chosen to provide additional insight into how students approach using a CAS calculator to solve a variety of Calculus problems, with the specific reason for selecting each of these individual problems discussed below.

Figure 6.7

Third Task-Based Interview Question for Participant D

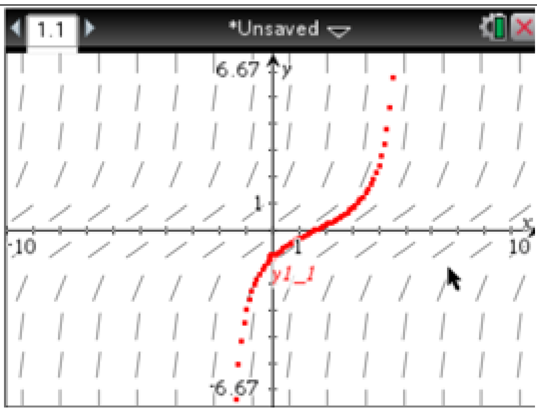
Question 3

3. Solve $\frac{dy}{dx} = \frac{1}{2}(1 + y^2)$ such that $y(0) = -1$. Write your answer in the form $y = f(x)$

Question 3 (version 2) given to participant D (Figure 6.7) was designed to see, firstly, if the student would produce a slope fields diagram, as shown in Table 6.14, as a visual mediator to assist in visualising the solution or for substantiating their algebraic solution was correct.

Table 6.14

Required TI-Nspire CX Commands and Resulting CAS Calculator Output for a Slope Fields Diagram with Solution Curve

Result obtained	**Calculator command**
	Go to a graphing screen, then select the **Graph Entry/ Edit → Diff Eq** menu. The differential equation can then be entered in this case as $y1' = (1 + y1^2)/2$ and the initial condition $(0, -1)$ can also be entered into this window. Press **Enter** to view the corresponding slope fields diagram, together with the curve of the solution.

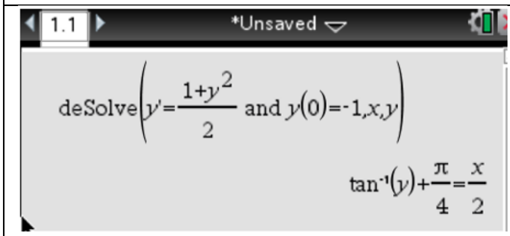
Secondly, it was intended to see if the student would use the **deSolve** command on the calculator correctly (Table 6.15) or whether she would be unsure how to do this and would instead take the more time-consuming process of solving the differential equation by-hand

(Table 6.16). Also, if the student used **deSolve** as instructed to in class, would it be used in a ritual way, by writing down the resulting output as the final answer, or would the student continue on to make y the subject, as generally required when solving an equation of this type? This problem was also chosen to see, when using CAS, whether it would be put into **radian** mode which is necessary to obtain the correct answer and whether the answer would be identified as incorrect if this was not done.

The solution paths shown in Table 6.15 and 6.16 have the minimal number of decisions required, based on the methods the students had been for solving these problems; therefore, the grain size of the task is 3 if CAS is available and 9 if required to solve it by-hand.

Table 6.15

Decisions to Determine Grain Size of CAS Calculator Solution to Question 3 (version 2)

Result obtained	**Decision actions**
	Use the command **deSolve**$(y' = \frac{1}{2}(1+y^2)$ and $y(0) = -1, x, y)$ to solve the differential equation for y
$tan^{-1}(y) = \frac{x}{2} - \frac{\pi}{4}$	Subtract $\frac{\pi}{4}$ from both sides.
$y = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right)$	Take *tan* of both sides to make y the subject.
	3 decisions

Question 4 (Version 1) in Figure 6.8 was chosen to again see if the student would produce a slope fields visual mediator to help in visualising the solution, as shown in Table 6.17. The form of the solution produced by the CAS calculator (Table 6.18) requires more steps (5) to solve for y than in the previous question (4). When working this question by-hand (Table 6.19), the final answer also appears differently, due to the fraction produced when solving for the constant C. I was also interested to see if she would attempt substantiation of the calculator answer by-hand, and if so whether she would simplify the final answer sufficiently to obtain the same result as that produced from rearranging the answer given by the calculator to make y the subject.

Table 6.16

Decisions to Determine Grain Size of By-hand Solution to Question 3 (Version 2)

Result obtained	Decision actions
$\frac{1}{1+y^2}\frac{dy}{dx} = \frac{1}{2}$	Separate the variables to get it in the form $g(y)\frac{dy}{dx} = f(x)$
$\int \frac{1}{1+y^2}dy = \int \frac{1}{2}dx$	Set up integration of both side (with respect to x)
$\int \frac{1}{1+y^2}dy = \arctan(y) + c_1$	Work out integral of the left-hand side
$\int \frac{1}{2}dx = \frac{1}{2}x + c_2$	Work out the integral of the right-hand side
$\arctan(y) = \frac{1}{2}x + c$	Rewrite the equation (with combined constant c)
$y = \tan(\frac{1}{2}x + c)$	Make y the subject
$1 = \tan(\frac{1}{2}(0) + c)$	Substitute in the initial condition $y(0) = 1$
$\tan(c) = 1, so\ c = \frac{\pi}{4}$	Solve for c
$y = \tan(\frac{1}{2}x - \frac{\pi}{4})$	Rewrite the solution of the DE, substituting in c
	9 decisions

Figure 6.8

Fourth Task-Based Interview Question Given to Participant D

Question 4

4. Solve $3x^3y' = y^4$ such that $y(2) = 1$

Table 6.17

Required TI-Nspire CX Commands and Resulting CAS Calculator Output for a Slope Fields Diagram with Solution Curve

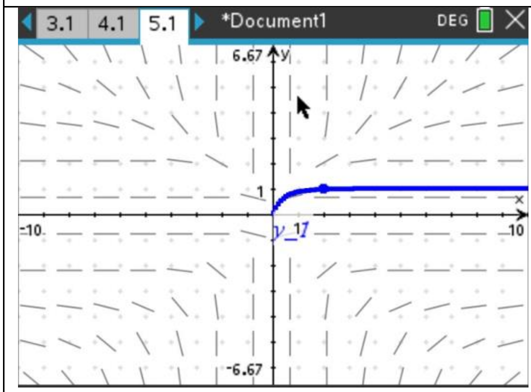
Result obtained	**Calculator command**
	Go to a graphing screen, then select the **Graph Entry/ Edit → Diff Eq** menu. The differential equation can then be entered, in this case as $y1' = \frac{y^4}{3x^3}$, and the condition $(2, 1)$ can also be entered into this window. Press **Enter** to view the corresponding slope fields diagram, together with the curve of the particular solution.

Table 6.18

Decisions to Determine Grain Size of CAS Calculator Solution to Question 4 (Version 2)

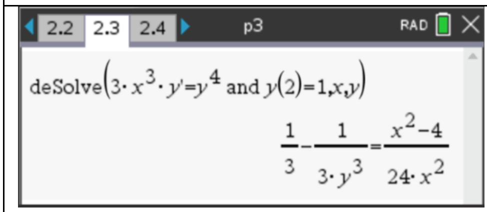
Result obtained	**Calculator command**
	**deSolve**$(3x^3y' = y^4 and\ y(2) = 1, x, y)$
$-\frac{1}{3y^3} = \frac{x^2-4}{24x^2} - \frac{1}{3}$	Subtract $\frac{1}{3}$ from both sides
$-\frac{1}{3y^3} = \frac{x^2-4-8x^2}{24x^2}$ $-\frac{1}{3y^3} = \frac{-4-7x^2}{24x^2}$	Rewrite the right-hand side as a single fraction
$-3y^3 = \frac{24x^2}{-4-7x^2}$	Take reciprocals of both sides
$y^3 = \frac{8x^2}{4+7x^2}$	Divide both sides by -3 to make y^3 the subject
	5 decisions

Table 6.19

Decisions to Determine Grain Size of By-hand Solution to Question 4 (Version 2)

Result obtained	**Decision actions**
$y^{-4}y' = \frac{1}{3}x^{-3}$	Separate the variables to get it in the form $g(y)\frac{dy}{dx} = f(x)$
$\int y^{-4}dy = \int \frac{1}{3}x^{-3}dx$	Set up integration of both sides (w.r.t. x)
$\int y^{-4}dy = \frac{y^{-3}}{-3} + c_1$	Work out integral of the left-hand side
$\int \frac{1}{3}x^{-3}dx = -\frac{1}{6}x^{-2} + c_2$	Work out the integral of the right-hand side
$-\frac{1}{3y^3} = -\frac{1}{6x^2} + c$	Rewrite simplified equation (with combined constant c)
$-\frac{1}{3y^3} = -\frac{1 + 6cx^2}{6x^2}$	Rewrite the right-hand side as a single fraction
$-3y^3 = -\frac{6x^2}{1 + 6cx^2}$	Take reciprocals of both sides
$y^3 = \frac{2x^2}{1 + 6cx^2}$	Solve for y^3
$1^3 = \frac{2(2)^2}{1 + 6c(2)^2}$ $1 = \frac{8}{1 + 24c}$	Substitute in the initial condition $y(2) = 1$ and simplify
$1 + 24c = 8$	Multiply both sides by $(1 + 24c)$
$24c = 8 - 1$ $c = \frac{7}{24}$	Solve for c
$y^3 = \frac{2x^2}{1 + 6 \times \frac{7}{24}x^2}$ $y^3 = \frac{2x^2}{1 + \frac{7}{4}x^2}$	Substitute in the value of c to find the solution

Or could write as $$y = \left(\frac{2x^2}{1+\frac{7}{4}x^2}\right)^{\frac{1}{3}}$$ or $$y = \left(\frac{8x^2}{4+7x^2}\right)^{\frac{1}{3}}$$	
	13 decisions

When solving this problem using a TI-NSpire CAS calculator of the type used at the time the student was completing the task, the solution path with the minimal number of decisions is shown in Table 6.18. This is also based on the expectation in this unit that, where possible, the solution of differential equations should be rearranged to make the dependent variable (y in this case) the subject.

Question 5 (Version 2) (Figure 6.9) was designed to see how students use and interpret the resulting solution produced by the CAS calculator. As shown in Table 6.20, it is different to what students would generate by-hand using the method of undetermined coefficients they were taught in class for solving differential equations of this type (Table 6.21). Therefore, whether the students would engage in a substantiation routine to try to reconcile the answer produced by CAS with what they would normally obtain by-hand was of interest.

Figure 6.9

Fifth Task-Based Interview Question Given to Participant D

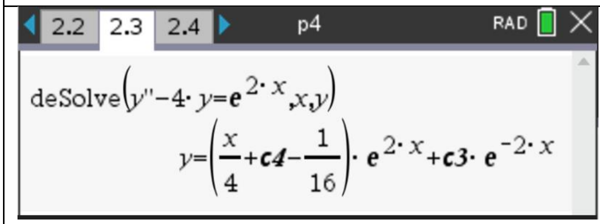
Question 5

5. Find the general solution of $\frac{d^2y}{dx^2} - 4y = e^{2x}$. Hence write down the complementary function and a particular integral.

Table 6.20 shows that only one decision is required if solving this problem using a CAS calculator, giving a grain size of 1. In contrast, if solving it by-hand, the minimal number of decisions required is much greater, with a grain size of 10.

Table 6.20

Decisions to Determine Grain Size of CAS Calculator Solution to Question 5 (Version 2)

Result obtained	**Decision actions**
	Use the command **deSolve**$(y'' - 4y = e^{2x}, x, y)$ to solve the differential equation for y
	1 decision

Tables 6.1 to 6.21 indicating the likely decision paths for solving each task show that, consistent with what was found in the previous chapter, the grain size of the tasks typically is much greater when working by-hand rather than with a CAS calculator. Table 6.22 summarises the grain sizes of each task, when working by-hand and when using CAS as part of the solution process; therefore, in cases where there were two or more possible solution approaches using CAS, the counts in the third column are for the method that required the least number of decisions to be made.

In most questions, this grain size was the same, regardless of whether a TI-NSpire CX or Casio Classpad 330 calculator was used. However, in question 3 (version 1) this differed for these two models of CAS, so the grain size is shown for each. These results indicate that question 3 required the most decisions and would also be likely to be the most *time-consuming* if solving it by-hand, while questions 5(a) and (b) required the least decisions and would correspondingly be considered the least time-consuming of the problems to solve by-hand.

Students' written responses and relevant parts of transcripts of their responses in the task-based interviews will now be analysed in section 6.3.

Table 6.21

Decisions to Determine Grain Size of By-hand Solution to Question 5 (Version 2)

Result obtained	**Decision actions**
$m^2 - 4 = 0$	Find the auxiliary equation associated with the homogeneous equation
$m = 2\ or - 2$	Find the roots of the auxiliary equation
$y_h = Ae^{2x} + Be^{-2x}$	Write down the complementary function
$y_p = Cxe^{2x}$	Find the general form of a particular integral
$y_p' = Ce^{2x} + 2Cxe^{2x}$	Differentiate y_p
$y_p'' = 2Ce^{2x} + 2Ce^{2x} + 2Cxe^{2x}$ so $y_p' = 4Ce^{2x} + 4Cxe^{2x}$	Find the second derivative of y_p
$4Ce^{2x} + 4Cxe^{2x} - 4Cxe^{2x} = e^{2x}$	Substitute y_p, y_p' and y_p'' back into the original differential equation
$4Ce^{2x} = e^{2x}$ So $C = \frac{1}{4}$	Solve for C
$y_p = \frac{1}{4}xe^{2x}$	Substitute the value of C into y_p
$y = Ae^{2x} + Be^{-2x} + \frac{1}{4}xe^{2x}$	Combine y_h and y_p to get the general solution of the differential equation
	10 decisions

Table 6.22

Grain size of each question given to students, when working by-hand or with CAS

	Grain Size		
Question	by-hand	with CAS	difference
1	6	1	5
2	13	2	11
3 (Version 1)	15	6 (Casio Classpad) or	9
		7 (TI-Nspire)	8
4 (Version 1)	7	2	5
5 (a) (Version 1)	3	3	0
5 (b) (Version 1)	2	1	1
3 (Version 2)	9	3	6
4 (Version 2)	13	5	8
5 (Version 2)	10	1	9

6.3 Commognitive Analysis of Students' Solutions and Narratives in Task-based Interviews

To assist in addressing Research Questions 2(a) and 2(b), the commognitive analysis of the participants' written work and their verbal and gestural responses supported by any actions on the CAS calculator in the interviews will now be presented for each of the task-based interview questions. I remind the reader of these research questions:

RQ2(a) How effectively do undergraduate Calculus students use their CAS calculator and use and interpret CAS output, especially when it is in a different format to what they would obtain by the methods of working by-hand they have been taught in class?

RQ2(b) What can we learn from a commognitive analysis of task-based interviews of students in relation to how they reflect on their answers in this situation?

An important factor potentially influencing university students' discourse is the *precedent search space* (Lavie et al., 2019; Viirman & Nardi, 2021) they are accessing. The students considered here, especially the three in first-year, were straddling two discourse communities: one from their upper secondary school education and one from their studies at university. This is associated with the *discursive footprint* (Biza, 2021) left by their previous studies, in this case, the discursive characteristics of how Calculus problems and the associated algebraic expressions have been signified and dealt with in different mathematical domains in the course of the students' studies, especially in their mathematical studies at school. In addition, they were also working within the discursive domain of CAS itself. A part, therefore, of analysing the students' narratives below is looking for any evidence of their accessing parts of their precedent search space that were associated with the discursive footprint left by their school studies of Calculus and Algebra. As well, any evidence of their making any transitions, via use of *metaphor* (Sfard, 2008) between school-based and university level mathematical *discourses*, or between these discourses and that of CAS, in producing *endorsable narratives* is sought. In addition, how the general nature of the discourse they produced when writing and talking about solving the set problems was affected by their use of CAS technology in solving these problems, will be considered.

The extent to which the students used *ritual* and/or *exploration routines* will also be examined, in relation to how effectively they were using and interpreting CAS in solving the tasks. In those tasks where *potential intrapersonal commognitive conflicts* were expected to arise (i.e., Questions 2 and 4), further analyses of these are undertaken, to determine if these students did indeed encounter such conflicts and, if so, whether they demonstrated any awareness that they were obtaining incorrect results.

6.3.1 Commognitive analysis of students' responses to question 1

6.3.1.1 Narratives around required calculator use

As described in section 6.2, Participants A, C and D had a TI-Nspire CX CAS calculator and Participant B used a Casio Classpad 330 calculator. As was shown in Table 6.1, in question 1 the two types of calculators produced very different output in answering the same question. In particular, the TI-Nspire CX calculator presented the answer in the form $\frac{2(x^2+1)^{\frac{2}{3}}}{3x^{\frac{1}{3}}}+\frac{4x^{\frac{5}{3}}}{3(x^2+1)^{\frac{1}{3}}}$, whereas the CASIO Classpad 330 presented it as $\frac{2.(3.x^2+1)}{3.(x^3+x)^{\frac{1}{3}}}$, which is similar, but not identical, to the form of the answer obtained by-hand when the chain rule is used (see Table 6.2). This was taught in the classes in the mathematics subject the students were currently studying (participants A, B and C) or had studied previously (i.e., participant D).

As shown in Figure 6.10, all four participants obtained the correct answer when using the calculator to answer this question. They all answered by *performing a direct, straightforward procedure* with their CAS calculator (Thomas & Hong, 2005, p. 219), in directly using the **derivative** command on the calculator to obtain the answer in one step. They all produced written command-based narratives regarding the menus required on the CAS calculators to obtain the derivative, using the discursive domain of CAS itself. Participants A and D stated that you go through the series of steps **Menu → Calculus → Derivative**, with participant A adding, “then input equation and press **enter**”. He had begun to write “function” instead of “equation” but crossed it out, suggesting his potentially temporarily moving more into the endorsed narrative of mathematics. Participant C wrote as a list of dot points “Go into **calculate** in scratchpad”, “Go into **calculus** then **derivative**” and “Then put the equation into the space provided” which is a more descriptive narrative of the screen environment in which the student is positioned when working out this problem on the CAS calculator. Participant B wrote the narrative of the required commands path in a more abbreviated, symbolic form as **2D → CALC →** $\frac{d}{dx}$**((x^3+x)^2/3)**.

6.3.1.2 Types of routines and CAS procedures used

This was a potentially ritual approach by the three first-year participants, in relying solely on the standard differentiation procedure from their calculator. Students A and B both noticed the CAS calculator answer was different from what would be the form expected with Student A, who had his calculator in **Approximate** (decimal) mode, saying it was more complicated, as this interaction with the interviewer (I) in the task-based interview shows.

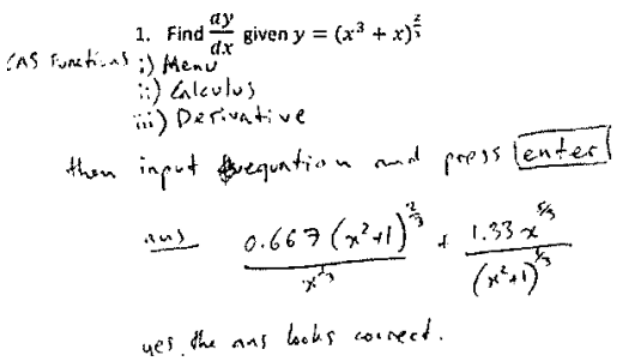
I: For the first one, you went to **Calculus** [menu] and the **Derivative** function and you put the equation and you got this answer and you thought that looked correct. Was that answer in the form you would have expected it to appear?

A: Ah, no, it looks a bit more complicated.

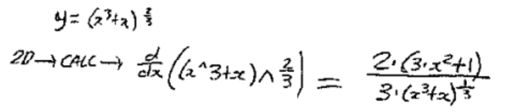
Figure 6.10

Written Solutions and Responses to Task-based Interview question 1 from Participants A-D

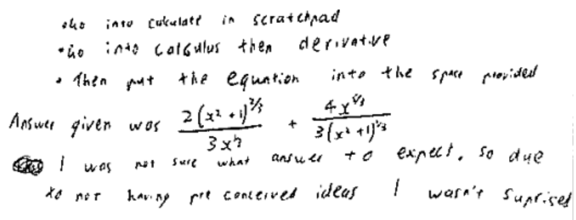
(a)



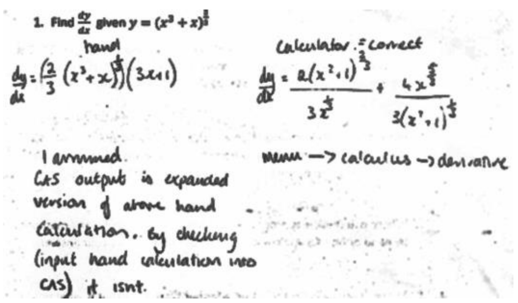
(b)



(c)



(d)



I: And how would you have usually solved this? Would you prefer to use the CAS calculator or would you usually do that by-hand if you were given a choice?

A: I would probably use the calculator because faster.

I: Then you think the answer looks correct even though it looks more complicated than the usual thing?

A: Yep.

I: Yep. Have you got any thought about why it might be presented like that? Rather than in the usual form?

A: Um, no I'm not sure why.

Student B, whose calculator was in **Exact** mode, was clearly capable of doing the differentiation "mentally" but was prepared for a calculator answer to be different to what he would "write in an exam or test".

I: So you've used the $\frac{d}{dx}$ command. Was that answer in the form you would have expected?

B: Let me see, yeah expected from the calc, yes. Wouldn't be the form I would probably write it like in exam or test, but it's about expected from the calculator.

I: What would be different about the way you would write it in an exam or a test?

B: Ah, just how I work it out. Um, I'd have $^2/_3$ outside then I would probably have $3x^2 + 1$ over cube root $x^3 + x$ (*using colloquial narrative*).

I: And do you think that answer looks correct?

B: Ah yeah, it looks good.

I: And if you were doing this yourself on an exam or elsewhere would you usually prefer to use the calculator for this problem or would you prefer to do it by-hand?

B: I would probably do it by-hand then check it with the calculator.

As indicated in Figure 6.10 (c), Student C wrote that he had no expectation of what the calculator output would be so was accepting of it.

Only participant D also attempted the question by-hand and compared the result with the answer she obtained on the calculator. When commenting on the results of both actions, participant D wrote that: "I assumed CAS output is expanded version of above hand calculation. By checking (*inputting hand calculation into CAS*) it isn't". Participant D appeared to be *attempting a substantiation routine*, by comparing the answer obtained with the calculator to that obtained by-hand. Unfortunately, participant D made an error in the answer worked by-hand using the chain rule, obtaining $\frac{dy}{dx} = \frac{2}{3}(x^3 + x)^{1/3}(3x + 1)$, whereas $x^3 + x$ should have been raised to the power of $-\frac{1}{3}$. This made it more difficult for her to see any correspondence between the two answers as they were no longer equivalent, due to the error. Participant D identified this inconsistency and that the answer obtained by-hand was incorrect, but did not know why as is confirmed in this excerpt.

I: Ok, so you've worked that out by-hand first of all? (*looking at answer to question 1*)

D: I tried to.

I: Yep. So, using chain rule by the look of that?

D: Yep.

I: Okay, so you assumed the calculator output was correct?

D: Yep, and then, only getting 1 on 3, the calculator is correct, ‘cause I wouldn’t ’ve got that (*using colloquial narrative*).

I: Yep, so that’s what your calculator output is, so you recognise the correspondence with that? (*pointing to hand calculated answer*)

D: Yep.

I: And then you say by checking (input hand calculation into CAS), it isn’t. So, what did you mean by that?

D: So, my hand calculation was not correct.

I: So, you believed the calculator, but you didn’t believe what you had done by-hand? Is that right?

D: Yep. (*Calculator as authority*.)

Furthermore, this faith in the authority of the calculator was a result of metacognitive task knowledge related to previous experience, saying, “I will always, or seem to always, get these wrong, so I just go straight to the calculator”.

The interview responses above show that all the participants used a *direct, straightforward procedure* (Thomas & Hong, 2005, p. 219) in the form of a *ritual routine* with their CAS calculators to obtain the answer. However, their attitudes towards use of CAS and their reasons for using it in this situation varied. Firstly, A and C were reasonably confident in their answer and they expressed a preference for using CAS to solve this type of problem, to save time (A and C). In contrast, Participant D expressed a low level of confidence in her by-hand answer (and in her ability to correctly answer problems of this type by-hand) and that she would therefore usually go straight to using the calculator to solve problems of this type, while Participant B also expressed a preference for solving a problem of this type by-hand, before checking the answer using CAS.

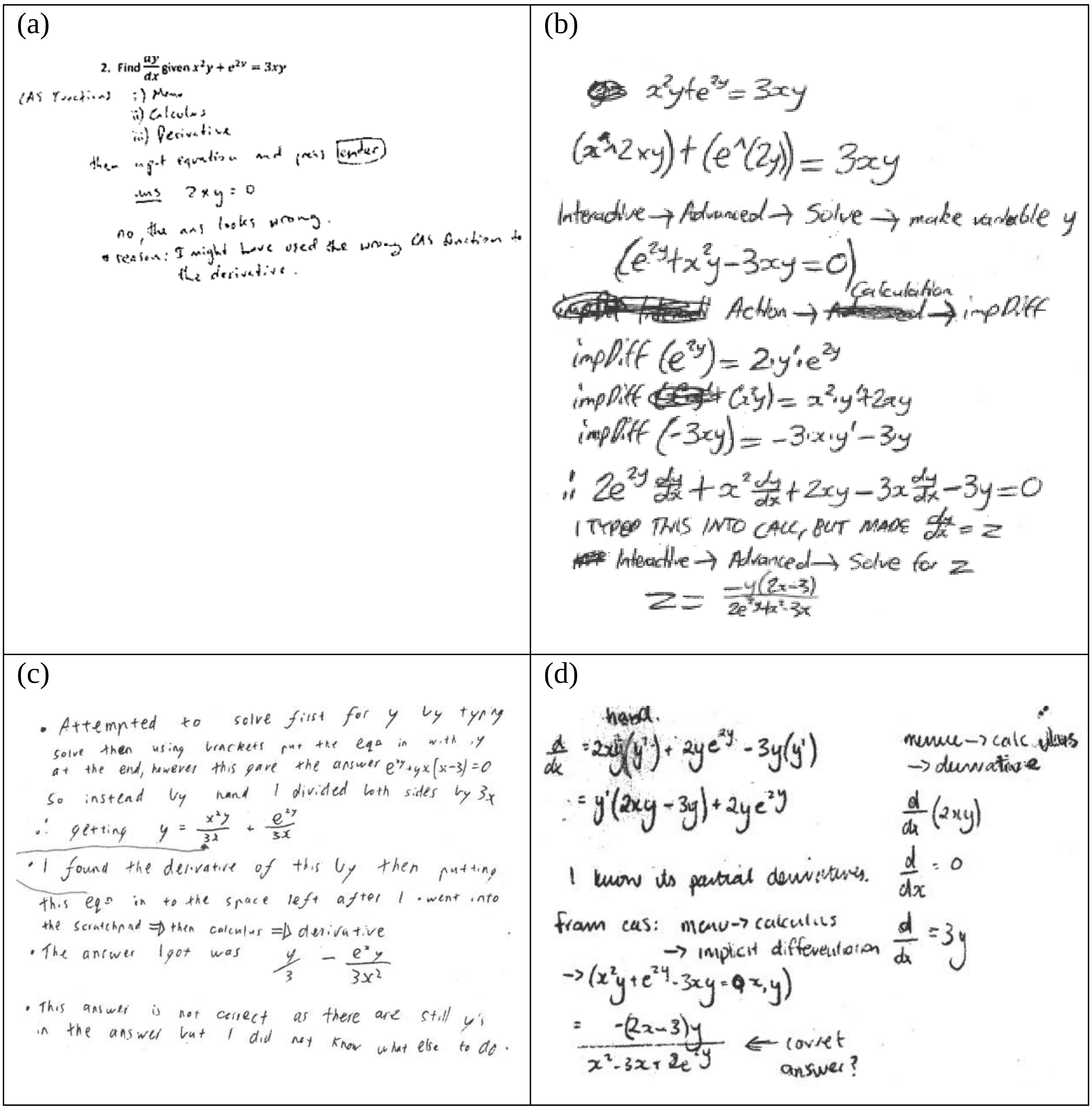
For this question, no potential commognitive conflicts were expected and none were realised in the students’ responses or interviews.

6.3.2 Commognitive analysis of students' responses to question 2

The written attempts made by each student are shown in Figure 6.11

Figure 6.11

Written Solutions and Responses to Task-based Interview Question 2 from Participants A-D



6.3.2.1 Narratives around required calculator use

Only participants B and D obtained the correct answer to this question. Participants A and C incorrectly used the **differentiation** command on their calculator that produces partial derivatives, which will be discussed further in the section below on commognitive conflicts. Therefore, their narratives on trying to produce the (full) derivative of the expression with respect to x were not endorsable.

As in question 1, participant B produced a *written narrative*, as shown in Figure 6.11(b), which could be seen as an *endorsed narrative* localised to the model of calculator being used. This showed steps that could be used to obtain the required derivative in a relatively abbreviated, command-based form, referring to commands to be used in the calculator menu environment, including the **impDif** command required to produce derivatives of the correct type. There is also on his first line of writing, an instruction to “make variable y” which is not a menu or command option but which is an *action-based narrative* in which he was advising what a person needed to then do in that step, to solve for y. In the second-last line of his narrative on how to find the derivative, he also referred to his own actions on the calculator as follows: “I typed this into calc, but made $\frac{dy}{dx} = z$.”

Participant D initially used an incorrect process/commands, giving a *non-endorsable narrative*. She then switched to a *menu/command-based endorsable narrative* of how to correctly obtain the required derivative as follows:

From cas: menu → calculus → implicit differentiation

→ $(\boldsymbol{x^2y + e^{2y} - 3xy = 0, x, y})$

6.3.2.2 Types of routines and CAS procedures used

There was no evidence of any of the participants attempting substantiation routines by-hand for this question after writing down their final answer attempt from using their CAS calculator. However, participants B, C and D changed their method for attempting to solve the problem after one or more initial steps, suggesting that even with what appears to be *ritual* behaviour, they were still also critically assessing their answers along the way, even if by “appearance” rather than by substantiation routines. For example, participants B and C both initially attempted to solve the equation $x^2y + e^{2y} = 3xy$ for y, which could be considered a *ritual* in applying the **Solve** routine on the calculator. Participant A’s written response also indicated that he judged his incorrect answer of $2xy = 0$ to look “kinda not right because it had y on the other side as well” (*colloquial discourse*) which is not the actual *cause* of why his answer was incorrect. This preoccupation with solving for y could indicate these participants were accessing their *precedent search space* from high school, where solving to make a variable (y in this case) the subject is a common routine in solving high school algebra problems. Participants B and C, however, then realised that this did not help in solving this particular problem. This also suggests potential recollection by the students that part of the process of implicit differentiation is rearranging the equation (which is required at the final step if working by-hand, to determine $\frac{dy}{dx}$).

The following excerpt of the interview with participant B illustrates his thought process in doing this, and how he then proceeded to use the **impDif** command to carry out implicit differentiation of each term, followed by, at the final step, reverting to using the **Solve** command on the calculator to obtain the result.

B: And I just gave it a crack, like try and solve for y and I was thinking that, if I got it in terms of y, I could easily do $\frac{dy}{dx}$ but the calc didn't like that, and pretty much just put that one that side and set it equals 0 (*using mathematics endorsed narrative*). So I implicitly differentiated with the calc, each of those three (*switching to the discursive domain of CAS?*).

I: So you did them individually term by term with the calculator?

B: Yeah, yeah just wanted to play it safe. I just assumed that y apostrophe was $\frac{dy}{dx}$ (*using colloquial discourse, but in the discursive domain of CAS itself*).

I: Yep.

B: Then I just did that myself but I wanted to get it, you obviously want to get it in terms of $\frac{dy}{dx}$, so I made that (*pointing to where he had written* $\frac{dy}{dx}$) equal z and then I typed that in (*pointing to the expression he has entered into the calculator, with* $\frac{dy}{dx}$ *replaced by* z *and using the* ***Solve*** *command to solve for* z) and solved for z (*using mathematics endorsed narrative*).

I: Right, okay, so that was an interesting one because you realised that wasn't quite working so then you went on to this sort of process. Did you think the answer you got there looks correct?

B: Doubtful, but it looks all right.

I: Yes, that looks pretty good, what makes you doubtful about that answer?

B: I don't usually do implicit differentiation with the calculator (*using endorsed narrative of tertiary mathematics*), I usually would do that one by-hand.

I: Yeah, and so you were a bit doubtful about it?

B: Yeah.

This demonstrated the student's awareness that, when using his CAS calculator, the **impDif** command should be used to find the answer, however the method he used was not the most efficient way of using this command. His cautious term-by-term approach probably reflected his unease with his calculator prowess in this particular process. As Table 6.3 in section 6.2 indicated, this command could be used to solve the problem in one step, whereas implicitly differentiating each term one at a time and needing to finally solve for $\frac{dy}{dx}$ greatly increases the number of decisions required to solve the problem, as demonstrated in Table 6.4, although still not as many decisions as when solving the problem by-hand (see Table 6.5). Participant B's comment that he would normally solve a problem of this type by-hand was consistent with the process he has used with the calculator, as it appears to be *ritual* behaviour in using the standard steps for implicit differentiation, in first differentiating each term and then solving for $\frac{dy}{dx}$, however this process was applied to the calculator input rather than being done by-hand. This is an example of using CAS to *perform a direct procedure*, but *implementing a new, CAS-based technique* for doing so (Thomas & Hong, 2005, p. 220).

Participant C's solution process commenced in the same way as for participant B, in first attempting to solve for y. He commented in his by-hand working that: "Attempted to solve first for y by typing **solve** then using brackets put the eqn in with y at the end" (*using the discursive domain of CAS itself*), stating that, however, this gave the answer $e^{2y} + yx(x-3) = 0$. Realising that this would not help him to find the answer, Participant C's subsequent thought processes, indicated by his written discourse in Figure 6.11(c), suggests that he became preoccupied with still believing that he needed to make y the subject, with his next written comment being: "So instead by-hand I divided both sides by $3x$, getting $y = \frac{x^2y}{3x} + \frac{e^{2y}}{3x}$" (*using mathematics discursive domain*). This step did not actually achieve anything in answering this question, as y was still present on both sides of the equation. He then carried out what he believed was differentiation of this expression, however, as will be discussed in the next section, he unknowingly encountered a commognitive conflict in using the incorrect differentiation command. This was demonstrated by his written comment that: "I found the derivative of this by then putting this eqn into the space left after I went into the scratchpad ⇒ then **calculus** ⇒ **derivative**" (*using the discursive domain of CAS itself*), which he reported gave the answer: $\frac{y}{3} - \frac{e^2y}{3x^2}$. In assessing this answer, as seen in Figure 6.11(c), participant C then commented that "this answer is not correct as there are still y*'s* in the answer (*not an endorsed narrative*) but I did not know what else to do".

This demonstrates Participant C's aforementioned misconception in believing that y must be the subject, with the comment on his final answer indicating a belief that y must not appear in (the right hand side of) the answer. In fact, the correct answer does also contain y and Participant C appeared unaware that he had both used the incorrect differentiation command and had also written part of the final answer incorrectly, in writing e^2y, whereas it should still be e^{2y}. As with Participant B, the approach Participant C took, in attempting to answer this question, was another example of attempting to use CAS *to perform a direct procedure* which included implementing a new, CAS-based technique. However, participant C was unsuccessful in this instance, partly due to not using the correct differentiation command and partly due to misunderstanding the final objective, which was to make $\frac{dy}{dx}$ the subject (not y, which was what he had suggested).

Participant D eventually obtained the correct answer, but initially used an incorrect method, in using the calculator's **derivative** command to attempt to obtain the answer. She then also wrote a line of working which indicated subsequent incorrect differentiation of each term by-hand:

$$\frac{d}{dx} = 2xy(y') + 2ye^{2y} - 3y(y')$$

In commenting on these first, incorrect steps, as the conversation below shows, she appeared confused about what the **derivative** command on the calculator was actually showing, stating twice that it cannot do a partial derivative, when that is actually what the calculator command produces. This suggests the student was encountering a *commognitive conflict* when using this command, which will be discussed further in the next section.

I: Yep. Okay, so looking at this next one (*looking at the student's answer to question 2*). So it looks like that was by-hand, and then you did it on the calculator.

D: Tried to do it on the calculator. I know I can't do partial.

I: So what did you think about the calculator answer, did you think that was correct?

D: I didn't think it was correct because I know the calculator can't do a partial derivative (*using tertiary mathematics endorsed narrative*).

Participant D was also not confident in her working by-hand in the second line of writing on the left in Figure 6.11(d), where she wrote an incorrect result of:

$\frac{d}{dx} = 2xy(y') + 2ye^{2y} - 3y(y')$ (*not an endorsed narrative in either upper secondary or tertiary mathematics*)

I: And what about the hand-calculation one, did you think?

D: No. Again I need, I'm better off when I have my rules.

I: That's right, yep, when not sprung, sort of? *(meaning when not knowing what types of questions would be asked and not having access to course notes or a formula sheet).*

D: Yep. So, I sort of tried what I could without a rule.

This conversation suggested this participant had a preference for being able to access and use a set of *ritual rules* for differentiation, of the type found in *formula sheets* provided to this student cohort when sitting their Calculus examinations.

The transcript below shows the part of the discussion where participant D talked in the interview about using the **impDif** command, which was the only part of her working for this question which was correct, but which, on its own, resulted in the correct answer due to the use of this command, in itself, being a one-step *direct straightforward CAS procedure* (Thomas & Hong, 2005). The resulting discussion below is also interspersed with her talking about the incorrect hand-calculation she had also attempted. She again also made reference to partial (differentiation), suggesting overall confusion and uncertainty about whether or not she was actually using the correct procedure, even when she eventually obtained the correct answer.

D: So then I went to that (*pointing to calculator*) and implicit diff there. (*using the discursive domain of CAS itself*)

I: So implicit differentiation? (*using endorsed narratives of tertiary mathematics*)

D: Yes.

I: And that's what you have to do, yep.

D: And have to remember, with partial (*not an endorsed narrative*).

I: Yep. Well what it is, it's interesting, because that's actually treating both x and y as variables, whereas this one is doing the partial derivatives here when you do that *(pointing to the first part of her answer attempt where she had used the differentiate command which produced partial derivatives of each term with respect to x)*.

D: Yep.

I: So when you did that (*pointing to first two lines of student's working*), you went **Derivative** (*using CAS discursive domain*), how did you do that there? You did term by term did you there? (*switching to mathematics discursive domain*)

D: Yep, just term by term to begin with (*using mathematics discursive domain*). But I knew it wasn't correct, because there ended up having to be the y in there (*pointing to the second line of student's working*) and I was like, okay, how am I going to do this?

I: How did you know it wasn't correct, sorry?

D: Because I knew that there had to be *y's* in there as well, and that it ended up having to have the y dash (*using mathematics discursive domain*).

I: Yep.

D: But without having a rule in front of me, it's hard to remember.

The following conversation regarding participant D's preferred solution method was also informative in relation to identifying the type of routine she preferred.

I: And which method would you prefer there if you were working that one out?

D: Probably calculator unless I had a rule in front of me. I like to do things by-hand, but when I don't have a rule in front of me, and when I can't remember how to do it, I, yeah, [go] straight to the calculator.

I: And if you had the rules?

D: If I had the rule, I'd probably try it by-hand and then check with the calculator, to see if I got the same one. If I didn't, I would go back and see where I'd gone wrong.

The above suggested a preference in participant D's general practice in problem solving for starting with a ritual routine for the problem solving process with the appropriate "rules" but also having a willingness or being motivated to then shift to a *substantiation routine* to check if the answer is correct, by working out a given problem both by-hand and on the calculator when possible.

It can be seen that none of the participants were very confident in their final answers, with Participants A and C stating outright that they thought their answers were incorrect and Participants B and D expressing doubt about the correctness of their answers. Participant D expressed a preference for using a calculator to solve problems of this type unless she had access to a *rule* (such as in a formula sheet). If she had a rule, she stated she would probably attempt such a problem by-hand and then check the answer with a calculator. Participant B expressed that he would usually do it by-hand due to less familiarity with the implicit differentiation command on the calculator and feeling that the calculator would not save time in solving it. Participant A would prefer to use CAS because it is "much faster." Participant C stated that he would probably use the calculator "but wouldn't really know," with this statement of his suggesting a general lack of confidence in solving a problem of this type.

6.3.2.3 Commognitive conflicts

One potential source of commognitive conflict referred to above that occurs in answering question 2 is in relation to the **derivative** command on the calculator. In standard classroom mathematical discourse, both at school and university, d represents a full derivative, while ∂ represents a partial derivative. However, on CAS calculators, the command d only calculates a full derivative if an expression involving just one variable is entered. If there are two or more variables, it will calculate a partial derivative, thus treating any other letters in the entered expression as 'constants'. This has the potential to produce a commognitive conflict, as the calculator is using and requiring the user to interpret the mathematical signifier d in a different way to that which is taught in standard mathematics courses (and which is also widely used in mathematics more generally) and which is therefore part of each student's *precedent search space*. In the resulting calculator output it shows, for example, $\frac{\boldsymbol{d}}{\boldsymbol{dt}}(\ )$ if differentiating with respect to t. For this question, I was looking to see if such a commognitive conflict would occur, because using the **derivative** command in question 2 will give the incorrect answer, as x and y are both variables.

The **derivative** command was incorrectly used by three of the four participants, but one of them (participant D) then realised, upon seeing the result, that the answer was incorrect and then adapted her working accordingly to eventually obtain the correct answer. This commognitive conflict was illustrated by, for instance, participant C's written comment that: "I found *the derivative* of this (*referring to a rearranged form of the equation provided in question 2*) by then putting this equation into the space left after I went into the Scratchpad ⇒ then **calculus** ⇒ then **derivative.**" This indicates he understood the calculator's differentiation command (and associated screen output) as signifying the usual full derivative of the expression (as opposed to what it actually represented, which was a partial derivative).

Another potential source of commognitive conflict associated with question 2 relates to how some CAS calculators name variables, and this is also model dependent. In particular, the TI-Nspire CX CAS calculator identifies an expression involving two or more consecutive letters as a distinct 'new' variable. This can generate a commognitive conflict between the discourse of the calculator (with its interpretation of symbolic visual mediators) and that of the students' classroom mathematical algebraic discourse from both high school and university. This is because in the mathematical discourse of algebra, when working 'by-hand', seeing a symbolic visual mediator that shows two consecutive letters, if they have elsewhere already been established to both be distinct variables (or constants), will signify their being multiplied together. For example, in question 2 when working with variables x and y, typing xy would usually signify "x multiplied by y", whereas on a TI-Nspire CX CAS calculator, it signifies the existence of an additional 'new' variable, named xy.

This question was analysed, to see if this source of error and commognitive conflict occurred for any of the 3 participants who were using a TI-Nspire CX CAS calculator in doing these tasks. On a Casio Classpad 330 CAS calculator, which was used by the remaining participant, writing two letters together does signify multiplication. Participants A and D both omitted the multiplication sign between x and y when typing the expression to be differentiated into the calculator, which led to incorrect results in question 2, as these two participants were using the TI-Nspire CX model of the calculator. This resulted in a commognitive conflict due to the calculator interpreting the input as signifying a separate variable, xy, while these two participants clearly assumed it signified "x multiplied by y". While both of them believed their answer to be incorrect, they did not know why.

6.3.3 Commognitive analysis of students' responses to question 3 (version 1)

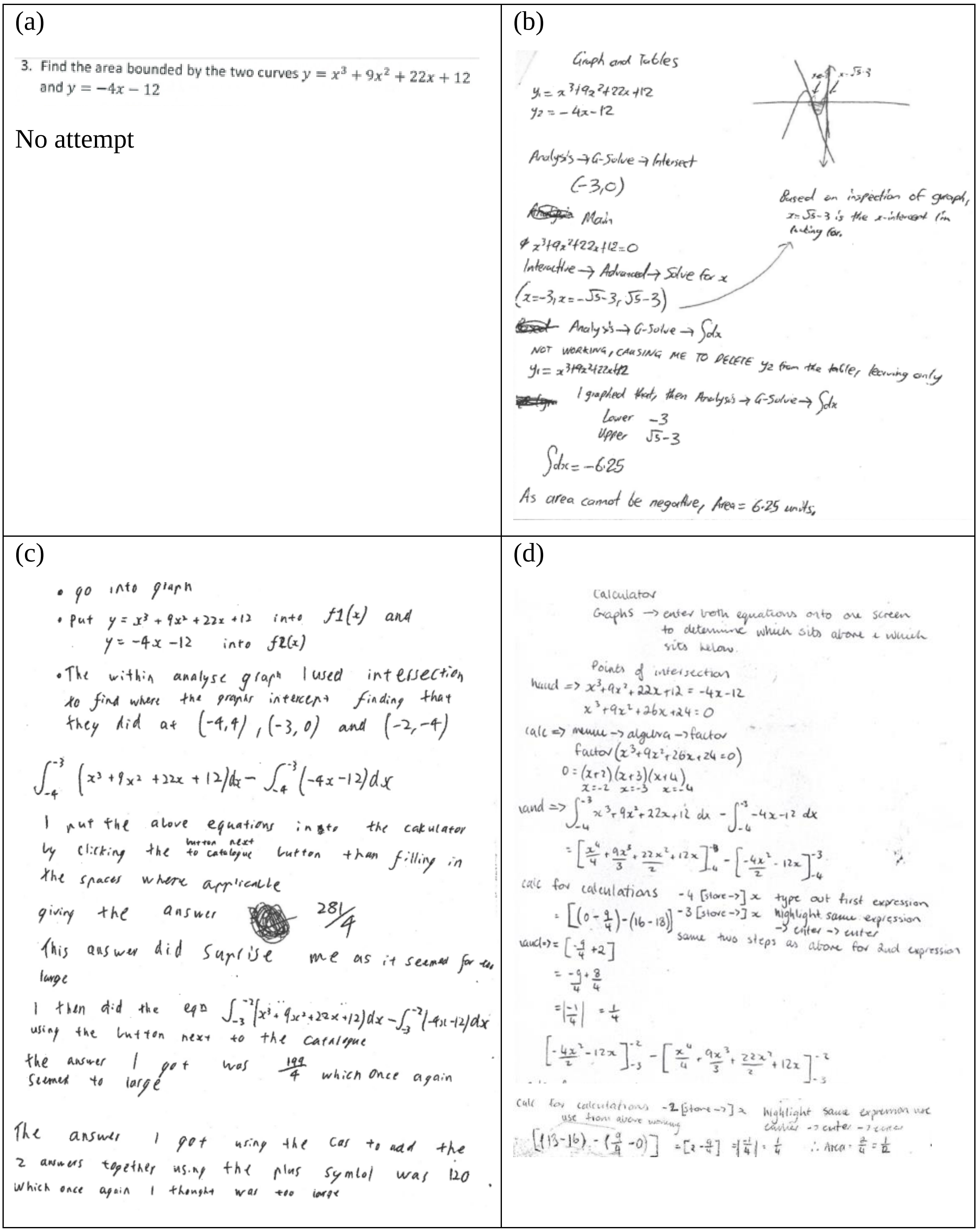
Figure 6.12 shows Participants B, C and D's attempts to respond to question 3 (version 1).

6.3.3.1 Narratives around required calculator use

Only three of the four participants (B, C and D) attempted this question. All three produced a graphical visual mediator on the calculator, showing the two curves. Participant D wrote an instruction that could potentially be considered as *a written endorsable narrative* in relation to this: "enter both equations onto one screen to determine which sits above and which sits below" (*referring to the relative position of the curves in relation to each other using CAS discourse*). Participant C produced written *command-based narratives*, using the discursive domain of CAS, writing: "Go into **graph**. Put $y = x^3 + 9x^2 + 22x + 12$ into $f1(x)$ and $y = -4x - 12$ into $f2(x)$. Then within **analysis - graph** I used **intersection** to find where the graphs intercept [*sic*]." Participant B also used the discursive domain of CAS, providing a narrative of the required commands path in a more abbreviated, symbolic form, interspersed with his working in attempting to solve the problem, which is consistent with the type of CAS command-based discourse produced by him for the other tasks. He first wrote **Analysis** → **G-Solve** → **Intersect** when attempting to find where the two graphs intersected, followed by **Interactive** → **Advanced** → **Solve for x**, and then **Analysis** → **G-Solve** → $\int \boldsymbol{dx}$ when attempting twice to find the required area. In the written working, participant B also produced the *endorsed narrative*, from the discourse of mathematics, that "area cannot be negative."

Figure 6.12

Written solutions and responses to task-based interview question 3 from participants A-D



Participant B also used the discursive domain of CAS, providing a narrative of the required commands path in a more abbreviated, symbolic form, interspersed with his working in attempting to solve the problem, which is consistent with the type of CAS command-based discourse produced by him for the other tasks. He first wrote **Analysis → G-Solve → Intersect** when attempting to find where the two graphs intersected, followed by **Interactive → Advanced → Solve for x**, and then **Analysis → G-Solve →** $\int dx$ when attempting twice to find the required area. In the written working, participant B also produced the *endorsed narrative*, from the discourse of mathematics, that "area cannot be negative."

6.3.3.2 Types of routines and CAS procedures used

Based on the comments made in the task-based interview, participant C zoomed in on the graphs to assist in viewing the area between the curves and then looked at the curves to assist in determining the required areas, as this interaction with the interviewer shows when describing finding the required areas bounded by the curves between $x = -4$ and $x = -3$ and between $x = -3$ and $x = -2$.

I: So why did you split it up in this way -3 to -4 and -2 to -3?

C: Because that's where the intercepts [*sic*] were and also that's where on the axis like, that said -3 to -4. I was like that will find, like that blue bit saying this is it and that'll find this bit here (*pointing to the area between the cubic curve and the x axis between -4 and -3)* and then I take away that red bit (*pointing to the straight line curve)* and then I'll be able to find the area (*referring to the area from -4 to -3 between the two graphs the student had drawn on his calculator, where the cubic was shown in blue on his calculator screen and the linear function was shown in red).*

I: Yes.

C: And then it was the same on the other side (*referring to the area between the curves from –3 to –2)*.

I: Yep, that makes sense, and so you zoomed in did you here, to get the picture like that?

C: Yeah, and I zoomed in on the other side as well, yeah (*using the discursive domain of CAS*).

I: Yep, so that bit's the –2, –3 bit for instance.

C: Yeah.

I: So there you were saying that this one here was (*pointing to the blue cubic curve on the student's calculator screen between –3 and –2*)?

C: It was like, further away from the axis than the red one, so I've done that first and taken away the other one.

These comments suggested potentially *exploratory substantiation routine* behaviour and also indicate that this student treated the graph as a *dynamic visual mediator*, by zooming in on it. Participant C also made *direct use of CAS within a mathematical process* (Thomas & Hong, 2005), first using CAS to sketch the graphs and then to find their intersection points by using the **intersection** command, working within the graphing window. He then wrote down the definite integrals required to find the two bounded areas, working these out directly using the calculator's **integration** command. He then added the two resulting areas to obtain the total area bounded by the curves (although, due to typing in one of the equations incorrectly when working out the definite integrals, the answer he obtained was incorrect). This shows that he used the CAS calculator for each individual step, but also needed to set up the correct definite integrals and to add the two bounded areas to obtain the final answer.

This participant also appeared to be critically evaluating his answers, evidenced by his stating in his written comments that, "This answer (*of 281/4 for the first area)* did surprise me as it seemed far too large". Similarly, he commented that the second area "seemed too large" and in relation to his final answer also commented that, "the answer I got using the CAS to add the 2 answers together using the plus symbol was 120 which once again I thought was too large" (*using mathematics discursive domain*). These comments suggest he was potentially attempting to substantiate his answer by linking his numerical answers to the small areas shown in the graphical visual mediator produced by his calculator, and consequently identifying an inconsistency, even though he did not know its cause. His final answer was incorrect, due to a typing mistake in entering the functions into his calculator, despite all his written working appearing correct. The result was that he obtained an answer of 120, when the correct answer was only ½, indicating good critical judgement that something was wrong with his final answer in it being so large.

When questioned, the other two participants (B and D) who attempted this question stated that they did not do any zooming in on the graph.

Participant D also produced the graphs of the curves, commenting on the need to see which curve was 'above' and which curve was 'below', as shown in this interaction with the interviewer.

D: So, in **graphs** the first one (*using CAS discursive domain*), so first of all I put both, um, equations in, to like graph them on my calculator on the same screen, just to see which one, like where they sat.

I: Yep.

D: So which one was above, which one was below. Um, and then I went back to finding the points of intersection by-hand *(using an integrated approach, as in Thomas & Hong, 2005)*

Participant D obtained the correct answer despite not zooming in and, like participant C, commented on the size of the value in the answer in judging its correctness (which was indeed correct in this case).

I: And do you feel that answer's correct? (*Referring to the student's final answer of* $\frac{1}{2}$)

D: Yes, because it looks pretty small (*using mathematics endorsed narrative*).

I: Yep, from looking at the graph. Did you have to manipulate or zoom the graph at all?

D: Nup. No, I just trusted that it was correct.

I: You could see the areas on it?

D: Yep. Oh, sort of just see the areas, but they were very small, so a half is.

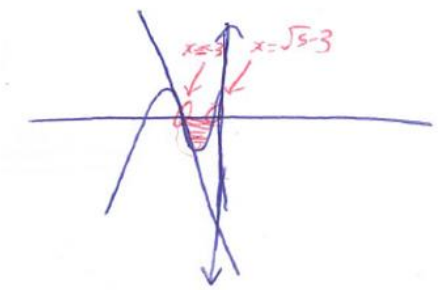
This suggested less exploratory behaviour than Participant C, but Participant D relating the perceived correctness of her answer to the size of the areas shown on her calculator screen indicated she was making a connection between the graphical visual mediator and the numerical answer she obtained. This suggests, as with participant C, a potential attempt to substantiate her answer by linking it to the small areas shown in the graphical visual mediator.

The above discussion indicates that Participant D demonstrated use of CAS in "performing a procedure within a more complex process, possibly to reduce cognitive load" (Thomas & Hong, 2005, pp. 221-222), with a "partnership evolving with CAS assigned a defined role within the overall solution process" (p. 221). She first graphed the curves, using the CAS calculator. Then, she set the equations of the curves to equal each other, simplifying the resulting expression by-hand to obtain a cubic expression equaling zero. She then used the **factor** command on the CAS calculator to factorise the cubic expression and used this to determine by inspection the solutions to the cubic equation, thus obtaining the x coordinates of the intersection points of the curves. She then wrote the definite integrals required to find the two bounded areas. In then evaluating these definite integrals, she stored the values for the upper and lower limits of each one, in the CAS calculator's memory and then accessed these as arguments of the resulting antiderivative functions to obtain $F(a) - F(b)$ in each case, in helping to evaluate the definite integrals more quickly than if done fully by-hand. She then added the two resulting areas to obtain the total area bounded by the two curves.

Participant B also did not zoom in on the curves and, based on the rest of his working, ended up focusing on finding the red area indicated in his hand-drawn sketch of what he could see on the calculator screen (see Figure 6.13). That is, participant B appeared to not 'see' the required areas between the curves, as he instead then proceeded to calculate the area shaded in his sketch, which was instead the area bounded by the curves and *the x-axis*. This suggests that, even though participant B obtained an incorrect result, he was again at least attempting to connect what he saw in the graphical visual mediator produced by his calculator with the numerical answer he obtained.

Figure 6.13

Sketch of the Curves and a Shaded Area, from Participant B's Working



The following interview excerpt indicated some of his approach and thought process in drawing the graphs and attempting to determine the required area between them, while, unlike the other participants, remaining working in the calculator's graphing screen window to do so. This discussion indicated that, after plotting both curves, he attempted to find where the graphs intersected by solving for intersection points in the graphics window that showed the graphs. However, he did not zoom in on the graphs or change the standard/default graphing window settings and as a result, did not see the actual required areas between the curves. At $x = -3$ the curves intersect with each other and also with the x-axis: participant B successfully found this point but, as the result of not being able to see the small area between the curves, he was then sidetracked into attempting to find the area bounded by the part of the cubic curve and the x-axis that he had shaded in his sketch in Figure 6.13.

B: Yeah, this one was a bit of a weird one. So, went to the **graph** menu and put them both in (*using CAS discourse*), and I know it's not a good drawing but it looked like that *(referring to his hand-drawn sketch of the graphs in Figure 6.13)*. So I thought, gotta find the difference in that *pointing to the graphs on his calculator screen)* … in that area in there. And then, so I went and found the intersection point of the two graphs, which was that one.

I: Using **gSolve** on the calculator (*using the discursive domain of CAS*)?

B: Yep, producing a unique 3 zero (*using the discursive domain of CAS*).

I: And was that directly from your graph that you did that? So that was when you were in the graphing window that you solved that?

B: Yeah. And then, I tried to do, like, root to find the other one but I think because there were two graphs my calc was screwing up on me (*using the discursive domain of CAS*).

I: So it would only find that intersection point there (*pointing to the x-intercept of 3*)?

B: Yeah. So I went back to the main menu because I just wanted to find where the first graph, so I did that with **zero** solve for x (*using the discursive domain of CAS*). It gave me these three. I knew it couldn't be that one because that's less than that (*pointing at calculator screen),* so I knew it had to be that one there, what I was looking for.

I: So what's interesting here is because of the resolution of the graph there actually are three intersection points there but it's fairly well hidden. We will keep going for now. So which one did you choose?

B: That one there (*pointing to one of the other x-intercepts of the graph:* $\sqrt{5} - 3$).

I: Why did you choose that one?

B: Because we already figured out that and that one I know would be… like because that's neg 3 minus root 5 *(pointing to the other x-intercept)* so I knew that would miss (*using* *colloquial discourse*). And then, yeah, I wasn't too confident with this question but I did **analysis, gSolve dx,** yeah, and with the lower and upper being. Oh, no I didn't even get to lower and upper because it just didn't work (*using the discursive domain of CAS*).

It is likely that the above problem occurred because of participant B selecting the point at $x = 3$ where the curves both intersect with each other and also with the x-axis, rather than his identifying that the first intersection was actually at $x = 2$. Participant B's next comments in the interview transcript that follows, about deciding to then delete the linear function $y2$ "because that was going down there anyway," indicates that failing to identify the areas then led him down the wrong solution path. The subsequent deletion of one graph removed any chance of obtaining the correct answer. This was the point where participant B then became diverted into finding one of the areas bounded by the cubic and the x-axis instead.

I: That's interesting, so that was the **gSolve** trying to do the integral (*using CAS discourse*)?

B: That didn't work, so I just decided to delete $y2$ from there, because that was going down there anyway, so just left that. I graphed it there **analysis- gSolve dx**, that was my lower and upper, I got that and (*using CAS discourse*) then area can't be negative so had it positive (*switching to mathematics discourse*).

I: So you're working it out from -3 to $\sqrt{5} - 3$, which you recognised is there. So that then you're basically working out that. So how did you interpret what that figure would give you? What sort of area do you think that was giving you in relation to your diagram (*using mathematics discourse*)?

B: Ahh, yeah that's something.

I: So you've shaded that's the area there?

B: The calculator didn't colour that area in (*using the discursive domain of CAS*).

I: And so, how do you interpret that, that it's the area under that curve?

B: Yeah, I interpreted it as that was the integral but obviously the calc doesn't know that area can't be negative (*transitioning to the discourse of mathematics*), so yeah.

The above comments suggest potential *ritual routine* reliance by participant B in relation to the **gSolve -**$\int dx$ command to calculate what he thought was the required area, combined with use of the standard constraint that students are taught from high school onwards that “area can’t be negative,” which is an *endorsed narrative* that he states in his written working and in his verbal explanation of what he did. Participant B could be considered to be *using CAS in a direct procedural fashion while also working in an inter-representational fashion* (Thomas & Hong, 2005, p. 220), by working within the graphing window of his CAS calculator to first graph the two curves and to then attempt to use the **Intersect** and $\int dx$ commands within the calculator’s graphing window to find the curves’ intersection points and bounded areas, respectively.

When interviewed, all three participants who attempted this question (B, C and D) stated they would use CAS to see what the graphs look like, with participant D also referring to it allowing her to see which one is the “upper curve and which is lower”, while participant C commented that graphing the curves “gives an idea” of the area involved and made it clear, in this instance, that his answer was wrong. This suggests the three participants were seeing the benefit in a question of this type of using CAS in an *epistemic role,* by taking advantage of its multiple representation capabilities when plotting the graphs, to assist in visualizing the required area bounded by them. Participants B and D also referred to preferring to use CAS to find the intersection points (B) or to factorise the associated equation (D) to help in finding them. Participant D was the only one who expressed confidence in her answer, which was justified as she was also the only one to actually obtain the correct answer.

For this question no potential commognitive conflicts were expected and none were realised in the students’ responses or interviews.

6.3.4 Commognitive analysis of students’ responses to question 4 (version 1)

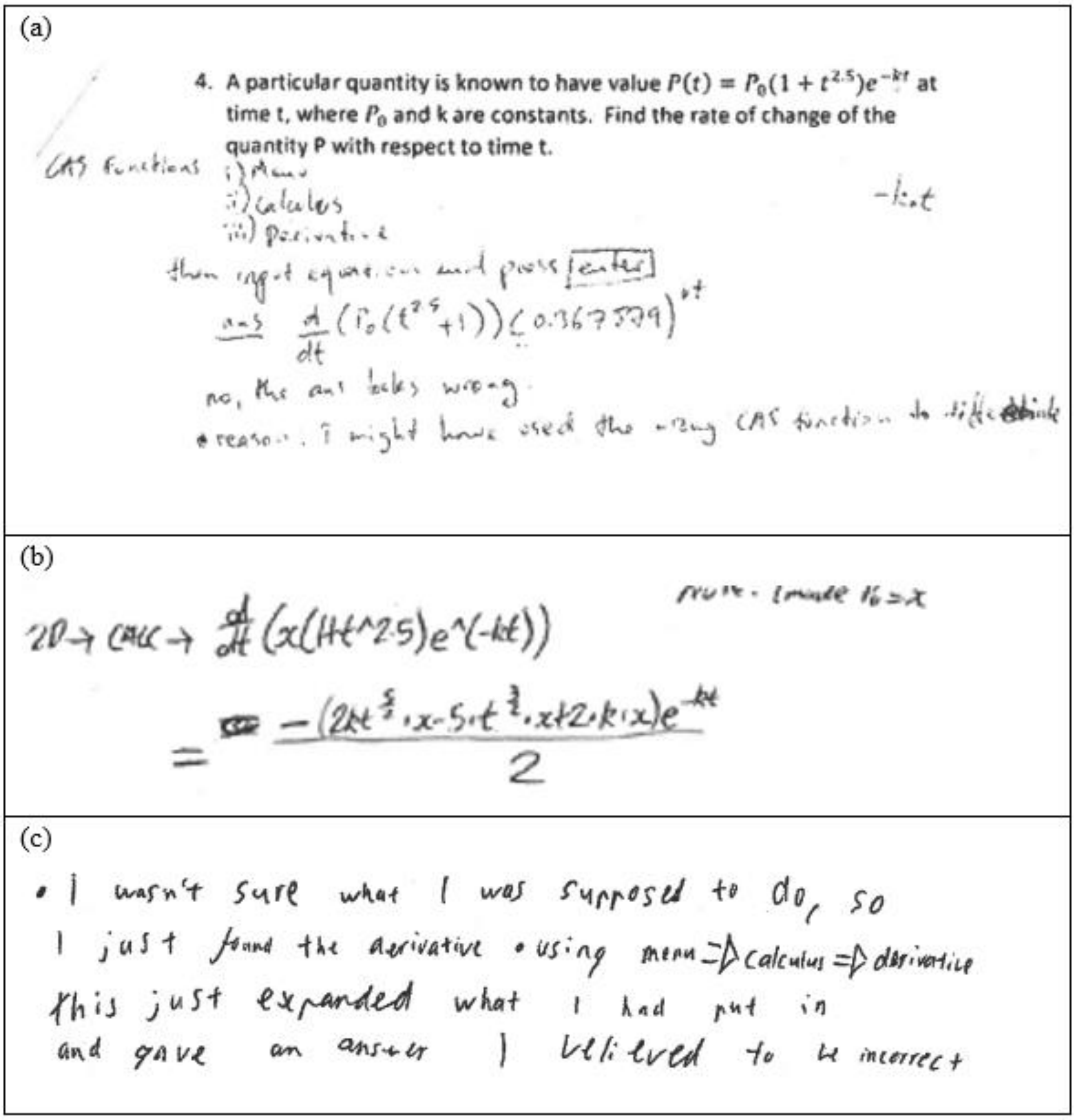
The participants’ attempts at question 4 are shown in Figure 6.14.

6.3.4.1 Narratives around required calculator use

In describing the required use of the calculator, the participants again produced narratives within the discursive domain of CAS itself. As seen in Figure 6.14(a), Participant A provided a *command-based narrative* of the same form as he did for question 1, again stating that the required series of menus are **Menu → Calculus → Derivative**, and that you should “then input equation and press **enter**”. While this process was correct, the actual function entered by participant A did not produce the correct result, due to two commognitive conflicts with the discourse of the calculator (described in the next section).

Figure 6.14

Written Solutions and Responses to Task-based Interview Question 4 (Version 1)



Participant B again gave the narrative of the required commands path in a more abbreviated, symbolic form as **2D → CALC → d/dt(x(1+t^2.5)e^(-kt))**, with this participant also noting that he had renamed P_0 ("I made $P_0 = x$"). Participant C said he wasn't sure what he was supposed to do, so he "just found the derivative using **Menu** ⇒

Calculus ⇒ Derivative", commenting that this gave an answer "I believed to be incorrect".

How the participants dealt with naming and entering the P_0 constant in this question was also of interest. In all three cases, in recognising that P_0 signified a single constant but having some doubts about entering 'P_0' for a variable name, they renamed it as a single letter when entering into the calculator, with participant A renaming it as A, participant B renaming it as x and participant C renaming it as p. All were sensible choices as these letters did not appear anywhere else in the algebraic function provided.

6.3.4.2 Types of routines and CAS procedures used

The approach taken by the three first-year participants was again potentially a *ritual routine*, as they all relied solely on the differentiation procedure from their calculator, without attempting any substantiation of their answers. That is, they used a "*direct, straightforward procedure*" (Thomas & Hong, 2005, p. 220) in one step with their CAS calculators, using the **derivative** command, to obtain an answer.

All also reported a lack of confidence in their final answers. Participant B was the only one to actually obtain the correct answer, and the following extract from the interview with this participant illustrates his lack of confidence in whether or not he had obtained the correct result.

B: Yeah, I didn't know how to do this one, so I kind of just gave it a punt (*colloquial discourse*). It says because P_0 and k are constants, so I decided to make P_0 equals x (*switching to mathematical discourse*). The rest I think I did what they are.

I: Yep, so you did $\frac{d}{dt}$ of that?

B: Yes, and then just went back, yeah.

I: And what did you think of that answer? Did you think that looked like what you'd expect to get?

B: Ah yeah, around about but I wasn't really confident, like. I think like on a test or whatever I would have done that as well, put it in a calculator and prayed for the best.

I: Yeah, you wouldn't have done that one by-hand?

B: No.

Participants A and C unknowingly encountered commognitive conflicts, which will be described in the next section. While both believed their answer to be incorrect, they did not identify the correct reason, with participant A suggesting that he "might have used the wrong CAS function to differentiate". As shown below in the transcript excerpt of the interview, he also *considered being more explorative,* by putting in a number to see how it behaved, but decided against doing this as being too specific to generalise to "the whole answer". Participant A also demonstrated some confusion about the nature of P_0 and k, describing them as *variables*, when they are actually *constants*.

A: Okay, it's here, I just put the equation but because there's more variable here you've got the P_0 and the k so I think that's not the right function for that. You need to use a different one.

I: So what's interesting here, it does look rather complicated, yes, with this P_0 and k. So P_0 and k are specified to be constants, so, because they are constants, that means they are not going to vary.

A: Yep.

I: And that means that on this calculator they can actually be treated in the same way as a number 3 or an 8 or anything else.

A: Yeah, I was going to try maybe putting a number, just give it a number and see what comes out (*considering using an exploratory routine*) and then thought that might not answer the question, so only give it an instance of that not the whole answer (*mathematics endorsed narrative*).

Participant C recognised that the answer he obtained was not the required answer, as shown by his written comment in Figure 6.14(c) that "this just expanded what I had put in and gave an answer I believed to be incorrect" and also commented that "I wasn't sure what I was supposed to do." The transcript of his interview also showed that he attempted to convert the answer to decimals in the hope that would somehow give the correct answer:

I: (*Looking at next line of student's calculator output, which showed an answer in decimal form)* And what did you do there, put it in decimal answer form?

C: Also tried **Control**, that (*pointing to the Enter key*) in case for some reason it got rid of that (*pointing to the first term of the calculator answer that was still given as a derivative*) but it didn't (*using CAS discourse*).

I: Yep, so you tried **Control- Enter** which basically gave the answer in decimals?

C: Yep.

This participant also expressed an apparent preference for *ritual* behaviour in solving this type of problem, in referring to having access to "rules" and formulae if given a choice about how to solve it:

I: So with that one, would you usually prefer to do that by-hand or on the calculator?

C: Ahhh, if I actually had the rule on me, I might prefer to do it by-hand, by like by the rules. But if I don't have the rules, I'd go calculator.

I: Yep, chain rules, do you mean that sort of thing? Or more for e?

C: Yeah, with e and also, oh, chain rule I'm pretty across, but still the e part, and if I had that I would do it by-hand. But because I didn't, if I didn't have the formula sheet I'd go with this (*pointing to the calculator*).

Both Participants A and B expressed a preference for using CAS to solve problems of this type. Participant C expressed a possible preference for working out the answer to problems of this type by-hand, if he had access to "rules" (such as a formula sheet), but otherwise for working it out using CAS.

6.3.4.3 Commognitive conflicts

In answering this question, the students needed to make use of the **derivative** command to solve this problem on the calculator (with respect to x). Unlike in question 2, however, use of this command in itself is not expected to produce a commognitive conflict, as in this case all the other letters in the expression provided (other than x) signify constants, meaning use of the **derivative** command will produce the correct result (provided the arguments of this command are also entered in a suitable format, as discussed below).

However, as in question 2, for those participants using a TI-Nspire CX CAS calculator, there was potential for a commognitive conflict resulting from entry of the term kt, if k and t were not separated by a multiplication sign. Participant A again made this mistake.

A second potential source of commognitive conflict in this type of question is the situation where the CAS calculator identifies the presence of a letter (or a letter followed by a number) which is then immediately followed by a bracketed expression with no mathematical operator separating them, as a *function*, with the bracketed expression being its *argument*. This is opposed to interpreting an expression of this type as *multiplication* of the term preceding the bracketed expression by the content of the bracketed expression, as is also common in algebraic mathematical discourse. This type of commognitive conflict also arose in both participant A's and participant C's working for question 4. Participant A entered $\frac{d}{dt}(A(1+t^{2.5})*e^{-kt})$ (replacing P_0 with A) while participant C entered $\frac{d}{dt}(p(1+t^{2.5})*e^{-k*t})$ (replacing P_0 with p), suggesting these participants were accessing classroom algebraic discourse from their precedent search spaces, when attempting this problem on the calculator. By-hand, this would be one valid way of writing down this initial step, suggesting these students were accessing their *precedent search space* of high school (and university level) algebraic discourse. However, omitting the multiplication sign after the letter A or p led to the calculator interpreting, for example, $p(1+t^{2.5})$ not as a *constant*, p, being multiplied by $1+t^{2.5}$, but as a *function* p, with *argument* $1+t^{2.5}$.

6.3.5 Commognitive analysis of students' responses to question 5

The participants' attempts at question 5 are shown in Figure 6.15.

6.3.5.1 Narratives around required calculator use

The participants again produced narratives within the discursive domain of CAS, when describing the steps they used to answer question 5. As seen in Figure 6.15, Participants A and D both provided a *command-based narrative* of the same form as their narratives for the other questions, this time correctly stating to go through the series of steps **Menu → Calculus → Integral**, with Participant A also adding that you "then input equation and press **enter**". Participant B commented that he had "changed setting from DEG to RAD" (which is required to produce the correct answer) and then once again produced a more *abbreviated symbolic narrative* of the form

2D → CALC → $\boldsymbol{\int (\sin(x)+\cos(x))dx}$.

Participant C described the steps he took as follows: "$\int(\boldsymbol{sinx} + \boldsymbol{cosx})\boldsymbol{dx}$, I put this eqn into the calculator after pressing the button next to the catalogue, then I put eqn using the keyboard typing $sin(x)$ and $cos(x)$".

The above showed that with the exception of the calculator commands, *sin(x), cos(x)* and the abbreviated form of the word *equation*, the participants' written descriptions of what they did again did not contain much specialised mathematical language. In the interviews, the only additional specialised mathematical terms used by the participants were *definite integral* (by Participant D) and *integral* (by Participant C). The latter occurred when the interviewer was mentioning the absence of the required constant of integration in discussing his answer to part (a). Participant C used the word *integral*, adding "I wasn't switched on there" in reference to his omission of *c* (the constant of integration), in answering the question.

6.3.5.2 Types of routines and CAS procedures used

All four participants wrote down the answer to the first part of the question as $-\cos(x) + \sin(x)$. This means that all participants obtained only a partially correct answer for part (a) where they were required to determine $\int(sinx + cosx)dx$. This is because they obtained *an antiderivative* of the correct type, but did not then add the required constant of integration. This indicates they all wrote the answer directly from their CAS calculator, with the omission of the required constant suggesting a *ritual* approach, in making an error that also commonly occurs when students are working by-hand. Participants A and D also used a *substantiation routine*, in also verifying that they obtained the same result for the definite integral in part (b) by-hand, but they did not notice the omission of the constant of integration in their answers to part (a). Using the guidelines of Thomas and Hong (2005), it can be said that Participants B and C, used a *direct, straightforward procedure* in one step for each of parts (a) and (b) with their CAS calculators, using the **integral** command, to obtain the answers. Participants A and D used a *direct, straightforward procedure* in part (a) in the same way as the other two participants, but they also worked out the answer to (b) by-hand and checked it using the CAS calculator (meaning that in (b) they *used CAS to check procedural by-hand work* (p. 221).

Participant B also suggested that he would normally obtain the answer by-hand by looking at a formula sheet and then checking the result on the calculator, suggesting he would usually attempt to substantiate the answer to a problem of this type.

I: And, would you usually do that by the calculator or by-hand?

B: That one's a pretty easy one to do by-hand so I probably would do it by-hand, that one

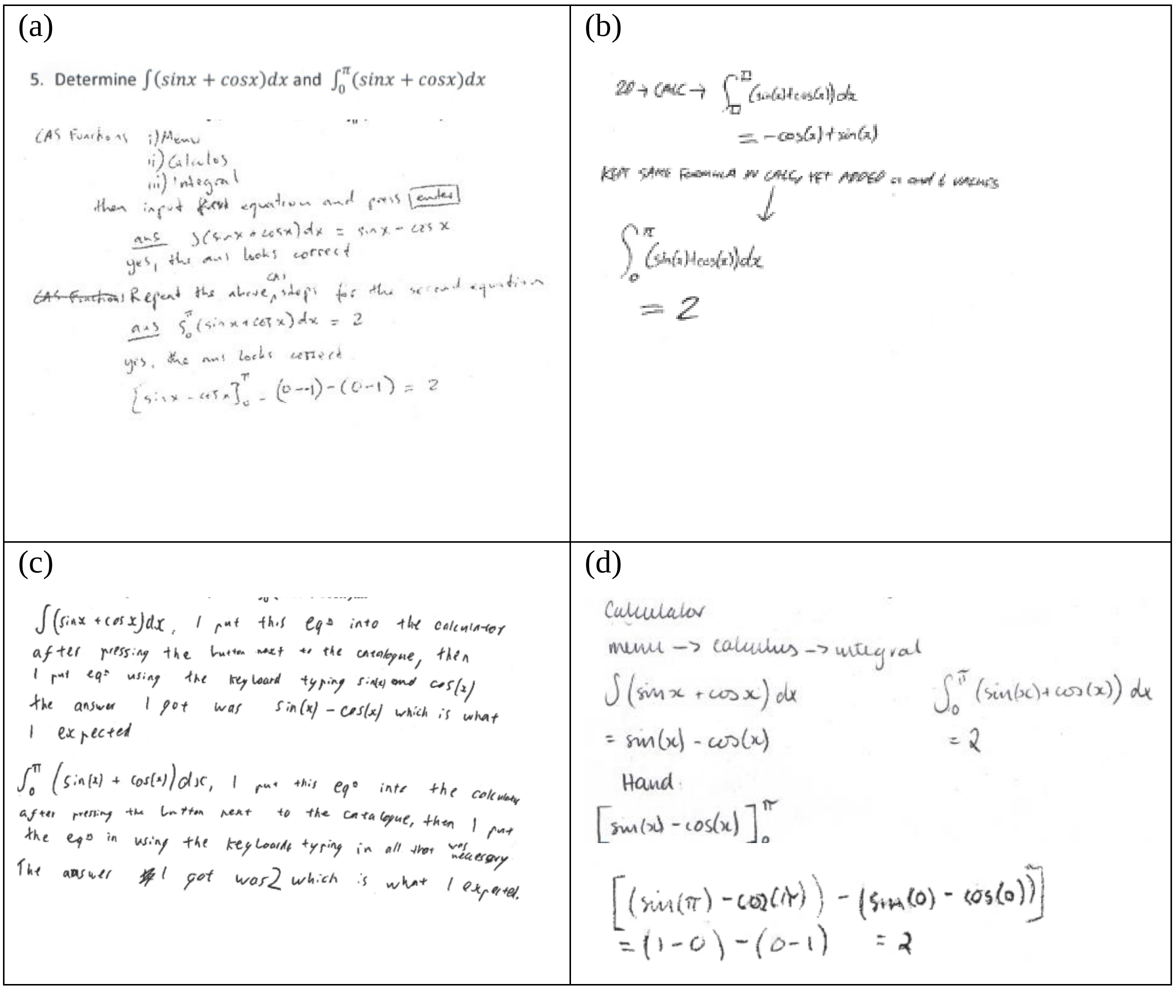
I: (*Referring to the second part of the question, requiring definite integral)* And then here you kept the same formula and that, so do you think your answer you got for that looks correct?

B: Ah, I think it looks pretty good. I'd probably do that by-hand then check with the calc.

For this question no potential commognitive conflicts were identified and none were realised in the students' written responses or interviews.

Figure 6.15

Written Solutions and Responses to Task-based Interview Question 5 (Version 1)



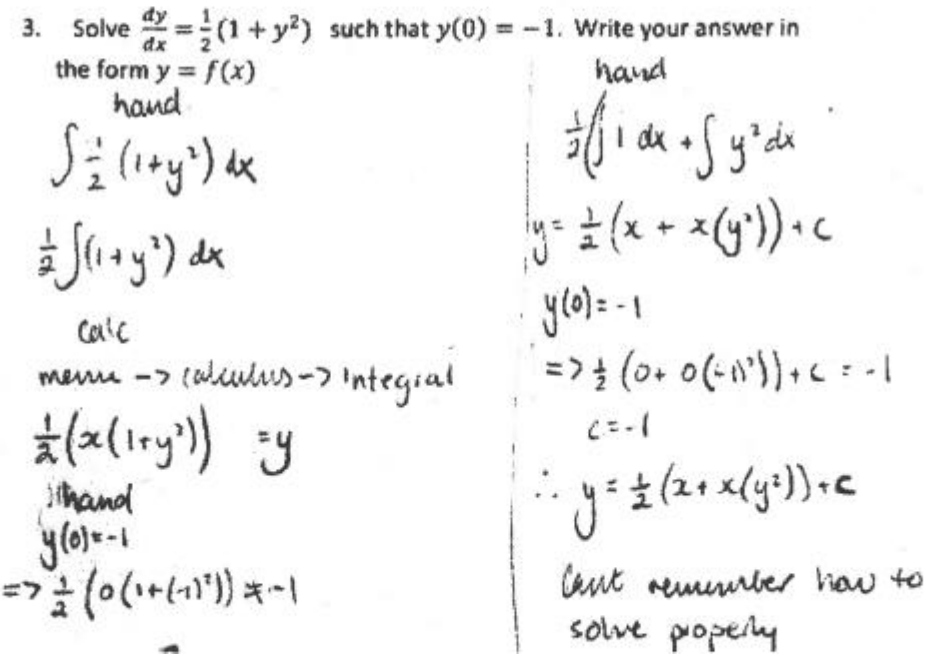
The other three participants also expressed a preference for solving a problem of this type by-hand, with Participant C also indicating having this preference because of this problem being “pretty easy” and with Participant A making a general statement that he would use “whichever method is faster”, identifying that in this case, that would be by-hand. Participant D also expressed a preference for usually working out problems of this type by-hand, and then using the CAS calculator to check the answer.

6.3.6 Commognitive analysis of Participant D’s responses to question 3 (Version 2)

Figure 6.16 shows Participant D’s answer attempt for question 3 (version 2).

Figure 6.16

Written Solutions and Responses to Task-based Interview Question 3 (Version 2)



6.3.6.1 Narratives around required calculator use

As shown in Figure 6.16, participant D provided a *command-based narrative*, in the discursive domain of CAS, of the same form as her narratives in previous answers to the other questions, in the form: **Calc. menu → calculus → integral**.

6.3.6.2 Types of routines and CAS procedures used

Participant D attempted both working with the CAS calculator and by-hand to solve the problem, but used an incorrect method in both cases, and hence did not obtain the correct answer. Her incorrect steps included attempting to integrate expressions involving y with respect to x, which included using the **integral** command on the calculator (instead of the required **deSolve** command). Her use of the **integral** command suggested potentially *ritual* behaviour, in recognising that the problem required solving for y, where y is an antiderivative of $\frac{dy}{dx}$. The right-hand column of her written working indicates that she used the correct general method for substituting the initial condition $y(0) = 1$ into the result she obtained which related y and x, so as to find the value of the constant c. However, the actual expression she obtained for y was incorrect, as the result of using incorrect methods, and she did not attempt to substitute the value she found for c back into the equation she had obtained, which related x and y.

The following transcript excerpt of the interview with participant D indicated her thought process in working on this task, including her confusion and attempt to try to solve the problem both with the calculator and by-hand, and her reliance on having access to a *rule* for solving this type of problem or a formula sheet when in test conditions.

D: So, I didn't quite know what the question was asking but I tried.

I: Yep, so there, when you're working that out on the calculator?

D: So I started to [do] by-hand and then I changed to calculator and then went back to [by] hand.

I: Yep.

D: But that wasn't right (*pointing to her final answer*).

I: And what sort of method did you try to use on the calculator to do that one?

D: Just the integral, but I know it can't properly (*colloquial discourse*). Again, without a rule.

Subsequent comments from participant D also suggested that in the absence of a rule, she was taking a trial and error approach or attempting different things and hoping one would work:

D: I knew it wasn't going to be right, I was just trying it.

I: Yep.

D: Try different things.

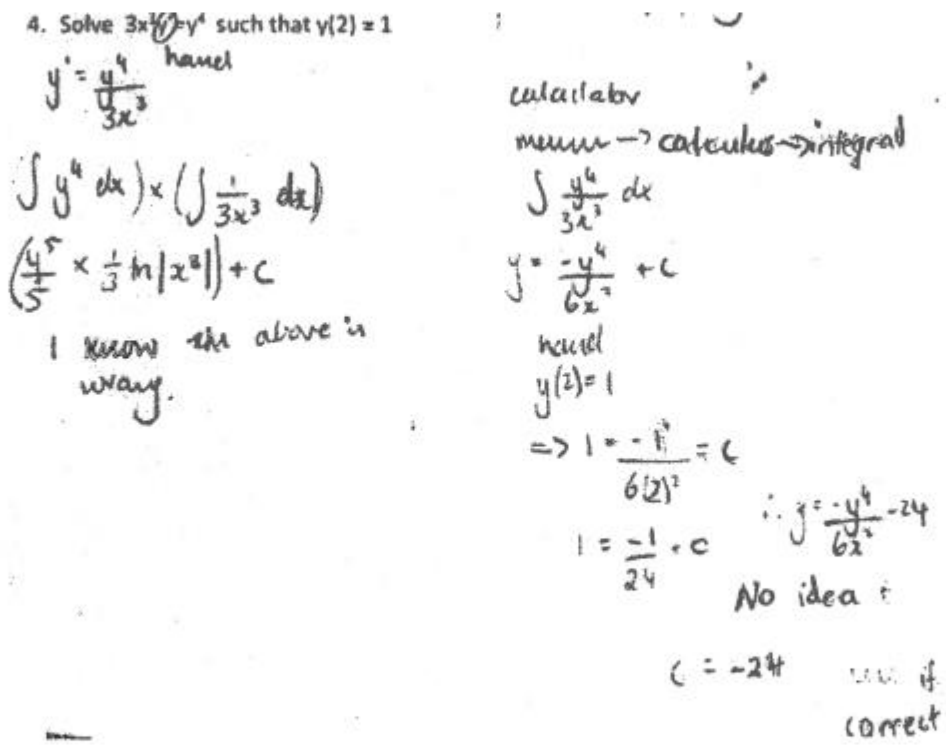
For this question, no potential commognitive conflicts were identified and none were realised in Participant D's written responses or interview. She was not confident in her answer and expressed a preference by working problems of this type by-hand, but with access to the relevant rules.

6.3.7 Commognitive analysis of Participant D's responses to question 4 (Version 2)

Figure 6.17 shows Participant D's answer attempt for question 4 (version 2).

Figure 6.17

Written Solutions and Responses to Task-based Interview Question 4 (version 2)



6.3.7.1 Narratives around required calculator use

As shown in Figure 6.17, Participant D provided a *command-based narrative* of the same form as the previous question, using the discursive domain of CAS in stating that you should use **menu** → **calculus** → **integral**.

6.3.7.2 Types of routines and CAS procedures used

As in the previous question, participant D was unable to obtain the correct answer, also stating in her written working that "I know the above answer is wrong." The following excerpt from her interview transcript indicates her approach in attempting the question.

D: Um, that one, I tried it by-hand to begin with.

I: Yep.

D: I knew that I wasn't doing it right though, I didn't remember how to split the fraction (*colloquial narrative*).

I: Yep.

D: So I just tried, to see what I got, knew it was wrong and then went to the calculator. Again tried, but because it's got the y in it, knew it wasn't going to be quite right (*colloquial narrative; referring to the final answer the student obtained and wrote down*).

I: So the y in it concerned you with that?

D: Yep, yep.

When she commented that "I didn't remember how to split the fraction," this suggests she had some recollection that it was a separable differential equation, however there are no endorsable narratives produced to suggest she knew how the variables then needed to be separated. At the end of her attempt, she wrote "no idea if this is correct", but her observation in the interview that "because it's got the y in it knew it wasn't going to be quite right" was correct, as in solving a separable differential equation for y, the final answer should only have y on one side of the resulting equation. As in her answer attempt for the previous question, her use of the **integral** command again suggests *potentially ritual behaviour* in recognising that the problem required solving for y, where y is an antiderivative of $\frac{dy}{dx}$. Part of the right-hand column of her written working indicates that, as in the previous question, she again used the correct general method for substituting the initial condition, $y(2) = 1$, into the result she obtained for y in terms of x to obtain a value for the constant c. However, the actual expression she obtained for y was incorrect, as the consequence of using incorrect methods, and she also once more did not attempt to substitute the value she found for c into the equation she had obtained, which related x and y.

For this question, no potential commognitive conflicts were expected and none were realised in the student's written responses or interview. Participant D was not confident in her answer and, as in the previous differential equations question, expressed a preference for solving problems of this type by-hand.

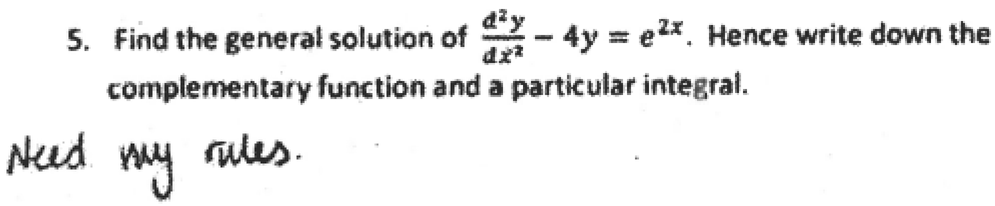
6.3.8 Commognitive analysis of participant D's responses to question 5 (version 2)

Use of rituals and/or explorations

As shown in Figure 6.18, Participant D did not attempt this question, but in the space provided instead wrote "need my rules", suggesting *potential reliance on rituals* when solving problems of this type at other times. When asked whether she would usually prefer to solve problems of this type by-hand or using CAS she stated that she would prefer to do it by-hand, "rather than I can't do it. Rather than rely on something else."

Figure 6.18

Written solutions and responses to task-based interview question 5 given to participant D



5. Find the general solution of $\frac{d^2y}{dx^2} - 4y = e^{2x}$. Hence write down the complementary function and a particular integral.

Need my rules.

When asked to elaborate on her statement that she needed her rules, Participant D said that she remembered doing that type of problem and would have been able to do it by-hand if she had access to the rule required. She also stated that she did not know how to solve this type of equation on the calculator and that, even if she did know, she would "much prefer to do things by-hand and just use the calculator as a backup."

6.3.9 Factors related to when the participants used CAS

In looking across the tasks attempted by multiple students, I examined their preferences for CAS and by-hand use, and the reasons they gave for their choices. Table 6.23 shows each participants' preference, to solve each of the first 5 problems given to them in the task-based interviews and reasons for their choice, based on their written and verbal comments in the task-based interviews.

When examining the reasons for the students' preference (CAS or by-hand) in solving problems of the types present in each question, it being quicker to solve a problem using one of these methods was the reason they provided in a few instances, expressing a preference for using the method they perceived to be less time-consuming. In some cases, this was a preference for using CAS, with Participant A stating this in relation to saving time in questions 1 and 2, and Participant C for question 1.

While not explicitly referring to CAS being faster, Participant C's comment regarding question 3 that it would be useful for graphing *quickly* to get an idea and Participant D's comment that she would prefer to use CAS for the otherwise *tedious* solving to find the intersections of the curves, suggested they were expressing a preference to use CAS, in part at least, to save time in those instances. A preference for working by-hand as being quicker was expressed by Participant B in question 2, with him stating it would be "too long to input" into CAS. Participant A expressed this preference for answering question 5 part (a).

Another reason the students expressed for preferring one method over the other was confidence in their ability to solve the type of problem. Most commonly, this was a preference for using CAS, due to not being confident in solving by-hand, with Participant B expressing this preference for question 4, Participant C for question 2 and Participant D for question 1. However, a preference for solving by-hand was expressed by Participant B for question 2 due to his not being very familiar with the required CAS command. A preference for solving by-hand was also given by Participants A, B and C for question 5, because they *were* confident that it would be "straightforward" or "easy" to work it out by-hand.

Use of CAS for visualisation of graphs was also referred to by all three who attempted question 3, as a reason for their having a preference for use of CAS for solving that type of problem overall (Participants B and C), or as part of an integrated strategy involving a preference for CAS for that aspect, and by-hand for some other aspects of solving it (Participant D).

In some cases, the participants expressed a preference from working by-hand, conditional on having access to a formula sheet or "rules," with Participant B referring to this for question 5, Participant C for question 4 and Participant D for question 2. There were also two instances, as described previously, where participants expressed a preference for solving problems by-hand, but then followed by using CAS to check their answers, demonstrating a preference for *checking of procedural work by-hand*.

6.3.10 Discussion of findings from the task-based interviews

The written responses to the task-based interviews indicated that the participants typically used command-based written narratives when describing their use of CAS. This could be because of the first written instruction for doing the tasks being to "write down all commands you use on CAS". There was evidence of them being able to switch, to a limited extent, from using the CAS discourse to colloquial or mathematical discourse in conversations with me and in their written responses. This indicated that they were clearly accustomed to making several discursive shifts in both written and spoken words. However, beyond that, their written discourse typically did not contain clear evidence of their discourse transitioning from high school to university level discourse, with the exception of Participant D who used endorsed mathematical terms as such "partial derivative" and "implicit differentiation". There was also limited verbal use of specialised mathematical words in the interviews. Instead, the participants frequently referred to mathematical objects as "it" or "that", while pointing to commands or outputs on their calculator, rather than using objectified discourse in naming them with specialised mathematical words such as, for example, "the graph" or "the equation" or "the function." As a result, there were few stand-alone statements in their written or verbal responses that would count as endorsed narratives in the community of mathematical discourse. There was no evidence found of use of metaphor in the students' discourses, either in transitioning from school level discourse to the limited amount of their university level discourse present, or from the discursive domain of CAS to that of mathematics. This is in part reflective of the limited amount of university level discourse present in their interviews and, in part, could also be because of the lack of objectified discourse when the students were writing and talking within the discursive domain of CAS.

In questions 2 and 4, where the potential for students to encounter intrapersonal commognitive conflicts with the technology had been identified, these did indeed occur with three participants encountering at least one such conflict in question 2 and two of them encountering at least one such conflict in question 4, indicating these students were accessing their precedent search spaces on classroom algebraic and Calculus discourse (for when solving problems by-hand). While these participants believed their resulting answers to be incorrect, none of them identified the cause of their errors. Two of these types of commognitive conflicts were due to the required discourse of CAS differing from the mathematical discourse of Algebra they were using. The other type of commognitive conflict encountered occurred due to the **derivative** command being used in an incorrect context, again due to this command and associated output notation signifying a different meaning when working within the discursive domain of CAS than when in the traditional discourse of Calculus. These commognitive conflicts highlight the importance of working within the correct community of discourse and making students aware where these discursive rules differ when using CAS.

The participants' behaviour in solving the problems frequently indicated use of ritual routines, with limited use of substantiation routines such as checking answers by-hand. The participant who showed the most evidence of attempting substantiation routines was the second-year participant (D), in attempting at least part of most of the questions by-hand as well as on the calculator. However, while only one participant zoomed in on the graph of curves produced in the *area between curves* question, suggesting *explorative* behaviour, all three participants who attempted that question produced a graph of the curves using their calculator and used this visual mediator to assist in determining the required area based on the relative position of the two curves. The interviews with these participants indicated that all found CAS useful for visualising the curves and two of them referred to the 'small' appearance of the area between the curves on the graphical visual mediator when asked if they thought their answer (for the calculated area between the curves) was correct.

The guidelines in Thomas and Hong (2005) were useful in further identifying and distinguishing between the different types of ways in which CAS was used by the participants. In most cases, the participants used CAS to perform a *direct, straightforward procedure*, frequently answering the question in one step. However, Participant B also demonstrated use of CAS to *perform a direct procedure*, but *implementing a new, CAS-based technique* for doing so, when answering question 2. The area between curves question also led to two additional types of CAS use by the participants, with Participant B *using CAS in a direct procedural fashion while also working in an inter-representational fashion* in producing and interacting directly with the graph of the two curves while, in answering that same question, Participant D demonstrated *use of CAS in performing a procedure within a more complex process (possibly to reduce cognitive load)*, by integrating use of CAS into parts of the solution, while also working out other parts of her answer by-hand.

Consistent with the findings of Mohammad (2019), all the participants expressed a preference for using CAS for questions like the most time-consuming question provided (question 3), where they were required to determine the area bounded by two curves. This question also had the biggest difference in *grain size*, with the minimal number of

Table 6.23

Preference for Use of CAS or By-Hand, for Each Participant on Different Questions

	Preference for Use			Reason
	CAS	By-hand	Both	
Q1	A			"Faster" speed with CAS
		B (then calc check)		[no reason given for preference]
	C			"Faster" speed with CAS
	D			Low confidence in solving this type of question by-hand based on previous experience
Q2	A			Time, especially in exam
		B		Too long to input and unfamiliar with **impDiff** command on calculator
	C			Unsure of question mathematically
			D	By-hand if have rule, use CAS if no rule or can't remember how to do
Q3[a]	A[b]			Motivated by what lecturer has said (doing by-hand in exam if required, otherwise prefer to learn how to do this on the calculator)
	B			For graphing and intersection points "there was one little spot there"
	C			Good to graph quickly to get an idea
			D	Use CAS for bits that that would be tedious algebraically and for identifying top and bottom curve. Other steps by-hand
Q4[c]	A			[no reason given for preference]
	B			Not confident how to do mathematically
			C	Use CAS if do not have formula sheet, to do exponential differentiation correctly; otherwise, do by-hand
Q5[d]	A (part b)	A (part a)		By-hand straightforward (for first part), would take longer to enter into CAS
		B		To just look at formula sheet and do by-hand is pretty easy
		C		Pretty easy
		D (then calc check)		[no reason given for preference]

Note. [a] Q6 for Participant D. [b] Participant A did not attempt Q 3. [c] Participant D given Q 4. [d] Q 7 for Participant D.

decisions required to solve it completely by-hand much greater than when using CAS. All four participants recorded a preference for doing the first part of the least time-consuming question by-hand, with this question part having the same grain size when solved by-hand as if it was solved using CAS.

Section 6.4 now follows, with a report on the results of a questionnaire given to first-year and third-year students around the time this study was conducted.

6.4 Students’ Attitudes Towards and Extent of Using CAS in their Mathematics Units

The analysis that follows relates to addressing Research Question 3, which I remind the reader of below.

RQ3 To what extent do undergraduate Calculus students use CAS and when do they believe its use to be most beneficial?

The purpose of analysing the questionnaire was to contextualise what the participants in the task-based interviews said in section 6.3, to the larger practices, discourses and attitudes of year 1 and year 3 students from the same cohort, studying mathematics units from the same course at tertiary level.

6.4.1 Survey on CAS usage and attitudes

Similarly to Stewart et al. (2005), in addition to examining students’ attitudes to, and use of, CAS overall, I was interested in comparing students’ attitudes and usage across different year levels. Different approaches and attitudes to using CAS can be affected by the nature of the subject (e.g., first-year vs third-year), as long-term exposure to, and experience of, CAS usage may also lead to greater appreciation of its merits and the overall increased complexity of mathematical tasks at a higher year level could also affect when and how students use CAS and their attitudes towards it. Within a given subject, the time of semester at which the survey was conducted could also potentially influence the distribution of responses, with shifts in discourse as students move from the periphery more centrally into the new discourse of tertiary mathematics, also potentially having an influence. To investigate this and to contextualise the results from section 6.3, students studying the first-year mathematics unit MATH104 and the third-year mathematics unit MATH310 were surveyed in 2015 on their attitudes to CAS and how they had made use of CAS in their studies of mathematics in these subjects. The first-year unit MATH104 was taken by all the participants whose written and oral responses were analysed in the task-based interviews in section 6.3. The MATH310 students had previously also studied MATH104. The 2015 students were given the surveys mid-way through the semester in which they were taking the subject. By that stage of semester, the first-year students had covered some Algebra and Calculus topics, while the third-year students already had extensive background knowledge of Algebra and Calculus and had covered work on Ordinary Differential Equations. A second cohort of MATH104 students were also surveyed in 2016 with the same questionnaire, but they were given the survey at the *end* of semester. This was because it was predicted that the slightly longer exposure to, and experience of CAS usage might lead to changes in students' attitudes and the extent of their CAS usage. Also, this was close to the time at which the task-based interviews discussed in sections 6.2 and 6.3 were conducted. The students who took part in the task-based interviews were from the 2016 cohort of MATH104 students who were surveyed.

The survey questions, together with the answer options provided, are provided in Appendix A. As shown in Table 4.4, the questions dealt with use of CAS, utility of CAS, preference for CAS or by-hand, and access to CAS in assessment. These will now be dealt with in order.

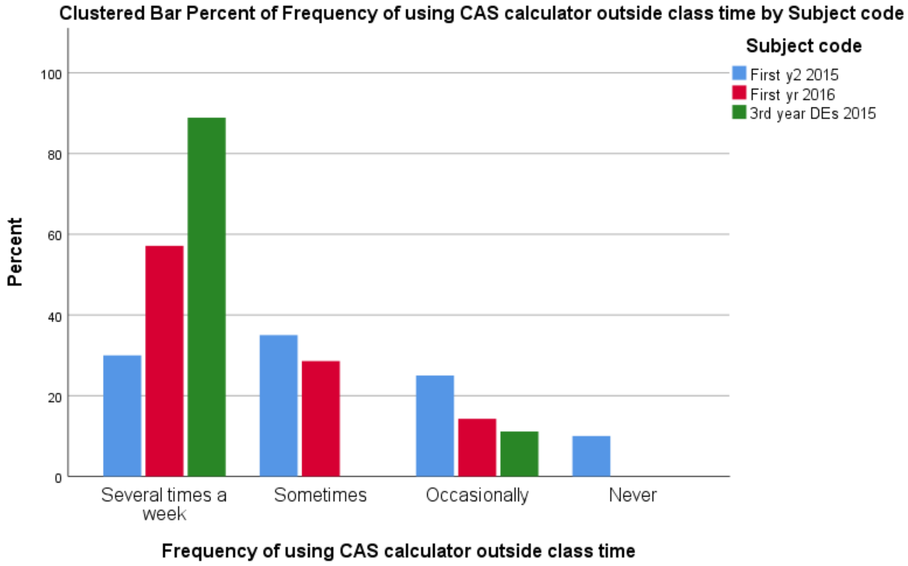
6.4.2 Use of CAS

An important aspect of students' usage of CAS calculators and software is their attitudes to towards CAS and the extent to which they choose to use it both in, and outside of, class time. As described in chapter 4, four of the questions in the survey related to aspects of the use of CAS, with the first three of these questions relating to the extent of the students using different types of CAS outside class, comparing use of handheld CAS calculators with CAS computer software. As can be seen in Figure 6.19, the third-year students were much more likely to report using CAS calculators several times a week outside of class time (89%) than either group of the first-year students, with those surveyed at the end of semester in 2016 being more likely to report this (57%) than first-year students surveyed in the middle of semester in 2015 (30%).

For reported usage of CAS computer packages outside of class time, Figure 6.20 shows the difference in the response distribution for first-year and third-year students was even more pronounced, with all 9 third-year students reporting using CAS computer packages several times per week outside of class. In contrast, only 10% of the first-year students surveyed in 2015 and 14% of those surveyed in 2016 reported using them several times a week.

Figure 6.19

Distribution of Responses for Extent of Use of CAS Calculator Outside Class



In addition, 50% of the first-year students in 2015 and 57% of them in 2016 reported never using CAS computer packages outside of class time. However, of the first-year students who reported 'never' using CAS computer software outside of class time, four of them actually wrote an option for 'type of software used outside class time' in question 3. Two mentioned Wolfram Alpha and two mentioned on-screen versions of TI software, so perhaps not recognising these were CAS computer packages.

When the three groups of students surveyed were combined, overall students were more likely to use CAS calculators outside of class time than CAS computer packages, with 52% of the students reporting using CAS calculators more than once a week as compared to 28% for CAS computer packages. Even more strikingly, only 4% of students surveyed reported never using CAS calculators outside of class time, whereas 44% reported never using CAS computer packages in their own time.

Wolfram Alpha was by far the most common type of CAS computer software used outside of class time with all of the third-year students reporting using this and 77% of first-year students reporting using it. One student who used Wolfram Alpha also reported using MATLAB and two others reported using TI-Nspire CX on the computer. Of the students who did not use Wolfram Alpha, one first-year student reported using Mathematica outside of class time, another first-year student reported using MATLAB and two first-year students reported using TI computer software.

Figure 6.20

Distribution of Responses for Extent of Use of CAS Computer Packages Outside Class

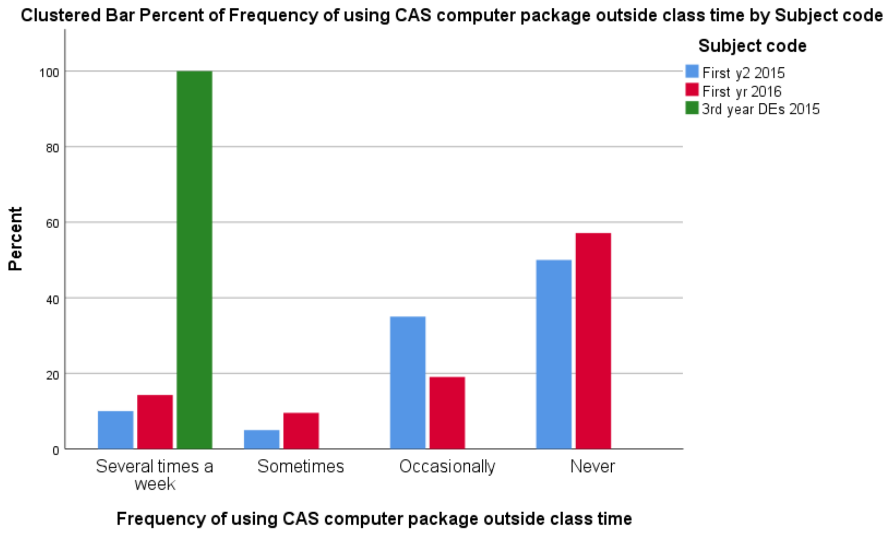


Table 6.24

Relationship Between CAS Calculator Usage and CAS Computer Package Usage Outside Class

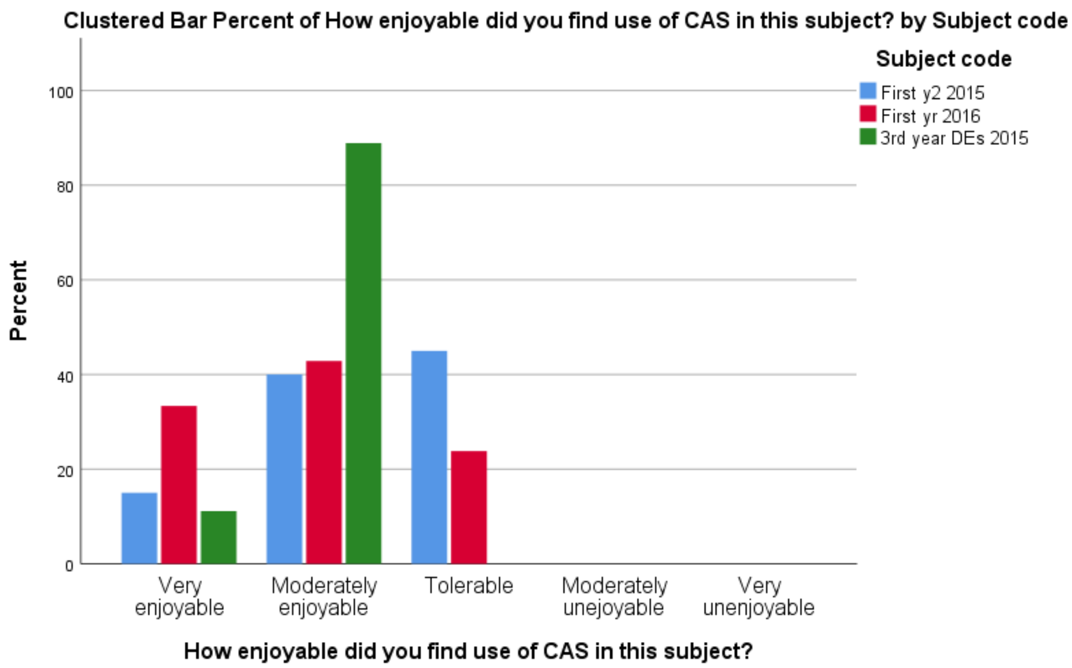
	Frequency of using CAS computer package outside class			
Frequency of using CAS calculator outside class	Several times a week	Sometimes	Occasionally	Never
Several times a week	92.9%	33.3%	27.3%	40.9%
Sometimes	0.0%	33.3%	36.4%	36.4%
Occasionally	7.1%	33.3%	36.4%	13.6%
Never	0.0%	0.0%	0.0%	9.1%

As can be seen in Table 6.24, students who reported using CAS computer packages regularly were the most likely to regularly make use of their CAS calculators outside class, with, for instance, 93% of those who reported using a CAS computer package several times a week outside class time reporting also using their CAS calculator several times a week outside class time. Only 41% of those who reported never using a CAS computer package outside class time used their CAS calculator several times a week outside of class time.

The final aspect of use of CAS examined was how enjoyable the students perceived use of CAS to be in the subject (Figure 6.21). The percentage of respondents who reported finding CAS to be at least "moderately enjoyable" to use was highest for the third-year cohort, with all describing it as either "moderately enjoyable" or "very enjoyable". For the first-year students, the cohort from 2016 were more likely to report this (76%) than the cohort from 2015 (55%) who had completed the survey earlier in the semester. None of the students in either group rated use of CAS in the subject as "unenjoyable".

Figure 6.21

Distribution of Responses for Enjoyment of CAS in the Subject by Cohort



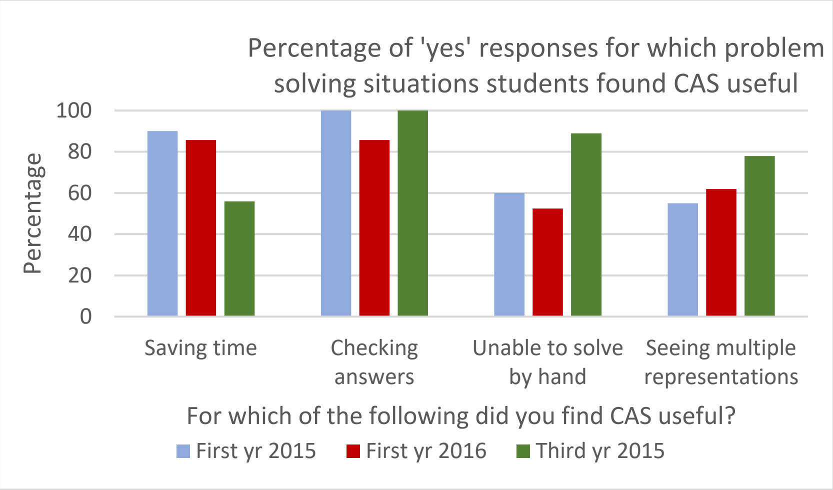
6.4.3 Utility of CAS

The first aspect of utility of CAS in the survey was looking at the ways in which the respondents used CAS in solving mathematics problems. As seen in Figure 6.22, the response distributions were different for the three groups of students, with the more recent first-year student cohort being less likely to use CAS to check solutions calculated by-hand (86% as opposed to 100% of the first-year and third-year cohorts in the previous year). This suggests potentially slightly less use of *substantiation routines* among the more recent first-year cohort but there was no clear reason apparent as to why this might be the case and so could be incidental to the particular cohort of students involved. The third-year students were much less likely to report using CAS to save time in solving problems

(56%) than either the 2015 or 2016 first-year cohorts (90% and 86%, respectively). In contrast, the third-year students were much more likely to report using CAS when unable to solve problems by-hand (89% compared to 60% and 52% for the first-year cohorts, respectively). In addition, the third-year students were also the most likely to have found CAS useful for seeing multiple representations of a solution (78% compared to 55% and 62% for the first-year cohorts, respectively).

Figure 6.22

Distribution of Responses for Utility of CAS in Different Problem Solving Situations



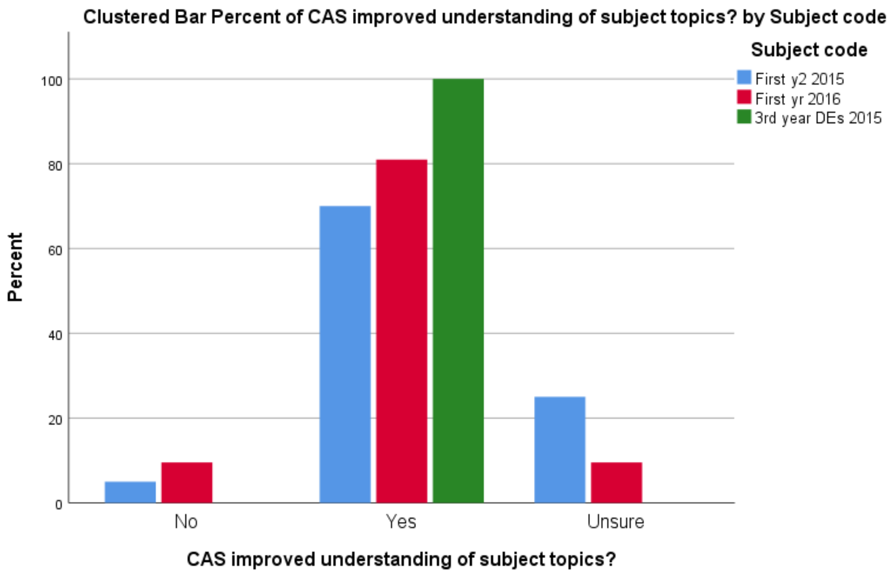
Students were also asked if there was anything else they found CAS useful for, but only one first-year student wrote "for decimal answers". There were no additional aspects mentioned by the remaining respondents.

Another important aspect of the utility of CAS was whether students believed it had improved their understanding of topics covered in the subject they were studying at the time. As can be seen in Figure 6.23, the majority of students in each group responded "yes", but while all of the third-year students believed CAS had improved their understanding of subject topics, the corresponding percentages of first-year students who believed this were lower (70% in 2015 and 81% for the first-year students who were surveyed later in the semester in 2016).

It was also interesting to see in which topic areas first-year and third-year students perceived CAS as useful. As can be seen from Figure 6.24, there was a similar distribution of responses for first-year and third-year students in both Algebra and Calculus, where the percentages of students in each group who found CAS useful in those topic areas were similar. In particular, 95% of the first-year students in 2015, 86% of the first-year students in 2016 and 89% of the third-year students found CAS useful in Algebra, while 95% of the first-year students in 2015, 91% of the first-year students in 2016 and 100% of the third-year students found it useful in Calculus.

Figure 6.23

Distribution of Responses of Perceived Improvement in Understanding of Subject Topics

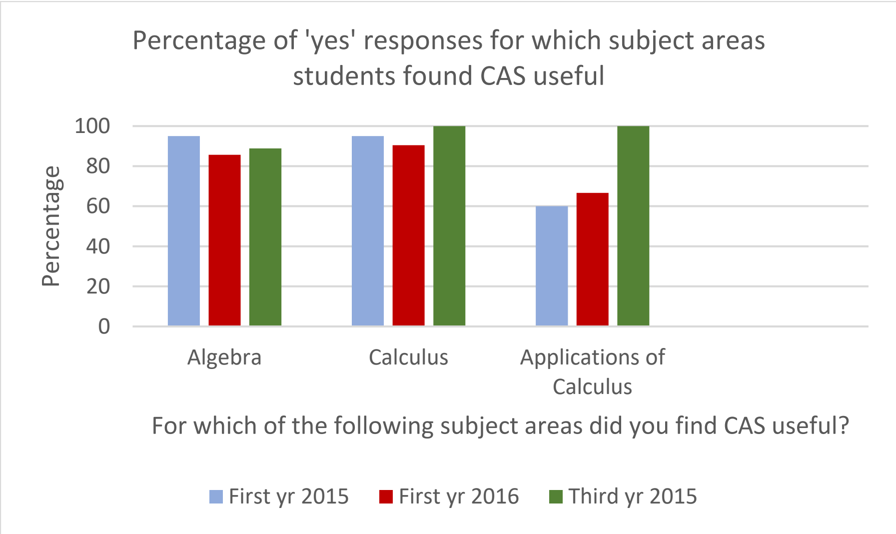


In contrast, while all of the third-year students found CAS useful in applications of Calculus, a much lower percentage of each cohort of first-year respondents reported this (60% and 67%, respectively). This could potentially be because these types of ‘applied’ Calculus problems typically become more common and also more complex by third-year, which a check of the types of exercise problems provided to students in the MATH310 course notes in comparison to the MATH104 course notes confirmed was indeed the case.

When looking at how useful the students perceived CAS to have been overall in the subject, as shown in Figure 6.25, there was also an increase in the percentage of first-year students who reported that they had found it “very useful” to have learnt CAS in the subject (67%, up from 45% the previous year), suggesting an increased appreciation of CAS after more time spent using it in the subject.

Figure 6.24

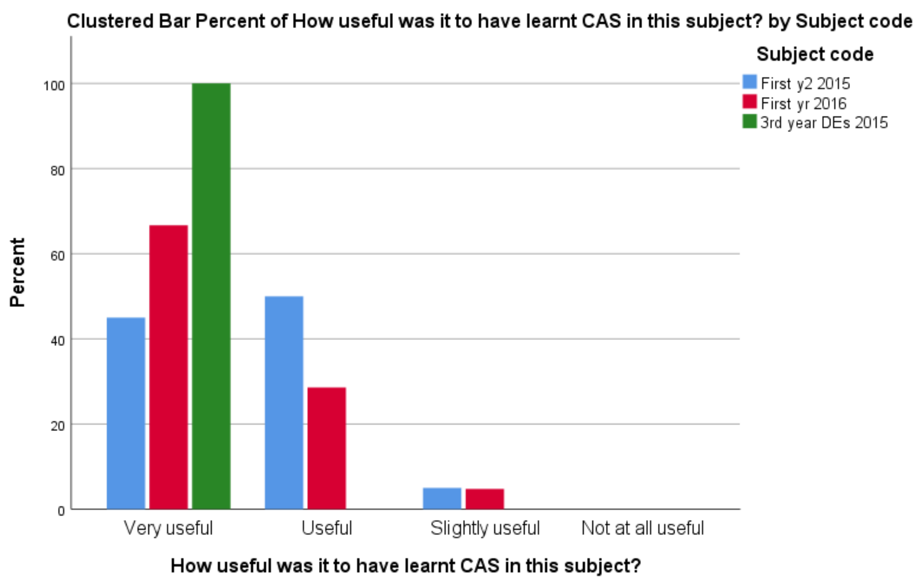
Distribution of Responses for Topic Areas Students Perceived CAS Useful



This is also consistent with there then being an increase to 100% of the third-year students reporting having found it “very useful”. The responses to this question also indicated that all students surveyed perceived at least some benefit of having learnt CAS, with none choosing “not at all useful.

Figure 6.25

Distribution of Responses for Perceived Overall Usefulness of Learning CAS in the Subject

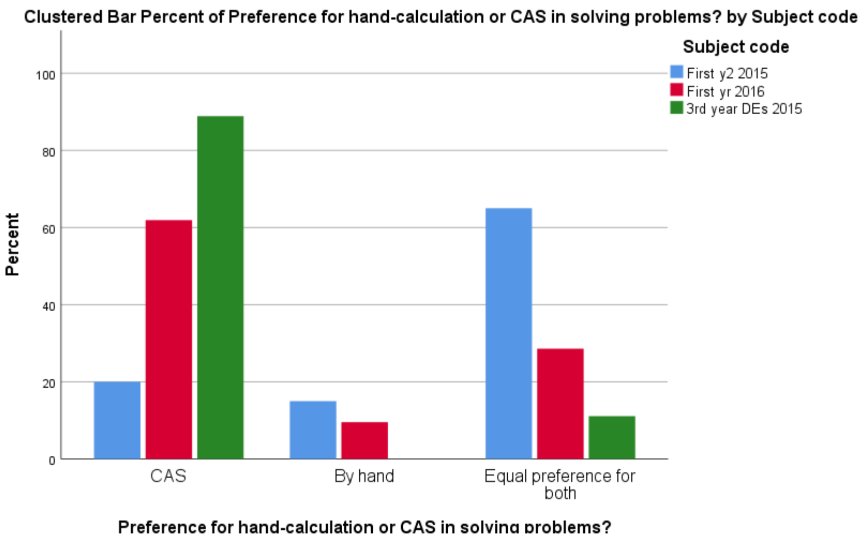


6.4.4 Preference for CAS or by-hand

When asked about preference for use of CAS, hand-calculation, or both in solving mathematics problems, Figure 6.26 shows that the percentage of respondents who chose CAS increased with the amount of time the students had been taking Calculus subjects, with the third-year students the most likely to select this option (89%), followed by the first-year students from 2016 (62%) who had been surveyed at the end of semester, with a much lower percentage of the first-year students from 2015 who were surveyed in the middle of the semester (20%). The percentage of participants who chose the "by-hand" option was lower in each case, with only 15 % of the first-year students in 2015, 10% of those in 2016 and none of the third-year students selecting this option. The remaining respondents expressed an equal preference for using CAS or working by-hand.

Figure 6.26

Distribution of Responses for Student Preference for CAS or By-hand Solving of Problems

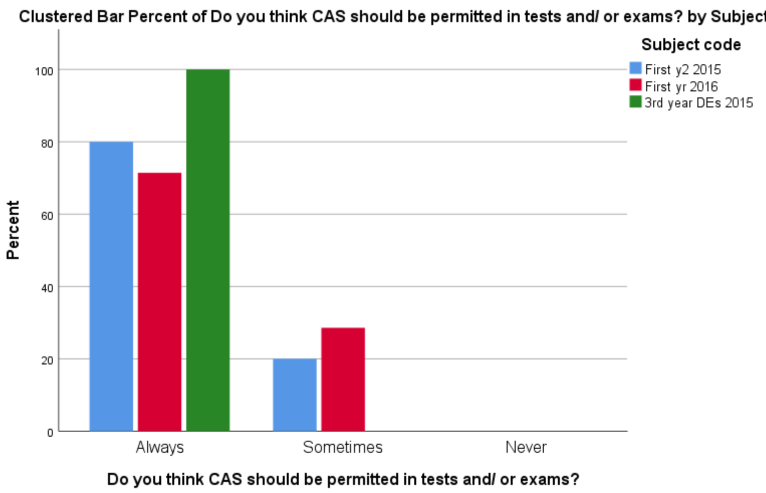


6.4.5 Access to CAS in assessment

As is shown in Figure 6.27, overall students surveyed had a positive attitude to CAS being permitted in tests and/or examinations, with all third-year students, 80% of the first-year students from 2015 and 71% from 2016 selecting "always" as their response. No students in either year level selected the option that CAS should 'never' be permitted in them.

Figure 6.27

Distribution of Responses Concerning Access to CAS in Assessments



6.4.6 Key findings and implications from survey for analysis of the task-based interviews

Overall, the responses to the questionnaire indicate that the respondents in the more recent first-year student cohort, who had been surveyed later in the semester than the previous first-year cohort, were more likely to report finding CAS enjoyable to use, to express a preference for using it over hand-calculation, and to use their CAS calculator outside of class time. The actual topic areas and types of situations in which they found it most useful showed a less consistent pattern across the two student cohorts. In comparison, the third-year cohort were the most likely to frequently use both CAS calculators and computer packages outside class time, and to use CAS for solving problems they could not work out by-hand, for seeing multiple representations of solutions and in solving problems in applications of Calculus. The third-year students were also the most likely to report that CAS had improved their understanding of subject topics, that they had found it very useful to learn CAS in the subject, to have a preference for using CAS over hand-calculations and to support it always being permitted in tests and examinations.

Almost all of the students surveyed in each group reported finding CAS useful in Calculus and Algebra topics, indicating that within the cohort of students the participants in the task-based interviews are from, students were very likely to find CAS useful in helping solve Calculus problems of the types the participants were given to solve in the sessions. That is, it indicates the interview participants are coming from a community of learning where use of CAS to solve problems of these types is recognised as useful and is also often carried out in practice.

6.5 Conclusion

In terms of difficulty and complexity of the questions used in the task-based interviews where CAS calculators were available and asked to be used, commognitive analysis of the mathematical discourse of the questions provided to the students revealed that the tasks typically had a much larger *grain size* if required to be solved by-hand than when solved using a CAS calculator (see Table 6.22), thus also increasing the level of difficulty of solving such questions by-hand. The questions asked in these tasks had relatively low levels of grammatical complexity, with question 4 given to the first-year participants showing the greatest logical complexity. The questions all contained objectified, alienated discourse, which was also typical of most of the examination and test questions analysed in Chapter 5.

The first aspect of addressing Research Question 2(a), regarding how effectively undergraduate Calculus students use their CAS calculator and the resulting CAS output, was seeing, for each question the students were given, the extent to which they appeared to be using ritual routines and the extent to which they appeared to be using exploration routines, including substantiation routines. The results showed that the students' work with a CAS calculator was often ritual in nature. Their *mostly ritual approach* was demonstrated by, for example, only two of the participants interacting with the graphical visual mediator produced in question 3 and by none of the participants adding the constant of integration to the output that relates to question 5 (a), which does not appear in the answer produced by the calculator but which is required when working out an indefinite integral. Their use of ritual routines was also demonstrated by none of them exploring in more detail the link between the form of the answer to question 1 generated by the calculator and the by-hand version using the chain rule. However, there was evidence of participants also in some cases attempting *substantiation routines* to check their answers, by also attempting to work out the corresponding results by-hand, even though this was sometimes not successful when errors occurred in their working. There was also one participant who, by zooming in on the graph of curves produced on their calculator in order to more clearly see the area between them, was also engaging in more *explorative* behaviour, when working on question 3 (version 1) and treating their calculator screen as a *dynamic visual mediator*.

A second aspect of looking at how effectively the students used their calculators and reflected on the results was examining situations where commognitive conflicts could occur between the discourse generated by CAS technology and the more ‘traditional’ classroom mathematical discourse that students were accessing from their precedent search spaces. In the two questions where I identified the potential for commognitive conflicts of this type to occur, students did indeed fall into these ‘traps’, with three participants in question 2 and two participants in question 4 encountering such conflicts and subsequently in their interviews demonstrating a lack of awareness as to why they were obtaining an incorrect answer in these instances.

In addressing Research Question 2(b), I was examining the discourse in the students’ written and verbal answers to the set tasks, to see the impact of CAS technology on the types of discourse they engaged in and to look for any evidence of transitional discourse from school discourse to the university discourse of mathematics. I was also looking for evidence of generalised mathematical discourse in the form of endorsed narratives, such as making general statements about the properties of mathematical objects or the relationships between them. I found that the students’ discourse was mostly not objectified, and, in most cases, focused on the *actions* the participants took in answering the set questions and how they talked about their answers, rather than their making more general statements or producing endorsed narratives about the more g*eneral* mathematical approaches being taken. This included the participants frequently pointing to mathematical objects on their calculator screen such as functions and graphs and referring to them in their verbal interview responses as, for instance, “that”, rather than assigning them names as mathematical objects such as “function”, “derivative”, “graph” etcetera. The presence of the discursive footprint of high school mathematics on their general Calculus discourse was evident, in the limited amount of transitioning to university level mathematical discourse. This lack of transitioning to university level discourse could also partly be influenced by the nature of the tasks they were set and the ones that they had difficulty solving. In particular, question 4 (version 1) which was given to Participants A, B and C and questions 3, 4 and 5 (version 2) given to Participant D were the questions that differed most from the type of problems they would typically be set to solve at secondary level, but they were also among the questions these participants had the most difficulty solving, with the only case in which a correct answer was produced among those particular questions being by Participant B, in answering Question 4, and with the participants in all these cases not being confident that their answer was correct. Their not being able to confidently solve those problems correctly meant they were less likely to produce university level discourse, in the form of endorsed narratives, in talking about their solution processes.

The commognitive conflicts the students encountered in questions 2 and 4 also indicated that they were affected by the discursive footprint of the general classroom Algebra and Calculus discourse commonly used when working by-hand. This suggests that such students are potentially straddling *three* discourses: those of high school, of university and also of CAS technology (which intersects with the other two discourses), with the discourse of CAS technology at times having different signifiers and also potentially leading to students typically producing less objectified, endorsed narratives when they are talking about their direct interactions with the technology in solving Calculus problems. It also means that they have to make several discursive shifts in both written and spoken discourse between these, which the participants in the task-based interviews appeared to be able to do without faltering. However, no evidence was found of their using *metaphor* in navigating between the school-level and university level mathematical discourses and that of CAS, which could also be associated with the lack of objectification in their written and verbal discourses in describing their approach to solving each of the problems set.

The participants used CAS in a variety of different ways in solving the set problems, with their most commonly using it to perform a one-step *direct, straightforward procedure*. However, as per the distinctions made by Thomas and Hong (2005), there was also one instance each of a student using CAS to *perform a direct procedure* while *implementing a new, CAS-based technique* for doing so; *using CAS in a direct procedural fashion while also working in an inter-representational fashion*; and including *use of CAS in performing a procedure within a more complex process (possibly to reduce cognitive load)* by integrating use of CAS into parts of the solution while also working out other parts by-hand.

In considering factors related to students' preference for using CAS or working by-hand, consistent with the findings of Mohammad (2019), a relationship was found between how time consuming students perceived problems to be and their preference for using CAS or working by-hand. All four participants expressed a preference for using CAS for questions they perceived to be relatively time-consuming by-hand, which was consistent with his findings. However, they also all recorded a preference for doing problems similar to the least time-consuming problem by-hand, regardless of their attitudes towards CAS, with reasons given including that they perceived that problem to be quick and easy to solve by-hand. In cases where the participants reported preferring to solve problems by-hand, there were four instances where they referred to this being if they also had access to supplementary material in the form of rules such as formulae.

In addressing Research Question 3, by surveying a larger sample of first-year and also third-year students on different aspects of their attitudes towards CAS, the results of the questionnaire indicated that the majority of the student respondents overall had a positive attitude towards CAS and that they found it useful in several general aspects of problem solving encountered in the Calculus units they were studying at the time of the survey, as well as reporting finding it generally useful in Calculus and Algebra. This is consistent with what was found for the participants in the task-based interviews, who were drawn from a similar cohort, who were most likely to use it for solving time-consuming problems and who all expressed a preference for using CAS in at least some of the Calculus problems they were set.

This chapter has advanced the thesis by looking at the research questions in relation to commognitive analysis of Calculus questions in which use of CAS can have an impact, including in providing answers in a different form to by-hand, reducing the number of steps involved to solve and, in two cases, in potentially generating commognitive conflicts with the technology. The commognitive analysis of the students' responses has allowed a detailed insight into how these participants approached answering such problems and the type of discourse they used when writing and talking about their answers and solution approaches. It also extends the types of characteristics that can be analysed in using this commognitive framework to analyse Calculus questions in which CAS can be used.

The next chapter will bring together these results with those from Chapter 5, where commognitive analysis of undergraduate Calculus examination questions was carried out, for their discussion in light of the literature on commognition, assessment and technology in Calculus. A final chapter will discuss broader conclusions that can be drawn from these results, implications for teaching, learning and assessment practice and possible areas in which this research could be extended.

CHAPTER 7

DISCUSSION

7.1 Overview

This thesis has focused on the impact of CAS on the discourse of Calculus at the undergraduate level, both in how examination questions are affected, and by how undergraduate students in a service course for the preparation of pre-service secondary mathematics teachers approach and talk about their responses to Calculus problems, when use of CAS is available. This focus on the impact of CAS in Calculus is due to this being a topic area which forms a large part of the content of first-year undergraduate mathematics units (Bressoud et al., 2016). At the same time Calculus is a topic that prospective upper secondary school mathematics teachers need to be very confident with due to its central role in the final years of several secondary school mathematics units. This more generally is an area where CAS is known to have an especially large impact (Hong & Thomas, 2015; Sevimli, 2016; Shahriari, 2019; Weigand, 2017), including when it is permitted in assessments such as tests and examinations (Buteau et al., 2014; Stacey & Wiliam, 2013). However, little study has previously been done on the *discursive* impact of CAS on Calculus examination and test questions, using a commognitive framework (Sfard, 2008; Morgan & Sfard, 2016). As will be seen in addressing the research questions in Section 7.2 below, using a commognitive framework for analysis can provide a rich, in-depth insight into key aspects of mathematics test and examination questions, in looking at both *mathematising* and *subjectifying* aspects of the discourse present in these questions, while rejecting the common assumption of the separability of the *form* and the *content* of a given question (Morgan & Sfard, 2016). Such an approach thus recognises that statements that may appear to many mathematicians to contain the same content mathematically, “may be seen by the student as anything but equivalent” (Morgan & Sfard, 2016, p. 98). Using a commognitive analysis framework can also add valuable insight into different aspects of how students interact with CAS when solving mathematics problems.

This Chapter will advance the thesis, by, in section 7.2 which follows, addressing each of the research questions in turn, in discussing the results of Chapters 5 and 6 respectively, and linking these findings back to the associated research in the existing literature on these topics. In particular, in section 7.2.1 the findings from Chapter 5 and the initial analysis of the task-based interview problems in section 6.2.2 are discussed, in relation to addressing Research Question 1 in looking at the usefulness and relevance of applying a commognitive framework to capture the complexity and level of difficulty of written test and examination questions. In section 7.2.2, I discuss how effectively undergraduate students use CAS, based on the findings from Chapter 6, in addressing Research Question 2(a). This includes a discussion of the types of routines (Sfard, 2008) the students were using in answering the questions they were given in the task-based interviews and the instances and types of commognitive conflicts they encountered and how they dealt with these. The types of solution approaches they used with CAS will also be reported, in relation to the categories of use identified by Thomas and Hong (2005). In section 7.2.3, I further discuss the findings from Chapter 6 in relation to Research Question 2(b), in examining how the students reflect on CAS output, especially through analysing the types of discourse they used, in terms of word use, both from the discourse of mathematics and of their CAS calculators, and looking for the production of any endorsed narratives, including any in which they incorporated CAS calculator commands into their resulting discourse. In section 7.2.4, I address Research Question 3 by discussing the responses from the student questionnaire in relation to students' attitudes towards CAS and the associated findings from the task-based interviews in which the participants gave reasons for their preferences for use of CAS to solve some problems and not others; providing insight into when they find its use most beneficial.

7.2 Discussion of Key Findings

7.2.1 Utility of a commognitive framework to capture complexity and difficulty of written assessment questions in undergraduate Calculus in a CAS enabled environment

Research Question 1: How can a commognitive framework be applied to effectively capture the complexity and difficulty level of written answer and multiple-choice examination questions asked in undergraduate Calculus units where use of CAS technology is available?

In Chapter 5, commognitive analysis of a selection of questions from first-year and third-year written Calculus tests and examinations from two universities was presented, in relation to addressing the first research question, with a focus on selecting examination and mid-term test questions where either CAS calculators were permitted or where symbolic or graphical visual mediators which contained CAS output were present. These included both short answer and multiple-choice questions.

Analysis of examination questions in relation to CAS technology has, in the past, focused on aspects including the extent to which CAS can be advantageous (Brown, 2003; Hong et al., 2000 and different classifications have also been made in attempting to judge its impact on the relative level of difficulty of such questions (MacAogáin, 2000). However, the results of Chapter 5 show that a *discursive* perspective (Sfard, 2008) provides an additional wealth of information, in allowing a systematic measure of the level of the *written* complexity of the questions, with systematic measures of grammatical and logical complexity (Morgan & Sfard, 2016) assisting with this.

The examination questions I analysed using a commognitive framework typically contained objectified discourse, with specialised mathematical words present from the discourses of Calculus, Algebra, Functions and Graphing. In measuring the grammatical complexity of the text, most of the questions analysed had relatively low grammatical complexity, but this increased greatly for the one question considered which was a context-separable application (Galbraith & Stillman, 2001) of differentiation, shown in Figure 5.9. Logical complexity of the written text of the questions, as per Morgan and Sfard's (2016) commognitive framework, was also found to be relatively low in most of the questions, with the exception again being in the application question referred to above. However, when analysing the complexity of *symbolic visual mediators* in Chapter 5, in applying my extension of Morgan and Sfard's (2016) commognitive framework to include a count of the number of distinct letters and arithmetic operators present, and to record the level of nesting of any composite functions, it was found that these additional measures of the logical complexity of symbolic visual mediators were informative in determining additional aspects of the logical complexity of the questions. This was because both applications questions and outputs provided by CAS software often contain both letters signifying variables and letters signifying constants, adding to the complexity of the resulting symbolic discourse. In both the application question described above and the standard applied question (Galbraith & Stillman, 2001) in the task-based interviews analysed in 6.2.2, there was greater grammatical and logical complexity than the other questions the students were given in each case. This was in part due to the greater logical complexity of the symbolic visual mediators that were present, particularly in the number of different letters signifying different variables and constants that needed to be considered. The presence of letters representing constants and parameters has, in previous research, been found to increase how difficult students find mathematics/Calculus problems to solve (Leigh-Lancaster & Stacey, 2022; Stacey, 2003). The grammatical complexity of these questions was also greater, especially in the case of the application question, in part because in describing an applied scenario which related to the visual mediators provided, more written detail was present. Applying Sfard's commognitive framework, together with my extensions regarding the complexity of symbolic visual mediators, allows additional insight into the overall complexity of such questions, especially in terms of the amount of information students need to process in order to be able to answer them correctly.

It was seen that CAS can also have an impact on the supplementary information included as part of some examination questions, through the presence of CAS output such as algebraic and/or graphical visual mediators. In turn, the distinguishing of visual mediators in questions involving CAS outputs (Alshwaikh, 2016) adds insight, with little study having been done previously of the logical complexity of such visual mediators in the context of *examinations*. The distinction made between narrative and conceptual diagrams (Kress & Van Leeuwen, 2006; Alshwaikh, 2016, 2018) was examined, in helping determine ways in which students might be expected to *position* themselves according to the interpersonal metafunction (Alshwaikh, 2018; Halliday, 1978; Morgan & Sfard, 2016), in approaching answering examination questions in which graphical and other diagrammatic visual mediators are present. In situations where visual mediators representing a graph of a function are present in examinations, these are usually in the form of conceptual diagrams, as they are diagrams in which human actions are absent, therefore indicating "atemporal objects or relationships" (Alshwaikh, 2016, p. 170), as in the examination question in Figure 5.15, which was analysed in Chapter 5. However, slope field diagrams are an exception to this, with two of these being present in the first order differential equations questions analysed in Chapter 5, from examinations given to third-year students. These slope field diagrams can potentially be considered as narrative diagrams in that the local slopes in them show the rate of change of one variable with respect to the other at different points. That is, they *offer* information (Alshwaikh, 2018; Halliday, 1985). In addition, cases such as the question in Figure 5.2, may invite some students to interact with the diagram by drawing on it to follow some of these local slopes. This could be done to, for example, determine one or more solution curves associated with the differential equation. Such visual mediators offer different modes of communication, which the examiner has deployed to convey meaning. Similar interactive behaviour by students was found in a previous study by Hyland et al. (2021), in a question where students were required to match a solution curve for a DE to a slope field. In that study, most students drew or superimposed the solution curve onto each slope field diagram provided, until they found the one where it fitted, indicating their treating the slope field diagrams provided in the question as narrative diagrams, responding to the perceived invitation to do so. However, it is also possible that students could use other strategies in solving such questions, such as substituting (x,y) values into the equation of each differential equation provided in the answer options, to calculate the slopes at those points and then comparing them with the slopes shown on each diagram to see which one matched. The results of Hyland et al. (2021) indicated some students also used this method. The question arises then if whether the diagram is narrative or conceptual depends on each individual student's positioning of themselves in relation to it and how they interact with it. I concur with Alshwaikh (2018) that "understanding what we communicate and what we want to communicate in diagrams is particularly important" (p.13), and even more so in the context of examinations. The second such question I analysed, from Figure 5.5 in Chapter 5, already had a dotted solution curve superimposed over the slope field. This suggested, as per the examples in Alshwaikh (2018), that in the solution process for solving the corresponding differential equation, this solution curve was *added* to the diagram, after first determining the local slopes in the diagram. That is, it suggested the visual mediator being presented as a *narrative diagram*, giving representation of a mathematical *activity* taking place over *time*, in first finding the slope fields and *then* superimposing the solution curve corresponding to a particular initial condition that must also be satisfied in the question to be answered.

There are also some examination questions which require the students themselves to produce graphical visual mediators, of functions of the form $y = f(x)$, where this can either be an explicit requirement in the instructions or optional, but beneficial, to a student in correctly answering the question. For example, questions involving areas bounded by curves are of this type, and one such question was present in one of the first-year examinations analysed. In that particular question, there was an explicit instruction to include a sketch of the graphical visual mediator that showed the region bounded by the two curves (see Figure 5.16). If produced on a student's CAS calculator, such a diagram could subsequently be treated as either a *static visual mediator* or a *dynamic visual mediator* (Antonini et al., 2020; Ng, 2016), depending on whether, or not, a given student interacts directly with it to see a the graph of a function more clearly by, for example, changing the calculator's **window** settings or by zooming in on the graph. Students' responses to a similar type of question was analysed in the task-based interviews, and, although in that case the students were not required to produce a sketch of the graphs, the three participants who attempted it all produced them using their CAS calculator, with one treated as a dynamic visual mediator as by zooming in on the curves, as was seen in section 6.3.3.

The results from Chapter 5 indicate that when CAS outputs are included as symbolic and graphical visual mediators, they can increase the complexity of a question. Consideration of the *interpersonal metafunction* (Alshwaikh, 2018; Halliday, 1978; Morgan & Sfard, 2016) regarding how students *position themselves* in relation to an examination question can also influence how complex an individual student finds some such examination questions. In particular, if a student is positioned as a *knower* of CAS outputs such as those produced by Wolfram Alpha, these will typically have less complexity for them, as they can, as was seen in Chapter 5, relatively quickly determine which lines of output they need to even consider, in order to answer the given question. In contrast, if a student is not a knower of such outputs, the amount of processing required in interpreting every line of output provided can in some cases be much greater, an example of which is shown in Figure 5.7, where there are some lines of output with high levels of logical complexity, but which do not even need to be considered to answer the actual question asked. This, in turn, means a student who is not a knower of this output, could potentially spend a lot of time reading and attempting to process lines of mathematical output that in some cases are complex, but that are redundant and do not assist in answering the actual question asked.

In determining the types of routines the students have been expected to carry out in examination questions, the socially situated context in which the students have been taught these questions is important. For example, in Thoma (2018), this was determined by interviews conducted with the subject's lecturers. In my study, to determine this, the course notes for each of the units considered were referred to, to see the usual expected solution process for these problems. It was found that the examination questions considered were mostly expecting ritual routines, with questions of the same type present in the course notes. However, in questions which could be solved both by-hand and using a CAS calculator, there was also an opportunity for *substantiation* routines, if for example, a student attempted the question by-hand and then checked it using their CAS calculator, or vice versa. The questions asked that invited more *explorative routine behaviour,* and that differed the most from the corresponding examples provided in the course notes, were the applications question in Figure 5.11 from the HMS112P examination and the slope fields questions in Figures 5.3 and 5.5 from the MATH314 and MATH310 examination papers, respectively. These two slope fields questions potentially invited *construction routines* if the students interacted with, or drew on, the graphical visual mediator provided and, in the second of these questions, in requiring production of written endorsable narratives about the local slopes and how they related to the associated differential equation.

When looking at *endorsed narratives* in the examination questions considered in this study, it was found that the discourse of the questions was frequently alienated, most commonly referring to either agentless relationships between mathematical objects or relationships in which one mathematical object could be seen as acting on another mathematical object. The only question suggesting *human* action was the context-separable (Galbraith & Stillman, 2001) application question in Figure 5.9, in which the depth of engagement with the context was very low due to the hot air balloon and its passenger described in the question having no actual role in determining the correct answer.

When examining the student autonomy aspect of *subjectifying discourse*, whether or not use of CAS was an option or could be beneficial was one aspect of this considered, as students have greater autonomy in choosing their solution path when there are more possible solution methods available. This is the case when students can choose whether to answer a question using CAS, by-hand or with a combination of both. None of the questions had an imperative instruction stating that students were *required* to use CAS, so in the examinations where use of CAS was permitted, students had autonomy regarding the choice of whether or not to use it in all the questions in which it could potentially be used to help in answering a question. However, using it would frequently reduce the level of difficulty in answering the question and is optimal in a timed examination situation.

The *grain size* (Morgan & Sfard, 2016) of tasks with and without use of CAS was also found to be an informative way of analysing examination questions, not only in determining the *student autonomy* aspect of how many independent decisions the students were required to make (Morgan & Sfard, 2016), but also as one aspect of measuring the associated *level of difficulty* of the examination questions. As was seen from the results of Chapters 5 and 6, the grain size of many of the set Calculus examination tasks or task-based interview problems is 2 when these tasks are solved using a CAS calculator, with one of these decisions in such cases being choosing the correct CAS function or command to use and the other decision involving then entering the correct arguments when using this command. Even in other questions that require more steps using CAS, the grain size is still usually relatively low, when compared to the grain size of when these tasks are required to be solved by-hand, which is in some cases much greater. Other questions such as calculating an area bounded by curves or maxima/minima problems take more steps to solve even when using a CAS calculator, as they are an *application* of a Calculus technique such as integration or differentiation, where that technique is used as part of a longer problem-solving process. Contextualised problems can also sometimes require more steps to solve, as in the example in Figure 5.11, in cases where they require several pieces of information to be synthesised, such as needing to substitute $z = \sin(vt)$ into the expression for T in the above example. Multiple-choice questions can also sometimes fall into this category, in cases where the answer produced by the CAS requires further rearrangement and/or simplification in order to match it to the correct answer option, as was seen in analysing the decisions required in answering the question in Figure 5.11.

Those questions that can be solved in one or two steps using a CAS calculator have, in the past, sometimes been classified as “CAS trivial” (MacAogáin, 2000), especially in cases where the questions require direct calculation of derivatives, integrals or solving an ordinary differential equation. However, such a classification does not take account of the grammatical/logical complexity of such questions and of any associated visual mediators. The results of this study have shown that application questions or other questions requiring the use of provided CAS output can require the processing of a lot more information than other standard Calculus examination questions, as seen in the previous discussion of the grammatical and logical complexity of such questions. A commognitive analysis framework is valuable in that it gives a rich, informative picture of the nature of the stories told about mathematical objects in an examination, including how both words and any associated visual mediators are used. It also analyses how the examination questions speak to students and the action that is subsequently expected of them in answering each question (Morgan & Sfard, 2016), taking account of the socially situated context in which the examination questions are set.

7.2.2 Effectiveness of undergraduate Calculus students' use of CAS

Research Question 2(a): How effectively do undergraduate Calculus students use their CAS calculator and use and interpret CAS calculator output, especially when it is in a different format to what they would obtain by the methods of working by-hand they have been taught in class?

In previous literature on the impact of CAS technology on students' work, considerations in determining how **effectively** students use CAS have included measuring whether they take full advantage of its capabilities relative to answering a given question, whether or not students use it correctly to obtain the correct answer and how thoroughly and accurately they interpret the results produced. For example, Pierce and Stacey (2004) looked at how fluently students used CAS syntax/commands, their flexibility with changing representations, and how well they interpreted CAS output. How students reflect on and manage answers produced by CAS, when they are not in the form the students would expect if working-by-hand, has also been analysed, for example, by getting students to work out the same set of problems by-hand and by using CAS, in cases where the resulting answers appear different using each method and to discuss their opinion on the accuracy of each resulting answer (Tonisson & Lepp, 2015). The majority of students in that study frequently did not provide an adequate analysis or exploration of the link between their by-hand answers and the corresponding answers produced by CAS (Wolfram Alpha) and often made incorrect judgements about the accuracy of each answer. Previous studies have also examined the extent and effectiveness of students' use of CAS technology and interpretation of the results when solving application problems, with Rogovchenko (2021) finding fourth-year Engineering students' use of Maple and Matlab to help in solving mathematical modelling problems involving differential equations was mostly "as a computational and verification tool", including some visualisation, but with it not becoming "a transformational or data collection and analysis tool" (p. 569).

Using a commognitive framework can provide additional insight into how effectively students use CAS, by classifying the types of *routines* they use in solving mathematics problems as explorations, deeds or rituals (Lavie et al., 2019; Sfard, 2008; Viirman & Nardi, 2021). In applying a commognitive analysis framework, the findings of

Rogovchenko's (2021) study were that mostly *ritual routines* were used by fourth-year Engineering students using Maple and Matlab CAS software to solve mathematical modelling problems, while they made use of more explorative use of routines in other aspects of the solution process where they did not use CAS. Applying a commognitive framework to the task-based interviews data for the first- and second- year student participants in my study was also informative, indicating that the student participants were also using mostly *ritual routines* in both their use of CAS and by-hand problem solving methods, which was demonstrated by their limited interaction with the graphical visual mediator produced in question 3, by none of the participants adding the required constant of integration in question 5, which does not appear in the CAS calculator output, and by none of them exploring in more detail the link between the form of the answer to question 1 generated by the calculator and the by-hand version using the chain rule. This could partly be due to the problems they were given to solve focussing on procedural rather than investigative skills for both CAS and by-hand work. Participants in some instances did, however, attempt *substantiation routines* to check their calculator answers by-hand. One participant, by zooming in on the graph of two curves produced on his calculator in order to more clearly see the area between them, was also engaging in more explorative behaviour, when working on question 3 (version 1), and treating the calculator screen as a dynamic visual mediator (Antonini et al., 2020; Ng, 2016).

Another important aspect of discursive analysis, which can lead to greater understanding of some of the sources of error in students' work, is being able to identify *sources of commognitive conflicts* (Sfard, 2008; Nachlieli & Heyd-Metzuyanim, 2022). As was seen in Chapter 2, one of the ways in which these occur is in situations where two interlocutors are communicating with each other but are using mathematical signifiers, which can be either words or symbols, in a different or inconsistent way. In turn, this results in signifiers from one discourse being used incorrectly or inconsistently in the context of another discourse, where they actually have a different meaning. Such commognitive conflicts can be interpersonal or intrapersonal (Kontorovich, 2021), with the latter occurring when inconsistencies occur within an individual's discourse. Commognitive conflicts have previously been identified between discourses in the transition from secondary to tertiary mathematics (Biza, 2021; Thoma, 2018) and between students' discourse and that of standard mathematical discourse as well as between the discourse of different scientific disciplines, such as Biology and Mathematics (Viirman & Nardi, 2021). However, the students in my study also engaged in discourse with their CAS calculator, as per Antonini et al. (2020), who looked at participants' discourse with dynamic interactive mediators produced by technology. In a similar vein, the CAS calculator output can be "considered an agent in the discourse" (Antonini et al., 2020, p. 7). Thus, this study has identified sources of commognitive conflict between Computer Algebra Systems technology itself and the classroom discourse of algebra, Calculus and mathematics more generally, with the presence and nature of these conflicts also sometimes being dependent on the model of CAS used (e.g., TI-Nspire CX vs Casio Classpad 330). The instances where these conflicts occur in this study can be considered to be intrapersonal commognitive conflicts between a student's written (and spoken) mathematical discourse and the discourse they have entered into their CAS calculator, which can have a different meaning. This can often be a source of error, with part of the cause appearing to occur because of students accessing the part of their *precedent search space* (Lavie et al., 2019; Viirman & Nardi, 2021) that is associated with traditional algebraic discourse from school (and university), in terms of how algebraic expressions are typically represented. In particular, in the context of algebra, omission of the multiplication sign if two letters or a letter and a bracketed algebraic expression are multiplied together (i.e., xy or $a(b + c)$ is a common, widely accepted way of writing such expressions in the usual endorsed communities of algebraic discourse, and is also the form in which the algebraic expressions appeared on the question sheet the students were given. However, when such expressions are entered into a CAS calculator, as noted in Chapter 6, TI-NSpire CX calculators interpret the omission of the multiplication sign between two letters as signifying the name of a new variable. Furthermore, both TI-NSpire CX and Casio Classpad 330 models of calculators (used by the students in this study) identify the omission of a multiplication sign between two letters and a bracketed expression as signifying a function named by the letter outside the brackets, with its argument being the expression inside the brackets. All three students who used a TI-NSpire omitted the multiplication sign between variables on at least one occasion across 3 task-based interview questions and two of the three students who attempted a question involving the expression $P(t) = P_0(1 + t^{2.5})e^{-kt}$ omitted the multiplication sign between the P_0 constant and the corresponding bracketed expression, in the task-based interviews. This, combined with their comments in the task-based interviews, suggests that, although the students made these errors, none of them identified their cause, in relation to these commognitive conflicts, suggesting a lack of knowledge about the nature of the symbolic mathematical discourse required to be entered into the calculator. This is consistent with the findings of Sangwin and Ramsden (2007), who found students made errors of this type when using Maple software, which also has this requirement regarding the presence of the multiplication sign in algebraic expressions of these types. An additional commognitive conflict was found to occur when the students were answering a question requiring implicit differentiation, between the standard Calculus discourse of differentiation, where a partial derivative is represented with the symbol ∂ and the corresponding symbolic discourse produced by CAS calculators, where the symbol d is used to signify partial differentiation. This again led to commognitive conflict for three of the four participants, with them again not being aware of the cause of this error. When the CAS output was unexpected, the students did not engage in much exploration to determine the reason, a finding consistent with findings of other studies such as Tonisson and Lepp (2015).

Applying Thomas and Hong's (2005) classification of different types of solution approaches that can be employed when using CAS, was seen to provide further insight into the nature of the students' use of CAS and how effectively they were using it, in solving the set problems. The most common approach they took was to use CAS to *perform a direct straightforward procedure*, typically answering a question in one step in such cases. This can be considered effective use of CAS in such instances, if that is all that is required to answer a given question and the correct answer is obtained. However, the reason for applying this classification was to also see if any of the participants used more complex processes using CAS, such as working with multiple representations or integrating use of CAS into part of a solution process. The finding of one instance where a participant demonstrated *use of CAS in performing a procedure within a more complex process*, in solving the question involving finding the area between two curves, indicated a more integrated solution approach in incorporating use of CAS into parts of the solution, while also working out other parts of her answer by-hand. Another participant solved that same problem by *using CAS in a direct procedural fashion while also working in an inter-representational fashion*, which is also indicative of some types of questions affording a greater variety of options in the ways in which they can be solved. This second participant did not obtain the correct answer, but the general type of use of CAS he employed would have been considered effective had he done so. Applying Thomas and Hong's (2005) classification also identified that same participant, in another question, was using CAS *to perform a direct procedure* while *implementing a new, CAS-based technique* for doing so, with the students' reported reason for doing so being because he was not sure what else to do. While it could be argued that the method he used in answering that question was not the most efficient, his use of CAS in this way could still be considered effective, due to his still being able to devise a technique for using it which obtained the correct answer to the problem.

7.2.3 Student reflection on answers in task-based interviews

Research Question 2(b): What can we learn from a commognitive analysis of task-based interviews of students in relation to how they reflect on their answers in this situation?

In investigating how students reflected on their answers to the questions in the task-based interviews, I was looking at the nature of their word use, for instances of their producing endorsed narratives and for any cases of their making transitions from school to university-level discourses, as well as at the general effect of CAS technology on the type of discourses they produced in reflection. There are differences in the types of word use and discourse used in the final years of high school and at university level. University level discourse will often be strongly objectified (Sfard, 2014), with the resulting alienated and reified discourse frequently present in endorsed narratives about the mathematical objects that students are learning about in university mathematics units. Students experience changes in mathematical cultures (Corriveau & Bednarz, 2017; Biehler, 2019) and discourses from high school to university, which can also produce changes in both their object-level and their meta-level learning (Sfard, 2008). The nature of these changes in mathematical cultures from high school to university is further influenced by different educational contexts (Thoma & Nardi, 2018), with concepts such as tangency (Biza, 2017, 2019, 2021; Biza & Zachariades, 2010) also having different properties and associated discourses across different topic areas in mathematics, adding to the importance of students recognising the context in which a given topic or mathematical construct is being used. The *discursive* footprint (Biza, 2021) is a useful concept in relation to this, which considers the characteristics of the discourse about a given type of mathematical object, across the different areas of mathematics taken by students where this object appears.

The expectations placed on students also differ in the transition from the final year of high school to university level mathematics in areas including in the rigour and level of detail expected in mathematical reasoning in a wide range of topic areas including, for example, proofs in set theory and probability (Thoma, 2018), local properties of functions in the context of Calculus (Bressoud et al., 2016) and the conceptual basis of derivatives in Calculus (Holton & Artigue, 2001). In turn, this leads to an expectation at university level that students will be engaging less in ritual routines and more in other types of routines including substantiation and construction routines. In this study, the nature of students' previous learning experiences, including the educational context in which they have learnt and are learning about topics in Calculus and associated fields such as Algebra and Geometry, were therefore taken into consideration as factors that could have an effect on how students position themselves when attempting to solve university level Calculus problems, and which could consequently have an effect on the nature of the resulting discourse which they produce.

In looking for evidence of transitional discourse from school discourse to the university discourse of mathematics, the first aspect I considered in analysing the task-based interviews was the nature and the extent of the students' specialised mathematical *word use*. There were two ways in which the students used specialised mathematical words: in the context of mathematics and in the context of the commands used on their calculators to solve the mathematical problems presented. I will consider each of these in turn.

In the context of describing the mathematics the students used in attempting to solve the set problems, the following specialised words (and phrases) were used verbally or in writing by at least one participant, that referred to naming mathematical **objects** specific to the discourse of Calculus: *integral, definite integral, derivative* and *partial derivative*, with the last of these terms (*partial derivative*) only being used by the second-year participant (D). The following mathematical words and phrases were used by at least one participant in describing the mathematical **processes** used which are specific to the discourse of Calculus: *chain rules, differentiate/d, implicit differentiation* and *implicitly*. Words and phrases referring to objects that also occur more generally in the associated discourse of Algebra were also used by the students, including: *coefficient, constants, variable, equation/s, formula, expression, power, symbol, both sides* and *term*, while the word *solve* was used in relation to mathematical processes in the field of Algebra and the word *algebra* itself was also used. Words naming mathematical objects which also occur in the general discourse of Functions and Graphing included: *area, axis, function, intercept/s, intersection, points, root, sin* and *cos*, while the word *graphed* was used to describe a process. The words *small* and *large* were used to describe the perceived size of the area between the curves in one question and the location words *above*, *below*, *upper* and *lower* were used to describe the relative position of the two curves in that question, with the terms *upper* and *lower* also being used in a second context, that of the limits of integration when finding the area bounded by the curves. Words used by at least one participant which name mathematical objects and which also occur in the broader discourse of arithmetic included: *brackets, number* and *decimal,* while the words used describing mathematical processes in arithmetic included: *add, divided, multiplication, plus, taken away* and *minused,* with the last two of these words indicating colloquial school-level discourse. The word *negative* was used in producing one of the few endorsed narratives present in the participants' work, with two statements by participant B that *area cannot be negative.* The general specialised mathematics words *calculation* and *rule* were also used in some of the participants' discourse. All of the above specialised mathematical terms used by the participants appear not only in university-level mathematical discourse, but also in high school discourse. That is, these words all appear in the intersection between these two communities of discourse, so the students were accessing these words from their precedent search space from secondary (and even primary) school in the context of their current university studies.

The students did not use much objectified mathematical discourse, and thus did not produce many endorsed narratives about the properties of mathematical objects or the relationships between them, with the only stand-alone endorsed narrative produced which made a *general* statement about relating mathematical objects being the previously mentioned statement by participant B, in the context of the question involving finding the area between two curves, that "area cannot be negative". I was also looking for any evidence of transitional discourse (Antonini et al., 2020) in the form of any statements that would not satisfy the requirements for scholarly mathematical discourse but that could have a relationship with scholarly mathematical discourse in what the students were attempting to articulate. For example, when, in answering question 2, participant B stated that: "you obviously want to get ***it*** in terms of $\frac{dy}{dx}$ so I made that equal z and then I typed that in and solved for z", in more formal mathematical discourse "get it in terms of $\frac{dy}{dx}$ " could have been replaced with "make $\frac{dy}{dx}$ the subject." . This is also an example of how in the few statements made by the participants, that had the potential to be endorsable narratives, mathematical objects were often not explicitly named with mathematical specialised words, but instead were often referred to as "it" or "that" in communicating with me as interviewer, clearly expecting that the meaning was recognisable by me as I was a member of the mathematical discourse community. The discourse used in such cases was also often at least partially colloquial in nature, such as, in his answer to the same question, Participant B stating that "if *I got it in terms of y* I could easily *do* $\frac{dy}{dx}$." This particular statement is also actually not a fully endorsable narrative in the context of this problem (question 2), as in that question there was no way to obtain an implicit expression of the form $y = f(x)$.

As well as the few narratives about mathematical objects having the limitations described above, they also suggested a rule-based, ritual approach to answering the questions, as was also apparent in the following verbal discourse from Participant D, in describing her answer to question 5 (version 1) when required to find the integral of $sin(x) + cos(x)$: "the integral of cos is sine and the integral of sine is negative cos", with this statement also disregarding the requirement to add two constants of integration.

When subsequently proceeding to look at the parts of the students' discourse which incorporated specialist mathematical words in the form of CAS commands used, I firstly found that when talking about their use of the CAS calculator, the following words from the discourse of Calculus were used by participants, in describing the menus accessed: *dSolve*, *calculus, derivative, integral* and *impDif,* while the words *gsolve, graph, intersect, root, zero* and *zoom* were used in the context of the discourse of graphing and working within the graphing window on the calculator and the word *solve* was used from the discourse of Algebra. The words *analysis, calc, calculate, control, function, interactive, menu, shift* and *scratchpad* were used from the discourse of the more general commands, menus and function keys on the calculator.

With very few endorsable narratives produced by the students using only the discourse of mathematics, it was expected that more endorsable narratives would be produced that had some of the above words from the discourse of the CAS calculator and its associated commands. As a part of this, I was also looking for any evidence of their using metaphor to transplant any words localised to the discourse of the model of CAS they were using (such as of the commands used or outputs produced) into their broader mathematical discourse. For example, this could have occurred if they had produced endorsed narratives about mathematics which included CAS-specific specialist word use, incorporating this into endorsable narratives about the relationships between the mathematical objects they were working with. However, most endorsable narratives produced by the students around how to access calculator menus and commands in solving some of the problems were only in abbreviated command-based form, for example, **Menu → Calculus → Derivative**, with only basic additional instructions sometimes being included, such as, "then input equation and press **enter**." Thus, the CAS calculator commands were not transplanted into endorsable narratives about mathematical objects in the discourse of mathematics in many instances, with the exception being the following written discourse from Participant C: "within **analyse graph** I used i**ntersection** to find where the graphs intercept finding that they did at (-4,4), (-3,0) and (-2,-4)". However, even this was not a fully endorsable narrative, due to the incorrect use of mathematical word "intercept" here, instead of "intersect."

These findings suggest that, in the task situation presented in the task-based interviews at least, that the student participants were not yet transitioning to the scholarly, university level discourse of mathematics through mechanisms such as metaphor, as no evidence of their transplanting specialised mathematical words nor CAS calculator commands into higher-level university mathematical discourse was evident for either the three first-year participants nor for the second-year participant. Consequently, no evidence was found of meta-level learning through the production by the students of any written or verbal endorsed narratives about mathematical objects and the relationships between them. Potentially due to their interaction with the CAS technology, the students did not produce many objectified endorsed narratives about mathematical objects and the relationships between them or mathematical processes involving them. Rather than naming the mathematical objects they were working with using mathematical discourse, they also frequently referred to mathematical objects as "it" or "that" when pointing to the objects they were discussing in communication with me. This could also be an effect of their describing how they used CAS technology in their solution process, as this often occurred when they were talking about commands entered or outputs produced on the screen of their CAS calculator. Previous studies have highlighted what their authors frame as weaknesses that can be present in the mathematical discourse of some pre-service secondary mathematics teachers. In commognitive theory terms this would be viewed in terms of the particular discursive routines they were actually using in different task situations, together with what that indicated about their interpretation of those task situations and the likely nature of their own precedent search spaces. For example, Van

Jaarsveld (2016) found, in a study of pre-service secondary mathematics teachers in South Africa, that some had “a poor command of the exact language of mathematics" (p. 150), which would partly relate to the precedent search spaces they were accessing in different task situations. In turn, this can mean a lack of engaging with mathematical discourse as a *mathematician*, which could occur for a variety of reasons and is also a concern with students taking mathematics units in other service courses. In investigating Biology students’ engagement with graphing routines in mathematical modelling tasks, Viirman and Nardi (2021) identified the students’ previous experiences with graphing as “highly influential precedent events” (p. 3) and they also found some instances of the students evolving from use of ritual routines to explorations with evidence of meta-level learning, as they progressed through the set tasks, although the extent to which this occurred varied across the student groups. Their study demonstrates an approach and considerations for engaging students in service courses of this type in mathematical activities designed to promote de-ritualisation and meta-level learning, as well as showing how commognition can be used to follow the students’ evolving discursive activity as they work on these tasks.

7.2.4 Undergraduate Calculus students’ use of CAS

Research Question 3: To what extent do undergraduate Calculus students use CAS and when do they believe its use to be most beneficial?

The purpose of the questionnaire analysed at the end of Chapter 6 was to answer this research question and to provide further insight in relation to addressing the other research questions, within the local context of the study, as these students are not typical or representative of undergraduate Calculus students in all universities and this is a caveat that must be applied to all the findings from this thesis. Overall, the responses to the questionnaire indicated that the longer the students had been taught with CAS during their university studies, the more likely they were to have a positive attitude towards it and to use it more frequently outside class time. Almost all of the respondents from both year levels surveyed (first-year and third-year) also reported that they had found it useful in Calculus and algebra topics. This suggests the participants in the task-based interviews, who came from this cohort, would have also been very likely to use it at least sometimes outside of class time and to perceive CAS as being generally useful in solving the types of problems that they encountered in the task-based interviews, suggesting they would be expected to have reasonably good proficiency with the CAS calculator in solving these types of problems. The frequency of responses for the different aspects of the utility of CAS suggest the students are most likely to find its use beneficial for algebra and Calculus topics, for checking solutions calculated by-hand and, in the case of the first-year students, also for saving time in solving problems and, for the third-year students, also for solving problems they could not do by-hand.

Examining the results of the task-based interviews to determine the reasons for the participants' preference for using CAS or working by-hand to solve problems of each type they were given, provided further insight into factors which can influence students' preferences. As with the questionnaire responses, the participants in the task-based interviews, in several instances, stated a preference for use of CAS for solving problems, or parts of them, when they perceived it would be more time-consuming to solve them by-hand. This is consistent with the findings of Mohammad (2019), while, in a few other instances, the students expressed a preference for solving problems by-hand, when they perceived that using CAS to do so would be more time-consuming. There were also two instances where the participants expressed a preference for solving a problem by-hand but where they stated they would then check the answer with CAS. Whether or not they felt confident about solving problems of particular types, availability of rules such as formulae and the presence of graphs of functions that could be visualised with CAS to obtain insight into their nature, were also reported as factors in determining whether or not use of CAS was their preferred solution method.

7.3 Conclusion

The results of this study show that commognitive analysis provides a valuable lens for analysing features of examination questions in which CAS is present, both when CAS calculators are permitted to be used in answering the question and when CAS screen output is present in symbolic or graphical visual mediators within examination questions. My extensions to the commognitive framework of Morgan and Sfard (2016), in analysing the logical complexity of graphical and symbolic visual mediators by also looking at the number of different letters and arithmetic operators used, help to determine the amount of information the students will need to process in answering an examination question where these are present. The aspect of *visual complexity* I have added to the framework, which looks at the number of lines of redundant CAS output present that are not required to answer a question containing such output, was found to also relate to the interpersonal metafunction (Alshwaikh, 2018; Halliday, 1978; Morgan & Sfard, 2016) regarding how students *position* themselves in answering an examination question, as the complexity of such a question for an individual student will, in part, depend on the extent to which a student is a knower of such outputs and if they are therefore selective in choosing to look at only the parts of the output which are actually required to answer the set question. Similarly, my documenting the different features present in graphical visual mediators and the number of graphs presented when considering *visual complexity* also adds further insight into the amount and complexity level of the information that needs to be processed by the student, with their positioning relative to an examination question again partly determining how complex and time-consuming they find this information to process. Distinguishing between whether graphical visual mediators were narrative or conceptual diagrams (Alshwaikh, 2016; Kress & Van Leeuwen, 2006) was also found to make a useful contribution in helping to determine how students might interpret graphical visual mediators of each type and position themselves in relation to interacting with such diagrams in approaching answering examination questions containing graphical visual mediators.

The analysis in Chapter 5 also shows that, while examination questions that either are permitted to be answered using a CAS calculator or that contain CAS output reduce the number of procedural steps required as opposed to when working by-hand, some of these questions are in other ways more logically, visually and grammatically complex. Using a commognitive analysis framework to analyse examination scripts where CAS technology is applicable can identify more aspects to consider in evaluating both the complexity of questions and how such questions might speak to students in terms of their expected response and positioning in relation to them.

Using a commognitive analysis framework to analyse students' responses to the task-based interviews in Chapter 6 added insight into how effectively students approach answering first-year Calculus questions when using a CAS calculator. A part of this was looking for sources of intrapersonal commognitive conflicts (Kontorovich, 2021), which occur when there are inconsistencies within an individual's discourse. In particular, I was looking for the occurrence of these when students were interacting with their CAS calculators in solving the set problems. As was described in section 7.2.3, there are several instances where the discourse that needs to be entered into a CAS calculator is different from the corresponding expressions that are commonly used in the community of algebraic discourse. Looking for commognitive conflicts of this type offered valuable insight into the discursive behaviour of the participants as they interacted with CAS technology, with three types of such commognitive conflicts being encountered in parts of the students' working. Part of the cause of these commognitive conflicts occurring appears to be because students are accessing the part of their *precedent search space* (Lavie et al., 2019; Viirman & Nardi, 2021) that is associated with traditional algebraic discourse from school (and university), which, as mentioned above, presents algebraic expressions differently, in some cases, from how they are required to be entered into the CAS calculator, in order to obtain the same result.

Looking at the types of routines the student participants in the task-based interviews carried out, especially identifying instances of their using ritual or explorative (substantiation and construction) routines (Lavie et al., 2019; Sfard, 2008; Viirman & Nardi, 2018), was valuable in gaining insight into the nature of their problem-solving approaches when using a CAS calculator to solve Calculus problems and this also related to how effectively they were using the calculator. One aspect of exploration students could potentially use when interacting with their CAS calculator was found to be treating graphical output as a dynamic visual mediator (Antonini et al., 2020; Ng, 2016), by interacting directly with the graph, for example by zooming in to see the required bounded areas between curves more clearly.

In analysing the nature of the participants' mathematical *discourse* in the task-based interviews, applying the concept of *discursive footprint* (Biza, 2021) was valuable in identifying the importance of considering the educational *context* the students in this study are coming from, especially in terms of their background mathematical knowledge and discourse from high school and their associated precedent search space (Lavie et al., 2019; Viirman & Nardi, 2021). This made possible the finding from Chapter 6 that the word use and associated discourses of the participants appeared to not have transitioned from the type of discourse they would have engaged in at high school into university-level mathematical discourse.

Sfard's (2008) theory of commognition, together with Morgan and Sfard's (2016) commognitive analysis framework with the extensions described in this chapter, is well suited to analysis of the effects of technology such as CAS, as technology use affects both the discursive characteristics of questions, such as increased complexity when needing to unpack complex written information (Morgan, 2016) and also students' discourse in answering such questions. Identifying sources of, and the nature of, commognitive conflicts between the discourse of students and the technology can assist in teaching practice and in interpreting sources of difficulty when students encounter these conflicts. Applying a commognitive framework was helpful in indicating that the discourse produced by the student participants in the task-based interviews contained very few endorsed narratives about mathematical objects, which could in part be due to the students becoming more focused on describing the technical commands used on their calculators to obtain their answers, and talking about their solution process by, in some cases, pointing to mathematical objects on their calculator screen rather than explicitly naming these mathematical objects using the discourse of Calculus. The analysis also indicated these students were not transitioning to university-level mathematical discourse, when talking about their answers to the questions, which could also in part be due to the typical motivation level of a student cohort who are being educated in service courses such as becoming secondary school mathematics teachers, with some such students not wanting nor identifying the need to join a tertiary mathematics discourse community.

CAS technology has continued to evolve since the collection of the 2015 - 2017 data used in this study, and this process is ongoing (Leigh-Lancaster & Stacey, 2022). This evolution has occurred both in its mathematical capabilities, including the addition of an increasing number of dynamic tools to CAS packages such as GeoGebra (Craig & Akkaya, 2022) and with more options for how some CAS packages can be engaged with discursively, as in the case of Mathematica (Nieto & Ramos, 2021). While the focus of this study has been on assessment *with* technology (Drijvers et al., 2016; Fahlgren et al., 2021), the greater presence of assessments *through* technology (Drijvers et al., 2016; Fahlgren et al., 2021) since the beginning of the pandemic in 2020 has increased the number of Mathematics examinations being conducted via computer which, in some cases, has involved student access to and use of CAS software in such examinations, such as Matlab (Betteridge et al., 2022). These changes since 2017, combined with associated changes in some teachers' pedagogical practices related to emergency remote teaching, may have also had some influence on current students' attitudes to CAS and the ways in which they engage in mathematical discourse when using it.

The next and final chapter of the thesis will look at broader conclusions that can be drawn from the results of this study, limitations of the findings and implications for potential future research in this field.

CHAPTER 8

CONCLUSIONS AND IMPLICATIONS

8.1 Overview

In this chapter, in section 8.2, I present a synthesis of the key findings from this study, together with their contribution to research in the field of commognition. I then, in section 8.3, outline some implications for practice, in terms of ways in which these findings could be applied and extended in the field of teaching and learning. Section 8.4 covers limitations in the current study, while section 8.5 covers possible ways in which this research, on applying a commognitive framework in looking at the effect of CAS technology on Calculus examination questions and how effectively students use and talk about CAS, could be extended. I then, in section 8.6, give some concluding remarks on this study and my experiences of learning about commognition.

8.2 Key Findings and their Contribution to Research on Commognition

CAS technology is constantly evolving and has its own distinct forms of symbolic and graphical discourses, which also vary according to the model of software being used and the form in which CAS is present (e.g., on a handheld calculator or on a computer). This distinctive discourse occurs both in the outputs produced by the different types of CAS and also in the inputs the user is required to make to produce these results. This discursive feature of CAS invites use of a commognitive framework to determine its impact in different settings and educational contexts; an area which has not previously been directly explored using commognition.

In applying a commognitive framework (Morgan & Sfard, 2016) to the analysis of examination questions in which CAS technology is involved, a key finding of my research, which relates to Research Question 1, is that the complexity of examination questions where outputs from CAS technology are included is partly dependent on how individual students are *positioned* in relation to answering such questions. This is because examinations “provide specific positions for students, that is, ways in which students may interact with the text and act within the practice” of university mathematics (Morgan & Tang, 2012, p. 242). In the case of examination questions which include CAS outputs, this positioning is influenced by the extent to which individual students are *knowers* of such outputs, which in turn will be influenced by the historically situated context of the extent and ways in which they have been taught about how to approach such questions. This will influence how complex they find such examination questions because, if they are knowers of such outputs, in many cases there will be parts they can either quickly disregard as not being required in answering the set question or which they can rewrite in a simpler form. For example, in the case of symbolic visual mediators produced by software such as Wolfram Alpha, there will often be several lines of output presented which are not required to answer the set question, such as in the output shown in Figure 5.7, and CAS calculator outputs will also often present constants in different, and sometimes more complex ways as, for example, in the symbolic visual mediator shown in Figure 5.9. The rules of the classroom community, in terms of how students are required to rewrite such expressions and constants, will also vary in different classrooms and educational settings, demonstrating the value of the commognitive perspective in considering doing mathematics as communicating in the discourse of a specific mathematical community. This is even more imperative at the secondary-tertiary interface, as students are in transition and often have to switch from one type of discourse to the other in a moment. Considering CAS using commognition, as an extra discourse to which students need to become a member, identifies how the presence of CAS complicates the situational context even more. How students interact with graphical visual mediators produced by CAS technology, especially those such as slope fields diagrams which can contain a large degree of visual complexity, will also be affected by student positioning in relation to these diagrams. The additional measures of logical complexity of symbolic visual mediators and visual complexity of graphical visual mediators I added to the Morgan and Sfard's (2016) commognitive framework, help bring out these features, together with the additions from Alshwaikh (2016) concerning whether graphical visual mediators are in the form of narrative or conceptual diagrams.

In looking at how effectively students use CAS calculators, in addressing Research Question 2(a), a previously unexplored aspect of this in commognitive research was the extent to which they managed potential intrapersonal commognitive conflicts *caused by the nature of the discourse of their CAS calculators*. Intrapersonal commognitive conflicts occur when there are inconsistencies in an individual's discourse (Kontorovich, 2021). This frequently happens as the result of individuals making errors, in, at times, using the discourse of a particular mathematical community incorrectly. However, when being taught topics in which CAS is used to solve mathematics problems, students are often required to straddle two discourses concurrently: first, the classroom mathematical discourse they are being taught which is used, for example, in describing mathematical results and when working by-hand and second, that of the CAS technology itself, which in topics such as Algebra, Functions and Calculus, does not always have the same rules. That is, they are concurrently interacting both with a community of classroom mathematical discourse and the discourse of their CAS calculator which, in some cases, have different rules. In other words, in applying a commognitive framework in looking for sources of *commognitive conflicts* in students' work in solving mathematics problems using a CAS calculator, it was found that participants are, in certain task situations, concurrently *required* to use symbolic mathematical discourse in a seemingly inconsistent manner. This is because the required usage is dependent on its *context* (in the context of standard mathematical classroom discourse or in the context of the discourse of the CAS calculator), which must be perceived by the user. In turn, intrapersonal commognitive conflicts are consequently seen to be occurring when participants continue to apply the rules they have learned in classroom mathematical discourse when engaging in symbolic discourse/communication in interacting with their CAS calculator. Such conflicts were found in this study across the discourses of Algebra, Calculus and Functions. So an equation or other symbolic visual mediator will, in some cases, be produced differently on a question sheet or when students are writing down their working on a paper, from the way in which they need to input it into CAS technology. In this thesis, examples of this have been found when using two models of CAS calculators. Being able to identify the different types of commognitive conflicts that can emerge from such use of CAS can contribute to making pedagogical recommendations regarding its application in different classroom contexts. Commognitive conflicts also sometimes occur in using other types of CAS software, which will be discussed further in the next section when looking at the implications of these results for practice.

Applying a commognitive approach, in looking for evidence of explorations and rituals when students are making use of their CAS calculators in solving mathematics problems, indicated mostly ritual behaviour by the students in attempting to answer the Calculus problems they were provided in the task-based interviews, with a few instances of attempting to substantiate answers by-hand after first doing them on their CAS calculator.

These occurred only when not many steps were involved in doing so, and only using further standard ritual routines they had been taught for differentiation and working out definite integrals by-hand. In the question involving finding the area between two curves, which was the only question requiring use of a graphical visual mediator, one participant's zooming in on the graph suggested the beginnings of potentially more explorative behaviour.

The application of commognitive analysis to looking at the task-based interviews in this thesis, in addressing Research Question 2(b), has provided an initial, detailed insight into the discourse of a cohort of undergraduate Calculus students interacting with CAS calculator technology and talking about the steps they took in doing so, and about their corresponding results. The influence of students' previous educational background and experiences carries over into their discursive practices, both in answering and talking about their answers to questions in mathematical tasks. In this study, applying the principles of commognitive analysis to looking at their written and verbal discourse has made it possible to see that their discourse in working on these tasks had much more in common with school level discourse than with the scholarly mathematical discourse that students at university level are expected to engage in. This raises questions about the possible reasons for this. Biza's early research (Biza, 2008; Biza and Zachariades, 2010), prior to her shifting to a commognitive analysis framework in looking at how students use and talk about tangents, led her to seeing "elements of what students had learned in previous years intertwined together" (Biza, 2021, p. 2), but she had also identified the need for further analysis to capture "the subtlety of the effect that these previous experiences had on students' responses." Her subsequent definition of *discursive footprint* (Biza, 2021) and her application of a commognitive framework, allowed for a much deeper insight into the discursive behaviour of the students in her study and a connection to their precedent knowledge from school. In my study, students' past experiences with both the usual classroom discourse of mathematics and in working with CAS technology are likely to be interacting, in contributing to the nature of their current discourse, with their level of motivation to join the community of more scholarly discourse also a likely contributing factor, as was discussed in Chapter 7. Consistent with what Biza describes, the detailed and systematic nature of analysis enabled by using a commognitive framework in my study in analysing the students' responses in the task-based interviews, has given me greater insight into the nature of the discourse of students from this cohort, when they are interacting with CAS technology. In turn, this knowledge can help in contributing to good teaching practice with CAS, as will be discussed in the next section, through an awareness of some current weaknesses and limitations in students' discourse which were identified from the analysis of the task-based interviews, as well as having the awareness of the historical and present mathematical and associated technological contexts in which such students are doing mathematics.

8.3 Implications for Practice

Commognitive analysis of examination questions allows not only for evaluating the effect of CAS on the difficulty and complexity level of these, but also for looking at "the ways that mathematical activity itself is altered by changing the language used to present it" in such questions (Morgan & Tang, 2012, p. 4). When students are allowed access to a CAS calculator (or other form of CAS technology) in such examinations, applying a commognitive framework to analysis of potential examination questions being designed and considered is valuable, in ensuring the questions are in fact assessing the skills required in the course, and to check that they are indeed formulated to produce or seek evidence of the types of activities by the student cohort answering them that align with the learning objectives of the subject.

Application problems and so-called "word problems" can have much greater complexity than those considered here, especially when set as assignment questions or class exercises. Even with a relatively small number of sentences in the questions analysed in this study, measures such as the logical and grammatical complexity are already shown to have increased considerably, with the resulting algebraic symbolic visual mediators also frequently increasing in logical/visual complexity due to there often being present both letters representing variables and letters representing constants. The commognitive framework allows a systematic measure of the complexity of the discourse of such questions, which in turn can inform good assessment practice in setting such questions with the intended "level of difficulty" for a given year level and classroom context.

Awareness of the interpersonal metafunction (Halliday, 1978), regarding how the students might position themselves in relation to examination questions involving outputs from CAS software, can also inform associated teaching practice, in terms of ensuring students understand which parts of the output are relevant to solving problems, together with an awareness of the context of the additional output, both in its general meaning and relevance. This is also true of computer outputs from statistical software that is not necessarily CAS-enabled. For example, statistical packages like Minitab and SPSS will often produce a lot of output when running procedures such as hypothesis testing and even descriptive statistics, but students will typically be positioned to only use a small proportion of the statistics provided, which are the ones needed in the context of answering the specific types of questions they are dealing with in the unit.

The identification, through use of the commognitive analysis framework, of potential commognitive conflicts occurring in relation to students' interaction with CAS technology, also has important implications for teaching practice, not only in raising awareness about the occurrence of these conflicts when students are using CAS calculators of the types looked at in this study, but also when using other types of CAS technology, including some CAS computer packages. For example, when using the Mathematica CAS software to input mathematical commands, square brackets are required to enclose the argument of a function, in place of the round brackets used in the standard mathematical discourse of functions. As seen in this study, such conflicts can occur both in relation to entering inputs into CAS technology and in interpreting the output it generates. The outputs produced by some types of statistical software, some of which are not even CAS software, also sometimes use words, symbols and abbreviations differently than in the standard discourse of statistics. For example, the statistical program SPSS uses the abbreviation *sig* in its outputs, to signify the *p*-value associated with an hypothesis test, but in classroom statistical discourse, some teachers use the phrase *sig level* as an abbreviation for signifying the level of significance at which an hypothesis test is being performed. Awareness of the potential for commognitive conflicts in situations like this can help inform teaching practice, to ensure that, where possible, the same terminology is not being used by instructors in two incompatible contexts. If it is inevitable, as in the case of discourse of functions and that of CAS calculators, it is important that students are instructed in the importance of understanding what certain words and symbols signify, in each of the contexts where they appear with distinct meanings. They also need to be made aware of the required syntax to be entered when using CAS in these situations (e.g., not omitting the multiplication sign when doing algebra on the TI-Nspire model of CAS calculator used in the current study), so that students know what they need to enter in order to obtain a correct result.

As is also common with people being educated for other professions, prospective secondary mathematics teachers will not always be aware of the importance and necessity of looking past ritual learning practices and this also has the potential to adversely affect their subsequent teaching practices in the longer term. The use of commognitive analysis allows an in-depth insight into their mathematical behaviour in task situations such as the task-based interviews conducted in this study. In turn, this can help inform teacher education to improve aspects of pre-service teachers' mathematical discourse that are identified as weak or ritual in nature.

8.4 Limitations of the Study

Below I present the limitations of this study, in terms of the data which was analysed and on the nature of the conclusions which could be drawn from the results.

1. The range of different types of examination questions looked at

These were limited to a selection from mathematics examination papers from two universities, with the majority of them from one of the two universities.

2. The types of CAS technology considered

Only two models of CAS calculators (TI-NSpire CX and Casio Classpad 330) were used by the student participants in this study and outputs from one additional type of CAS software (Wolfram Alpha) was analysed, with these all considered over a limited period of time (from 2015 to 2017). This is important to keep in mind, as CAS and other digital technology has continued to evolve rapidly since then, in terms of its capabilities and user interfaces in, for example, some of its graphing capabilities.

3. The sample size for the task-based interviews

For the task-based interviews, only one student cohort from one university were invited to participate and the resulting sample size was small (only 4 participants). This was due to difficulty finding enough student participants to volunteer to take part in the study.

4. The amount of detail recorded in the task-based interviews

The task-based interviews only took place at one time point and without participant interaction with the investigator while they are working on the tasks, nor were they asked to work aloud or to have their actions recorded using video. This limited the amount of insight that was possible into some of the participants' thought processes when they were working on the set tasks and producing their written responses and attempts at the set problems.

5. The limited amount of knowledge of the students' past educational backgrounds

When understanding the discursive behaviour of participants using a commognitive framework, having insight into the nature of the precedent search space (Lavie et al., 2019; Viirman & Nardi, 2021) they are likely to be accessing is informative, in seeing the evolution of their discourse and understanding contributing factors to its current form. In the case of the student cohort considered in this study, the nature of their precedent search space will be partially dependent on whether the pre-requisite Mathematical Methods (or equivalent) was the only mathematics subject they studied in year 12 or whether they also did the more advanced Specialist Mathematics subject, whose discourse is more mathematical and where the level of complexity of mathematics problems is also greater. I did not request that information about the participants' previous educational background because it was only towards the conclusion of this research that I realised that it would have been useful.

8.5 Implications for Future Research

The analysis of examination scripts and task-based interviews could be extended to cases where other types of CAS are used, including software programs like Mathematica, where the required inputs are relatively complex and in some cases again require different syntax to what would be required in the standard discourse of algebra and functions if working by-hand, as was described section 8.3. This could also include cases where students are given more opportunities to engage with the dynamic elements that CAS use can afford, both when using CAS calculators and when using CAS packages such as Mathematica and GeoGebra that have additional Dynamic Geometry features such as sliders, which also make it possible to vary the values of parameters while interacting with graphical visual mediators. Studies that include analyses comparing the impact and uses of different types of CAS, using a discursive perspective, could also be conducted.

In future commognitive research involving task-based interviews, information should also be collected from each participant regarding the type of course they are doing (e.g., a service course, such as for future secondary mathematics teachers, or a more specialised mathematics degree), their future academic and career aspirations, and their current view of the role of mathematics in their life. This should include determining whether each student taking part sees mathematics as relevant and beneficial to them in the long-term, with motivation to transition to the scholarly discourse of mathematicians, or if they just see it as a tool to be utilised in the short term, to help them successfully complete their current course.

In task-based interview problem solving sessions involving use of CAS, recording digital videos of the participants working during the session in future studies would provide valuable additional information, as has been done by others including Viirman and Nardi (2021), as this would give a greater depth of insight into the effect of CAS technology on students' discourse when they are working on a range of mathematics problems, in capturing more of their intermediate actions when there are engaged in the problem-solving process, which they might not document in writing or comment on verbally in task-based interviews. In looking for evidence of de-ritualisation, there is the potential for future group studies (Tetaj, 2021; Viirman & Nardi, 2021), with more complex problems, recording all parts of the discussion and also potentially allowing input or questioning from the instructor during the problem-solving process.

The information derived from the task-based interviews and summarised in Table 6.23, regarding reasons for each participants' preferences for when to use CAS in solving the set problems from this study, could be used as the basis for formulating questionnaire items for future studies. These studies should also be scaled up to include a greater number of participants.

As discussed in Nardi et al. (2021), longitudinal studies that employ commognitive analysis of students' discursive behaviour are starting to emerge. This approach could also be valuable in future research following on from this thesis, in investigating students' discursive behaviour and exploratory behaviour in relation to using CAS technology at various time points over their period of studying mathematics, including looking for evidence of any de-ritualisation, and to see the extent to which the different types of commognitive conflicts identified in the "snapshot account" from this study between the technology and standard mathematical discourse are resolved, and to what extent they persist.

Some related questions for such research could be:

1. How do different types of CAS software affect the level of difficulty and discourse of Calculus examination questions, when their outputs are present in examinations?
2. How does undergraduate students' discourse when using CAS change over time, in learning to solve different types of Calculus problems?
3. How does a student's chosen future career (e.g., becoming a secondary mathematics teacher) shape the discursive practices in which they engage?

8.6 Concluding Remarks

Researching and applying a commognitive framework has given me a greater appreciation of the complexity and diversity of factors that influence the nature of how we assess mathematics, talk about it and the importance of students' precedent knowledge and the nature of the community of mathematical discourse in which they are studying, and how this interacts with CAS technology and its own rules of discourse.

APPENDICES

APPENDIX A: Student CAS Questionnaire

1. In studying for this subject, how often did you use a CAS calculator outside of formal class time?
   - ☐ Several times a week
   - ☐ Sometimes
   - ☐ Occasionally
   - ☐ Never

2. In studying for this subject, how often did you use CAS computer software outside of formal class time?
   - ☐ Several times a week
   - ☐ Sometimes
   - ☐ Occasionally
   - ☐ Never

3. What type/s of CAS computer software did you use outside of formal class time? Tick all that apply.
   - ☐ Wolfram Alpha
   - ☐ Mathematica
   - ☐ Maple
   - ☐ MATLAB
   - ☐ Other (please specify) ______________________________

4. For which of the following, did you find CAS useful? Tick all that apply.
   - ☐ Saving time in solving problems
   - ☐ Checking answers that you had already attempted by-hand
   - ☐ Solving problems where you did not remember (or understand) how to do them by-hand
   - ☐ For seeing multiple representations of a solution (e.g., graphical)
   - ☐ Other (please specify) ______________________________

5. Do you believe use of CAS has improved your understanding of topics covered in this subject?
   - ☐ Yes
   - ☐ No
   - ☐ Not sure

6. For which topics covered in this subject did you find CAS useful? Tick all that apply.
   - ☐ Algebra
   - ☐ Calculus
   - ☐ Applications of calculus (e.g., Taylor Polynomials)
   - ☐ None

7. Overall, how enjoyable did you find use of CAS during your study of this subject?
   - ☐ Very enjoyable
   - ☐ Moderately enjoyable
   - ☐ Tolerable
   - ☐ Moderately unenjoyable
   - ☐ Very unenjoyable

8. Overall, how useful to you think it was to have learnt how to use CAS for the topics covered in this subject?
   - ☐ Very useful
   - ☐ Moderately useful
   - ☐ Slightly useful
   - ☐ Not at all useful

9. If given a choice, for problems in this subject that could be solved either by-hand or using CAS, which method would you prefer to use?
   - ☐ CAS
   - ☐ By-hand
   - ☐ Equal preference for both

10. For future mathematics subjects, do you think students should be permitted to use CAS calculators in tests and/ or exams?
    - ☐ Always
    - ☐ Sometimes
    - ☐ Never

APPENDIX B: Task-Based Interview Instructions and Tasks

"Use your CAS calculator to solve the following problems. Write down all commands you use on CAS to do so. In each case, also comment on whether the CAS output is what you expected. If it is not what you expected in some cases, how do you think it could be reconciled with what you expected? Where you think it appropriate, also consider/ demonstrate any methods by-hand or on CAS you could use to check or interpret your answer." [Participants had access to their own CAS calculator and were given the following tasks with ample space to answer.]

Question 1 for Participants A, B, C

1. **Find $\frac{dy}{dx}$ given $y = (x^3 + x)^{\frac{2}{3}}$.**

Question 2 for Participants A, B, C

2. **Find $\frac{dy}{dx}$ given $x^2y + e^{2y} = 3xy$.**

Question 3 for Participants A, B, C

3. **Find the area bounded by the two curves $y = x^3 + 9x^2 + 22x + 12$ and $y = -4x - 12$.**

Question 4 for Participants A, B, C

4. **A particular quantity is known to have value $P(t) = P_0(1 + t^{2.5})e^{-kt}$ at time t, where P_o and k are constants. Find the rate of change of the quantity P with respect to time t.**

Question 5 for Participants A, B, C

5. **Determine $\int(\sin x + \cos x)\,dx$ and $\int_0^{\pi}(\sin x + \cos x)\,dx$**

Question 3 for participant D

3. **Solve $\frac{dy}{dx} = \frac{1}{2}(1 + y^2)$ such that $y(0) = -1$. Write your answer in the form $y = f(x)$.**

<u>Question 4 for participant D</u>

4. Solve $3x^3y' = y^4$ such that $y(2) = 1$.

<u>Question 5 for participant D</u>

5. Find the general solution of $\frac{d^2y}{dx^2} - 4y = e^{2x}$. Hence write down complementary function and a particular integral.

APPENDIX C: An Extract from Alshwaikh's Analytical Framework showing its Structure

	Property of the Discourse	Specific Questions Requiring Analysis	Indicators in Verbal Text	Indicators in Visual Text
a) How is the nature of mathematics and mathematical activity construed?	**Specialisation**	To what extent is specialised mathematical language used?	• vocabulary used in accordance with mathematical definitions • 'conventional' expressions • mathematical symbols	• 'conventional' mathematical diagrams, charts, tables, graphs and labelling systems
	Further properties include: objectification, alienation, logical structure, status of mathematical knowledge			
b) How are the learners and their relationship to mathematics construed?	**Agency**	What kind of activity is the learner expected to engage in?	• 'thinker' or 'scribbler' processes ascribed to the learner (e.g., imperatives, *you*...)	
	Further properties include: authority, formality			
c) What role does the text play?	**Construction of knowledge**	Does the text present facts or develop arguments? What is assumed?	• thematic progression • Given-New	• information value • Given-New (horizontal) • Ideal-Real (vertical) • Centre-Margin
	Further properties include: topic, structure			

Source: Alshwaikh & Morgan (2013, p. 72)

APPENDIX D: Alternative Solution Processes for Questions Analysed in Chapter 5 in Helping to Determine Grain Size

Table D1

Decisions for Solution of Q1, MATH203 Test A Using Alternative Approach

Result obtained	**Decision actions**
Second order	Determine the order of the differential equation
Option C eliminated	Eliminate answer options that do not include this
Non-linear	Determine if it is linear or non-linear
Options A and E eliminated	Eliminate answer options that do not include this
Ordinary	Determine if it is ordinary or partial
Option D eliminated so B is the correct answer	Eliminate answer options that do not include this
	6 decisions

Table D2

Decisions for Solution of Q9, MATH314 Test A, 2017 (Alternative Approach 1)

Choice taken	Decision action
Second order DE	Identify that the DE presented is second-order
Complementary function must have two arbitrary constants	Identify that complementary function of a second order DE of the type shown will have two arbitrary constants
Option B: $Ae^{-2x} + Be^{-x}$	Select the only answer option that has two constants
	3 decisions

Table D3

Decisions for Solution of Q9, MATH314 Test A, 2017 (Alternative Approach 2)

Choice taken	Decision action
$r^2 + 3r + 2 = 0$	Determine the auxiliary equation associated with the DE
$r = -2$ or -1	Solve the auxiliary equation for r using factorisation or the quadratic formula
$y_h = Ae^{-2x} + Be^{-x}$	Determine the form of the complementary function, given the auxiliary equation has 2 real roots
Option B	Match with the correct answer option
	4 decisions

Table D4

Decisions for Solution of Q12, MATH314 Test A, 2017 (Alternative Approach)

Result obtained	**Decision actions**
$m^2 - 4 = 0$	Find the auxiliary equation
$m = 2\ or - 2$	Find the roots of the auxiliary equation
$y_h = Ae^{2x} + Be^{-2x}$	Write down the complementary function
$y_p = Cxe^{2x}$	Find the general form of a particular integral
$y_p' = Ce^{2x} + 2Cxe^{2x}$	Differentiate y_p
$y_p'' = 2Ce^{2x} + 2Ce^{2x} + 2Cxe^{2x}$ so $y_p' = 4Ce^{2x} + 4Cxe^{2x}$	Find the second derivative of y_p
$4Ce^{2x} + 4Cxe^{2x} - 4Cxe^{2x} = e^{2x}$	Substitute y_p, y_p' and y_p'' back into the original DE
$4Ce^{2x} = e^{2x}$ So $C = \frac{1}{4}$	Solve for C
$y_p = \frac{1}{4}xe^{2x}$	Substitute the value of C into y_p
Option C	Select the correct answer option
	10 decisions

Table D5

Decisions for Solution of Q1, MATH104 final examination, 2015 (Alternative Approach 1)

Choice taken	**Decision action**
$f(0) = 0$	Evaluate $f(x)$ at 0
$f'(0) = 0$	Evaluate the first derivative at 0
$f''(0) = 2$	Evaluate the second derivative at 0
$f'''(0) = 0$	Evaluate the third derivative at 0
$f''''^{(0)} = -4$	Evaluate the fourth derivative at 0
$p_4(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f''''(0)\frac{x^4}{4!}$	Identify the general required form of the Maclaurin polynomial
$p_4(x) = 0 + 0 + 2\frac{x^2}{2!} + 0 - 4\frac{x^4}{4!} = x^2 - \frac{x^4}{6}$	Substitute in the values of f and its derivatives at x = 0
	7 decisions

Table D6

Decisions for Solution of Q1, MATH104 final examination, 2015 (Alternative Approach 2)

Choice taken	**Decision action**
$f'(x) = \sin(x) + xcos(x)$	Find the first derivative of $f(x) = xsin(x)$
$f''(x) = 2\cos(x) - xsin(x)$	Differentiate again to find 2nd derivative
$f'''(x) = -3\sin(x) - xcos(x)$	Differentiate again to find 3rd derivative
$f''''(x) = xsin(x) - 4cos(x)$	Differentiate again to find 4th derivative
$f'(0) = 0$	Evaluate f(x) at 0
$f'(0) = 0$	Evaluate the first derivative at 0
$f''(0) = 2$	Evaluate the second derivative at 0
$f'''(0) = 0$	Evaluate the third derivative at 0
$f'''^{(0)} = -4$	Evaluate the fourth derivative at 0
$p_4(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f''''(0)\frac{x^4}{4!}$	Identify the general required form of the Maclaurin polynomial
$p_4(x) = 0 + 0 + 2\frac{x^2}{2!} + 0 - 4\frac{x^4}{4!} = x^2 - \frac{x^4}{6}$	Substitute in the values of f and its derivatives at x = 0
	11 decisions

Table D7

Decisions for Q13(a), MATH104 final examination, 2015 (Alternative Approach)

Choice taken	**Decision actions**
	Plot the two curves
	Shade (or otherwise identify) the area bounded by the curves
$x = -1\ or\ x = 1\ or\ x = 2$	Find the x values where the curves intersect (either by solving $x^3 - 2x^2 + 5 = x + 3$ algebraically or by using a graphing intersection tool on the calculator
$\int_{-1}^{1} x^3 - 2x^2 + 5 - (x + 3)dx$	Write down the integral for the area of the leftmost bounded region (taking account of which is upper curve)
$\int_{-1}^{1} x^3 - 2x^2 + 5 - (x + 3)dx = \frac{8}{3}$	Evaluate the resulting integral
$\int_{1}^{2} x + 3 - (x^3 - 2x^2 + 5)dx$	Write down the integral for the area of the second bounded region
$\int_{1}^{2} x + 3 - (x^3 - 2x^2 + 5)dx = \frac{5}{12}$	Evaluate the resulting integral
$\frac{8}{3} + \frac{5}{12} = \frac{37}{12}$	Add up the two areas to get the total area enclosed by the curves
	8 decisions

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