

# Empirical Evidence for Niss' *Implemented Anticipation* in Mathematising Realistic Situations

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Mathematisation of realistic situations is an on-going focus of research. Classroom data from a Year 9 class participating in a program of structured modelling of real situations was analysed for evidence of Niss's theoretical construct, implemented anticipation, during mathematisation. Evidence was found for two of three proposed aspects. In addition, unsuccessful attempts at mathematisations were related in this study to inability to use relevant mathematical knowledge in the modelling context rather than lack of mathematical knowledge, an application oriented view of mathematics or persistence.

Mathematical applications and modelling of real world situations are receiving increased emphasis in several curricula internationally at the moment (e.g., Ministry of Education, 2006). The teaching and learning of applications and modelling has been the subject of on-going research for many years (Blum et al., 2007; Kaiser et al., 2011; Niss, 2001). Two of the areas receiving on-going attention have been the mathematisation (i.e., translation into mathematics) of the idealised problem formulated from the real situation and the reverse process, de-mathematisation (i.e., interpretation of mathematical outputs of modelling in terms of the real situation). Recently, Niss (2010) has added to the theoretical models informing research in these areas.

The notion of mathematising has been promoted by the mathematics frameworks for PISA 2003, 2006, 2009 and 2012 assessing the mathematical literacy of 15 year olds roughly at the end of compulsory schooling. In the earlier frameworks what seemed to be presented as mathematisation was the entire mathematical modelling cycle (see OECD, 2009, p. 105). The mathematical modelling cycle is described as a key feature of the draft PISA 2012 mathematics framework (OECD, 2010) but mathematising is less prominent. It is one of seven fundamental mathematical capabilities that underpin the framework. Mathematising is taken to mean the fundamental mathematical activities that are involved in "transforming a problem defined in the real world to a strictly mathematical form...or interpreting or evaluating a mathematical outcome or a mathematical model in relation to the original problem" (OECD, 2010, p. 18). The latter would be termed "de-mathematisation" by Niss (2010). Mathematising is thus seen in PISA as one of the cognitive capabilities that can be learnt through schooling so as to enable students to understand and engage with the world in a mathematical manner. The purpose of this paper is to demonstrate whether or not there is empirical evidence for one of the main explanatory constructs of Niss's model of mathematisation processes (Niss, 2010, p. 57).

## Theoretical Framework

Researchers in the area of applications and modelling often use diagrams of the "so-called" modelling cycle to discuss what appears to be happening at a task and mental level during modelling. These are mere simplifications but they are a useful means of communicating amongst international researchers. The diagram used by Niss (2010) is reproduced in Figure 1. This shows two quite disparate domains in this particular representation of modelling: the extra-mathematical domain (i.e., the real world situation

and the modeller's idealisation of this) and the mathematical domain. Others such as Stillman (1998) argue for the importance, particularly in the context of schooling, of also including in these diagrams a representation of the blending of the real world and mathematical world. Idealisation occurs through making assumptions and identifying elements in the situation which are of interest which are then formulated, that is, specified into a problem statement which may take the form of a question. The mathematical domain includes the mathematical model that has been made of the situation, and mathematical artefacts (such as graphs and tables) that might be used in solving the mathematical model. The idealised situation is mathematised through a process of translation into mathematics and similarly mathematical outputs need to be de-mathematised, that is, interpreted in terms of the idealised situation and the real situation which was the launching point for the modelling in the first place.

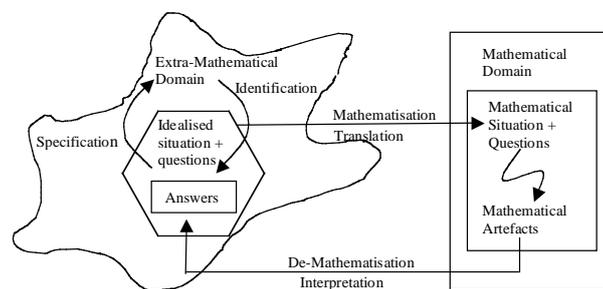


Figure 1. Modelling processes (after Niss, 2010, p. 44)

In order to produce a theoretical model of the mathematisation process, Niss (2010) introduces the construct, “implemented anticipation” (p. 54). Successful mathematisation, according to Niss, involves anticipating what will be useful mathematically in subsequent steps of the cycle and implementing that anticipation in decision making and carrying out actions. Firstly, the idealisation of the real situation from the extra-mathematical domain involves implementing decisions about what elements or features are essential as well as posing any related question or statement of the problem in light of their anticipated usefulness in mathematising. Secondly, when mathematising this formulation of the problem situation the modeller needs to do this by anticipating mathematical representations and questions that, from previous experience, have been successful when put to similar use. Thirdly, when anticipating these mathematical representations, the modeller has to be cognisant of the utility of the selected mathematisation and the resulting model in future solution processes to provide mathematical answers to the mathematical questions posed by the mathematisation. This involves anticipating mathematical procedures and strategies to be used in problem solving after mathematisation is complete. Thus successful implemented anticipation involves a three step foreshadowing process.

To be able to successfully use implemented anticipation in mathematising a real or realistic situation modellers need to: (1) possess relevant mathematical knowledge, (2) be capable of using this when modelling, (3) believe a valid use of mathematics is modelling real phenomena, and (4) have persistence and confidence in their mathematical capabilities (Niss, 2010, p. 57). Clearly, this is a challenging process. It is reasonable to expect that modellers, especially new modellers, would experience this challenge and have difficulties related to the three foreshadowing aspects of implemented anticipation. These difficulties

might be explained by these four requisites. Both successful and unsuccessful attempts at modelling or applying mathematical knowledge to real situations are opportunities for developing deeper “metaknowledge about modelling and mathematisation, in particular” (Schaap, Vos, & Goedhart, 2011, p. 145) and thus should be the foci of any study of mathematisation.

Researchers (e.g., Galbraith & Stillman, 2006; Schaap et al., 2011; Stillman, Brown, & Galbraith, 2010) have found evidence of beginning modellers in secondary schools having difficulties with mathematising because of impeding formulations of the problem statement. However, no one to date has attempted to use Niss’s model in analysing classroom data. In this paper we will attempt to use the model as the basis for our analysis of data from a year 9 class of beginning modellers who had participated in a program of quite structured modelling over one year. To operationalise the mathematising construct for research purposes we take as its starting point the formulated statement of the problem situation. This may or may not be formulated as a question. The end point of mathematising will be the mathematical model.

### The Study

As part of the RITEMATHS project, a series of three modelling tasks were used in a class of 21 Year 9 students. The data used in this paper relate to the implementation of the last of these tasks, *Shot On Goal*. Names used are pseudonyms. In brief, the task uses a soccer context (see Figure 2) where the modelling problem involves optimising a position for an attacking player to attempt a shot on goal whilst running parallel to the sideline (for details, see Stillman et al., 2010). The teacher anticipated that students might have difficulty with task formulation so he used a nearby soccer field to provide an outdoor demonstration at the beginning of the lesson sequence. The teacher used a rope parallel to the sideline (i.e., perpendicular to the goal line) as a run line for several students to run down and stop when they had the best shot on goal. One student then stayed on this run line, marking the average of their estimates. The process was repeated for a run line closer to the goal. Students then discussed the effect on the average position for the best shot before returning to the classroom to begin the task. They worked in 7 groups of 2 to 4 students. The teacher allocated each group a particular distance for their run line from the near goal post.

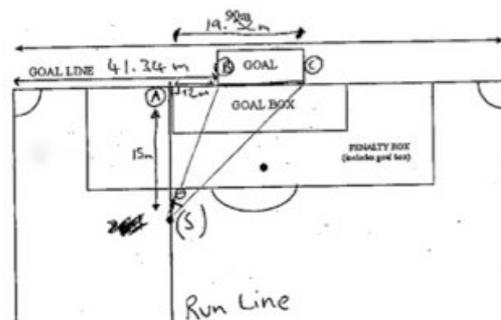


Figure 2. Student diagram of *Shot on Goal*

The main part of the task (Tasks 1 to 10) offered structured scaffolding. Two questions at the end included no details of how to mathematise or approach them mathematically (see Figure 3). Fifteen students attempted part or all of these. Tasks 11 and 12 are the focus here. The task was implemented over three lessons on consecutive days, and these final tasks were

attempted during the third lesson. Data collected which is relevant to the focus of this paper consisted of transcriptions of 3 video-recordings and a further audio-recording of groups working on the tasks, individual task scripts from 15 students and interview responses of nine of these students (see Appendix for example questions).

**TASK 11 – CHANGING THE RUN LINE**

Investigate whether the position of the spot for the maximum shot on goal changes as you move closer or further away from the near post. [Collect data from other students results to help you see if there are any patterns in the position of spots for the maximum angle.] What does the relationship between position of the spot for the maximum shot on goal and the distance of the run line from the near post reveal?

**TASK 12 – CHANGING THE RULES**

Soccer is often a low scoring game. Some have suggested that it would be a better game if the attackers had more chance of scoring, so the width of the goal mouth should be increased. Others claim it would be a more skilful game if the goal keeper was given more of a chance to stop goals by reducing the width of the goal mouth. Investigate what effect changing the width of the goal mouth would have on the position of the maximum shot on goal for the run lines and give your recommendation.

*Figure 3. Tasks for mathematising at end of Shot on Goal*

The research questions addressed in this paper are:

1. Is there evidence for the existence of Niss’s *implemented anticipation* in mathematisations occurring in the classroom?
2. Do Niss’s four requisites explain unsuccessful mathematisations?

To analyse the data student responses to Tasks 11 and 12 were classified as (a) mathematisations showing (i) successful or (ii) unsuccessful implemented anticipation (b) qualitative statements (i) identifying a relationship between relevant variables or (ii) identifying variables but not supported, in either case, by any mathematical objects or representations and (c) incomplete as only raw data with no translation into mathematics or interpretations recorded. When a(ii), (b) or (c) classifications were given, interview data and video and audiotape data were scrutinised in detail for explanations. These were then compared to Niss’s four requisites for successful implemented anticipation.

## Results and Analysis

Fifteen students from 6 different groups attempted Task 11. Group 5 was a long way behind other groups because of difficulties with formulation of the task (see Stillman, et al., 2010). Two students from Group 7 were also behind because of difficulties they were experiencing using their graphing calculator. These students did not attempt tasks 11 or 12.

Ned from Group 6 recorded his results of collecting data for Task 11 in a partially ordered table which he called a “commentary table”. He correctly identified the three relevant variables “run line to goal post”, “dist along RL”, and “Angle” (“Max angle” used in recording the raw data). There was an error in the data which Ned and Len created themselves and recognised at the time of data collection as they decided Group 4’s angle for a 14 m run line distance maximised at 18 m as the data were recorded to only one decimal place and thus the angle was the same for 16 to 19 m. They did not correct this datum as they felt the rest of the data supported their conclusion.

After tabulating the data in this fashion from Len’s systematically organised two column recording of the data and scrutinising it for less than 15 seconds, Ned exclaimed:

Ned: Check it out. The distance along the run line you have to be is, with the exception of-f-f 14, ah, is 3 more than the distance, what the distance is from the run line to the goal post. [Video, Group 6]

This was correct. In recording his interpretation of the data Len added: “It also proves our theory that the closer you come to goal, the closer you have to be on your run line to achieve maximum angle.” Neither student recorded this symbolically. However, when interviewed Len, wrote “ $y = x + 3$  where maximum angle is =  $y$  and distance from near post =  $x$ ” in response to being asked to write his answer algebraically. The group’s work was classified as successful implemented anticipation (ai).

Ozzie and Jaz of Group 2 used correctly ordered tables with Jaz also identifying and labelling the variables in the columns correctly on his script. Ozzie concluded:

The closer to the post, the higher the maximum angle you can achieve, but you have to run in [along the run line] closer to an extent. If [the run line is] further away from post, the goal is largest from far [meaning the maximum angle occurs further out]. [Script, Ozzie, Group 2]

This was typical of the conclusions of all three group members. When asked in interview if he was able to write his answer algebraically, Ozzie responded: “I think you could but I would have to think about it.” However, he recognised the response from Group 6 as being a linear function that produced a straight line graph. In interview, Jaz claimed to have seen “there was a relationship, three metres” from the start of the task. When asked if he could have expressed this algebraically he said he “could have” but his written conclusion (as above) was trying to capture this. He elaborated:

Jaz: Yeah, yeah. Like the closer you get, the closer your run line is to your goal post the bigger angle, you get a much bigger maximum angle but you have to run towards the goal further but I think it is every 3 metres you have to run in by figuring out all of this [seems to be describing: position of maximum angle on run line = run line distance from post – 3]. [Interview, Jaz, Group 2]

When asked to show Group 6’s model algebraically, he used a diagram and  $d$ , the distance from the near post to the run line, to show the maximum shot angle occurring on a run line at  $d + 3$ . He wrote  $y = 2d + 3$  and labelled  $d + 3$  on his diagram also as  $y$ . When asked what  $y$  equalled, he replied, “ $y = 2d + 3$ ” but his explanation described  $y = d + 3$ . The work from these two Group 2 members was classified as unsuccessful implemented anticipation (aii) as their ordered table did not help produce a mathematical answer although the representation was correctly anticipated as potentially allowing this. The response of Sven from Group 7 also used an ordered table but no conclusion was drawn. It was classified similarly as (aii). This student appeared to run out of time.

Molly and Christine (Group 4) collected data from five groups and systematically recorded these in two-celled rows across the page. When they attempted to translate these, together with their own results for a run line of 14, into an ordered mapping of *run line distance from near post*  $\rightarrow$  *shot distance down run line*  $\rightarrow$  *maximum angle of shot*, three mappings had *run line distance*  $\rightarrow$  *shot distance down run line* reversed. The results in this format were interpreted as showing: “In most cases the further away the run line from the goal post the smaller the angle [meaning maximum angle] and the larger the distance down the run line.” This work was classified as unsuccessful implemented anticipation (aii) as the mapping does not support this conclusion even though the representation is correctly anticipated as a useful tool to do so.

The four members of Group 4 all produced qualitative descriptions as did both members of Group 1 and the third member of Group 2. Simon, Max and Lori (Group 4) showed only a relationship between the run line distance from the goal post and the maximum angle (bi). Rose (Group 4), Raza (Group 2), and Jim (Group 1) described two relationships between relevant variables (bi), for example:

The relationship reveals the angle of “shot on goal” gets bigger as you get closer to the goal. This is effective for both “position of the spot for the maximum shot on goal” and “distance of the run line from the near post”. [Script, Rose, Group 4]

Ahmed (Group 1) merely identified the variables, “the further from post” and “distance on the run line to a certain point”, as playing “a crucial factor in getting the best angle” (bii). The work of Rod (Group 7) was classified as incomplete (c) as it merely listed data. The video record showed he ran out of time as he was recording data at the end of the lesson.

Eleven students from 5 different groups also attempted Task 12. The others ran out of time. The mathematisation required that the students realise that the representations and models they had used earlier for finding the angle of the shot using their particular run line distance from the near post and a standard soccer goal width of 7.32 m could be used to capture the changed conditions created by varying the width of the goal. Work from four groups (1, 2, 6 and 3) was classified as showing successful implemented anticipation (ai).

Groups 1 and 2 used their original equations and then changed the widths in the formula so they could graph the three functions showing size of shot angle versus distance along run line on their graphing calculator screen (Figure 4) to find where maximums were occurring and make comparisons. They recorded examples of the functions they had used with the exception of Ahmed who merely recorded goal mouth width, run line distance for maximum shot angle and size of the maximum angle for three specific cases. In addition, Ozzie showed where the changes in the algebraic model came from in the real context.

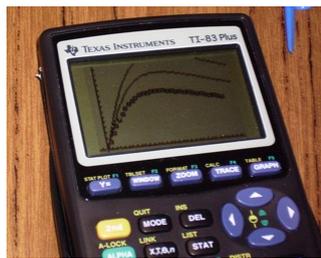


Figure 4. Group 1’s graphs for goal widths of 5, 7.32 and 10m for a run line from near goal post of 10 m.

For example, as Group 2’s distance of their run line from the near post was 12 m and they narrowed the goalmouth to 5 m, their function became:

$$y = \tan^{-1}(17/x) - \tan^{-1}(12/x); 17 \rightarrow 12 + 5 \text{ [Script, Ozzie, Group 2].}$$

Ozzie then observed that: “if the goal’s width is smaller, the total maximum of the ‘angle on goal’ is smaller. You must also go in closer. The max angle was  $9.93^\circ$  at 14 metres.” They then looked at the case of the goal being 10 m in width. Neither group made a definite recommendation.

Group 6 also used their previous formula varying it to generate numerical data for two cases in their calculator LISTs. They then compared with their original data set for their run line distance. Finally, they produced two tables of ordered data for goal widths of 8 m and 6 m. These showed distances of 1 to 5 metres from the goal line along the run line and corresponding angles. These were labelled “examples” but it was clear from the video that distances much further down the run line were examined which allowed them to see where the angle was becoming a maximum. Their observations and recommendation were:

Based on calculations, widening the goal increases the distance away from the goal required to achieve the maximum angle. However, the overall shot angle is increased. The reverse is also true,

Narrowing the goal means the distance from the goal is less, but the shot angle is decreased. I would recommend shortening the goal width, so players can get closer to the goal and still have a large chance of scoring. [Script, Ned, Group 6].

Group 3 again used an ordered mapping of *shot distance down run line*  $\rightarrow$  *angle of shot* for several shot distances along their particular run line for a goal width of 9 m and then 6 m (although the latter was not stated by either student). The mapping was correctly recorded on both scripts. Angle data were generated by changing formulae in the calculator for a larger then smaller goalmouth. Only Christine noticed that the angle was reaching a maximum then decreasing but she mentioned this only for the smaller goalmouth case: “if you decrease the goalmouth to make it smaller, the angles grow but become smaller”. The group recommendation, recorded on Christine’s script as Molly had already filled her page, was: “Make goals larger for more scoring shots”.

Simon and Max (Group 4) made only a qualitative statement indicating “changing the width of the goal would change the angle”. Max added that “the bigger the width of the goal the bigger the angle” and the reverse of this, indicating it was a direct relationship. The former was classified as identifying variables (bii) and the latter as identifying a relationship (bi) but both were without mathematical support other than collected data.

With respect to explanations of students’ unsuccessful mathematisations, confidence and persistence were not considered to be playing a part for the 15 students who attempted these tasks. All students were persisting on the task as a whole at the end of the three lessons (i.e., 165 minutes). Students who were unable to complete because of time pressure or for whom there was no interview data of relevance were eliminated from the analysis leaving the work of nine. Evidence of students’ appreciation of modelling real world phenomena and references to reality in mathematics was gleaned from the video and audio interactions in their group and through responses to interview questions. Only Raza was negative towards the utility in the real world of the mathematics they were using saying his father was 40 years old and used algebra only once since school. Rose and Lori were more interested in mathematical exercises but conceded others such as sports people could use mathematics to inform decisions. Ned remained ambivalent towards the need for a task context to be real or not although he appreciated mathematics being in some sort of context. Thus, appreciating the modelling of real phenomena was not seen as discriminating between successful or unsuccessful mathematisation. As Task 12 did not prove to be discriminating either, it was also eliminated from the analysis. To see if students whose responses to Task 11 received classifications other than (ai), possessed the relevant mathematical knowledge of linear relationships, their responses in interview to being asked to describe and graph such a relationship revealed that all but Jaz did this correctly. Jaz’s response was, however, a linear function. Rose used the relation correctly but only in terms of mappings of coordinate points. Clearly, being able to use this mathematical knowledge in a modelling context, not possessing the knowledge itself, was the difficulty for those students who were unsuccessful in mathematising Task 11.

## Discussion and Conclusions

As the tasks were already formulated as mathematical statements or questions by the task setter, all the successful mathematisations could be said to have involved students identifying relevant variables, foreshadowing representations that would be useful in identifying relationships between variables and producing mathematical answers, and realising that the representations and models used earlier could be used to capture changed

conditions. This is thus empirical evidence for the last two aspects of Niss's implemented anticipation (Niss, 2010). A more complex task requiring student input to formulation would be required to demonstrate the first aspect.

Where evidence was available, unsuccessful attempts at mathematisations in a classroom context were in this study related to inability to use relevant mathematical knowledge in the modelling context rather than lack of the relevant mathematical knowledge per se, an application oriented view of mathematics or persistence on the task. As this was only the third in a series of modelling tasks the students had attempted as their first experience of modelling, it is not surprising that this was the most discriminating of Niss's four requisites (2010) for successful implemented anticipation in mathematisation.

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## Appendix: Selected Interview Questions

- Q10.1 Could you have written an algebraic model for your answer to Task 11?
- Q10.2 One group [Group 6] said that the position of the spot for the maximum shot on goal was 3 metres more than the distance of the run line from the goal post. What type of mathematical relationship is this?
- Q10.2.2 Draw me a graph to show it.
- Q11.1 Do you like doing challenging tasks like this in maths? Can you elaborate on that.
- Q11.1.1 What types of maths tasks do you prefer?
- Q11.2 What is the purpose of tasks such as Cunning Running and Shot on Goal?
- Q11.3 Do you like the fact these tasks are set in a real world context?