Concept Maps: Implications for the Teaching of Function for Secondary School Students

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This paper reports analysis of the concept maps of the understanding of function developed in secondary school constructed by seven experienced secondary mathematics teachers who were part of a larger study. The concept maps were analysed according to (a) key notions related to the definition of function, (b) process or object view of function, and (c) identification of the importance of working within and across representations. The findings suggest a teaching emphasis that might not be supportive of students developing a deep understanding of functions.

This paper presents part of a larger study involving teachers and Years 9-11 students of mathematics in six Victorian schools. The teachers were participants in a three year research project involving the use of technology in the teaching and learning of mathematics for the development of deeper understanding (see http://extranet.edfac.unimelb.edu.au/DSME/RITEMATHS/). The focus here is related to answering the following research question: What allows teachers to perceive particular affordances of technology-rich teaching and learning environments (TRTLE's) and act on these to develop student understandings of functions and the development of higher order thinking? (see Brown, 2005, for details). One method of data collection related to addressing this question involved asking the teacher participants in the study to create a concept map.

Concept Maps

Concept mapping is attributed to Novak, who recently described concept maps as "graphical tools for organising and representing knowledge" (Novak & Cafias, 2008, p. 1). They were developed in 1972 as part of Novak's research seeking "to follow and understand changes in children's knowledge of science" (p. 3). Whilst originally used mainly in science, there is much evidence to suggest increasing use of concept maps in mathematics (e.g., Afamasaga-Fuata'i, 2007; delos Santos & Thomas, 2005; Hasemann & Mansfield, 1995; Mwakapenda, 2003; Williams, 1998). Concept maps were initially used by teachers to more effectively present knowledge "with the intention being to *map* something from the *outside* world *into* the student's mind" (Hasemann & Mansfield, 1995, p. 45). Later, students generated their own concept maps, either in a scaffolded or open fashion. Subsequently, they have been used as a research tool to gain insight into the understandings and knowledge of the concept mapper, as is the case in the study reported here.

Student generated concept maps have been considered by many (e.g., delos Santos & Thomas, 2005) "to be an externalisation of conceptual schemas" (p. 378) and hence a useful tool to help identify students' current conceptual schemas and changes in these over time. Concept maps are often used to "trace students' understanding" (Hasemann & Mansfield, 1995). For example, Afamasaga-Fuata'i (2007) and delos Santos and Thomas (2005) used pairs of concept maps produced by each student, (undergraduate and final year

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In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1). Palmerston North, NZ: MERGA.

school mathematics students respectively) to “indicate both growth and transformation of their continuous conceptions” (delos Santos & Thomas, p. 381).

Williams (1998) suggests that “concept maps are a direct method of looking at the organisation and structure of an individual’s knowledge within a particular domain and at the fluency and efficiency with which the knowledge can be used” (p. 414). In her study, the function concept maps constructed by professors (experts) were compared to the function concept maps of university students studying first year calculus. In contrast to many researchers who use elaborate numerical scoring schemes in assessing concept maps (e.g., Afamasaga-Fuata’i, 2007; Hough, O’Rode, Terman, & Weissglass, 2007) or rubrics (e.g., Bolte, 1999), Williams argues that for her data “typical scoring schemes did not appear to be valid” (p. 416) and she focused on the concept map as an integrated whole and interrogated her data for differences between the function concept maps of the experts and those of the students. She found function concept maps created by students tended to include trivial or irrelevant information, and were algorithmic in nature. In addition, "none of the experts demonstrated the students' propensity to think of a function as an equation. Instead, they defined it as a correspondence, a mapping, a pairing, or a rule." (p. 420).

Generally, the participants in studies using concept mapping as a tool initially work together in exploring concept maps. Mwakapenda, for example, describes his study on concept mapping and context in mathematics education with university mathematics students where **“**after being introduced to concept mapping and the processes involved in constructing concept maps, students were asked to construct a concept map to show how [given] concepts were related” (2003, p. 193). Similarly, in the study by Hasemann and Mansfield the grade four and six “students were shown an example of a concept map prepared by a student on the topic of fractions. The features of the concept map were pointed out to the students, including examples of positive and negative propositional links joining concept names [also, as is often the case] the students were then given 11 concept names to use in the construction of a concept map about parallel lines” (1995, p. 61)

Method

Seven experienced mathematics teachers from four schools were asked to create a ‘concept map of function’. Unlike studies discussed earlier, no information about, or examples of, concept maps were provided. It was assumed that experienced mathematics teachers had knowledge of concept maps. None of the participants asked for elaboration of the task. The purpose of this particular task was to get some sense of what the teachers believed were the key aspects involved in the teaching and learning of functions in secondary school mathematics. It is not assumed that all of each teacher’s knowledge of functions is represented by their concept map. Rather each concept map is used as being somewhat representative of the teachers understanding of functions and their subsequent emphasis on the various aspects of functions in their teaching. Although the teachers were asked to complete the task individually, TD1 and TD2 (Teacher 1 and Teacher 2 from School D) worked on the task together, although TD1 also presented his own concept map. Six concepts maps were completed on the day, in addition to the two mentioned, TA1, TB2, TB5, and TC1 all completed a concept map. TB1 completed his a short time later.

The concepts maps of TB1 and TB5 are presented in Figures 1 and 2 respectively. As reported, specific numerical scoring systems are often used to analyse concept maps.

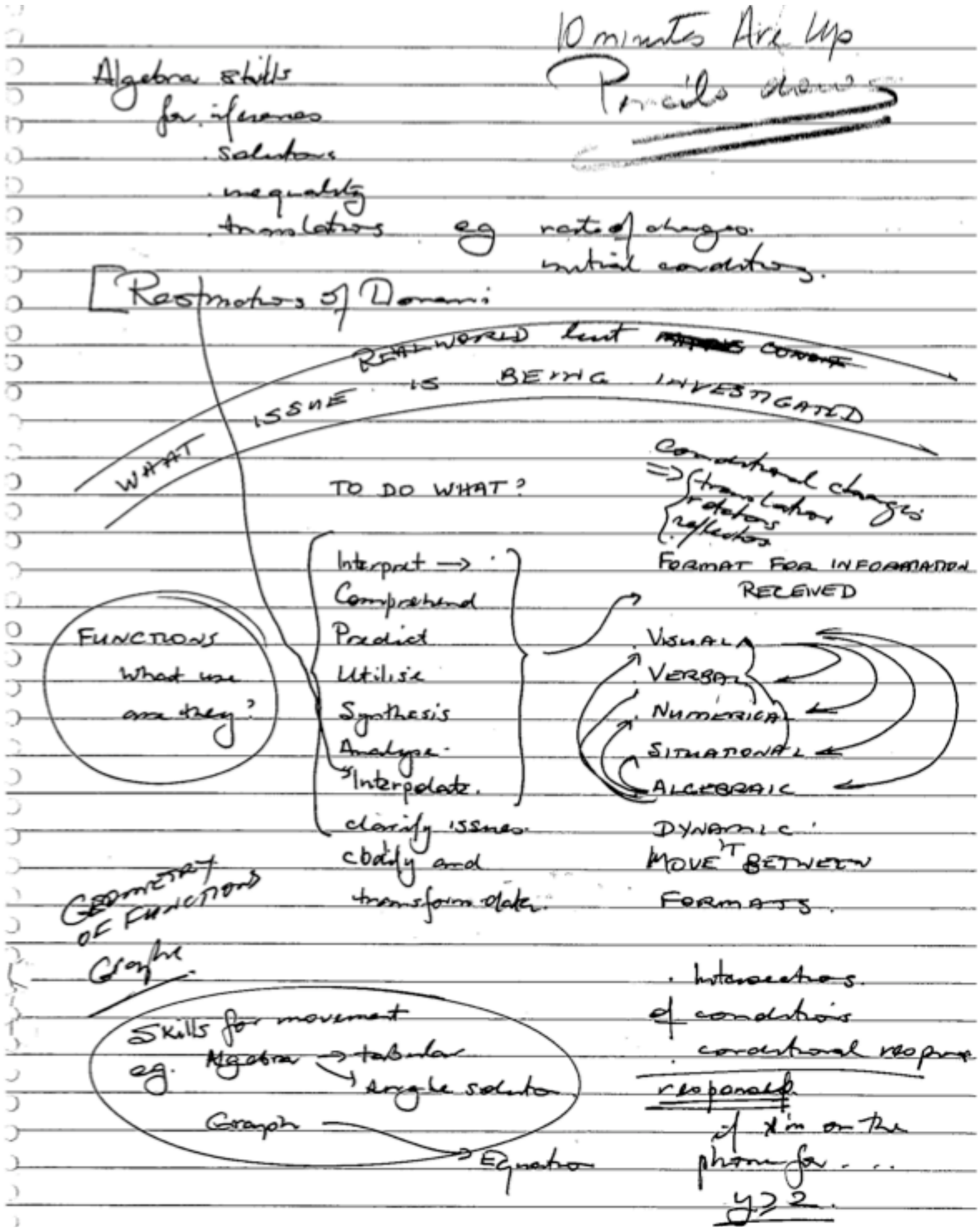

Figure 1. Concept map by TB1 of understanding of function developed in secondary school.

However, this author is in agreement with Williams that this was not the appropriate course of action to take with these concept maps. So how do we analyse such diagrams, not withstanding the context in which they were constructed? The purpose here being to get some sense of what the teachers believed were the key aspects involved in the teaching and learning of functions in secondary school mathematics.

Scanning across the concept maps showed varying degrees of complexity and density, and variation in ways the ideas are presented. Some were presented as a list, others as a ‘simple’ sequence of ideas, and yet others as a ‘complex’ connected web (e.g., Figure 2). The central idea under investigation is the concept of function. Hence, it was decided to begin with key notions of the definition of function that could be expected to be developed across Years 9-11 of secondary school.

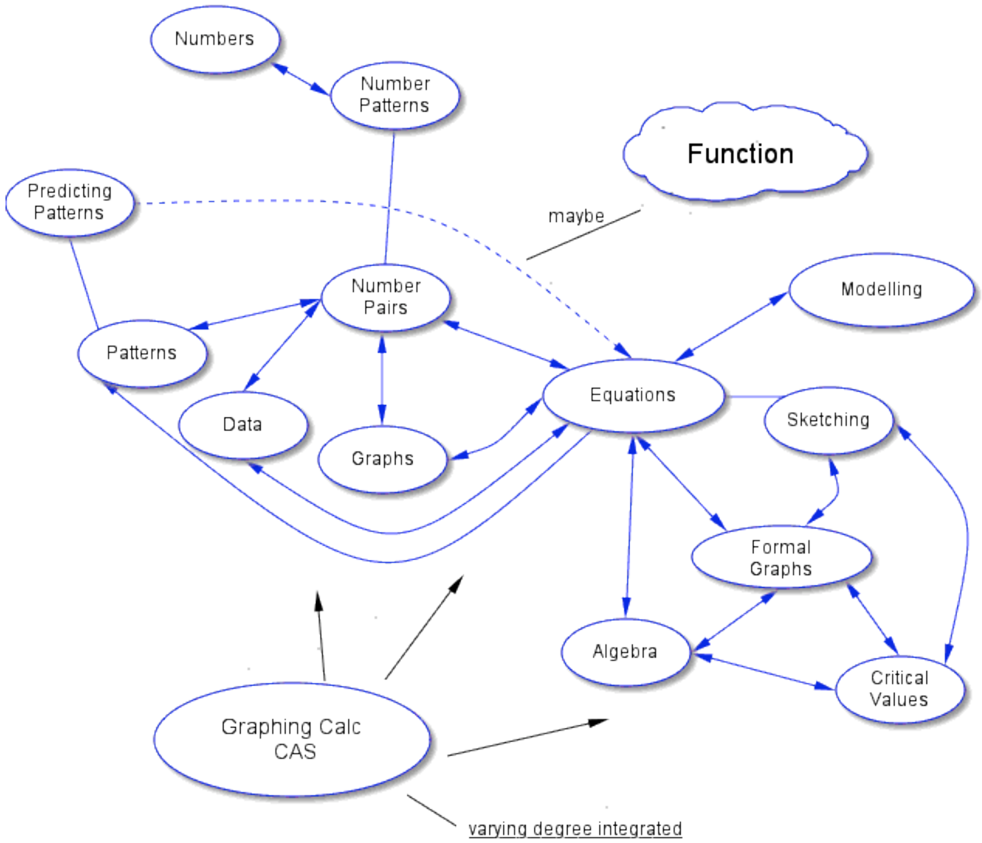

Figure 2. Recreation of concept map by TB5 of understanding of function developed in secondary school.

Results

Analysing the Concept Maps: Key Notions - Definition of Function

Tall (1992) describes the evolution of the complex function conceptions as including "the geometric image of a graph, the algebraic expression as a formula, the relationship between dependent and independent variables, an input-output machine allowing more general relationships, and the modern set-theoretical definition" (p. 497). For a more detailed elaboration of recent relevant research literature on functions, see Brown (2007)

Building on Tall's description, a framework was developed and used for the initial analysis of the teachers' concept maps. The concept maps were examined considering each notion in turn. Both the framework and the identification of elements of this in each of the teacher's concept maps are presented in Table 1. In the initial analysis of the teachers' concept maps, with respect to the key notion related to the definition of function, differences and similarities in the responses were evident. None of the concept maps included all of the key notions, listed in Table 1. Interestingly, none of the concept maps indicated that different representations allow different aspects of function to be more obvious or explicit or that they may make the functional relationship more understandable. The concept maps with the greatest number of these notions, namely five, were constructed by TC1 and TD1, although one of the key notions in TD1's concept map was inferred. The concept map presenting the fewest of the key notions [two] was constructed by TB2, with the concept map of TB1 containing three (two being inferred) of the key notions.

Table 1

Understanding of Function to be developed in School Mathematics: Key Notions Identified

	Teacher						
Key notions	TA1	TB1	TB2	TB5	TC1	TD1	TD1& TD2
Two groups are involved	✓	×	✓	✓	✓	✓	✓
Can be represented in different ways	✓[a]	✓[b]	✓[a]	✓[c]	✓[c]	✓[c]	✓[c]
A relationship exists between the two groups	×	×	×	×	✓	✓	✓
Relationship explicit / needs to be discovered	×	×	×	I	×	I	×
Relationship may be described by a pattern.	✓	I	×	✓	✓	✓	✓
Families of functions identified	✓	×	×	×	×	×	×
Different representations make different aspects of the pattern or explicit	×	×	×	×	×	×	×
Different representations may make the pattern more understandable	×	×	×	×	×	×	×
A dependent relationship	×	I	×	×	×	×	×
A special dependent relationship	×	×	×	×	✓	×	×

Note. ✓ = present in concept map, × = not present in concept map, I = Inferred from concept map, [a]Two representations indicated (algebraic and graphical); [b]Five representations indicated (visual, verbal, numerical, situational, and algebraic); [c]Three representations indicated (algebraic, graphical, and numerical).

Analysing the Concept Maps: Process, or Object

Secondly, the object-process-representation schema of Moschkovich, Schoenfeld, and Arcavi (1993) was considered. Whilst, it was not possible to map the concept maps to this schema, it was possible to identify which of the concept maps show both an object and a process view of functions and which show consideration of ideas related to working within a single representation as well as across representations. Sfard (1991) notes that a process view of a function is at a lower level than the object level, hence there are links here with the intentions by teachers for students to engage in higher order thinking. Furthermore, this suggests it is important for teachers to consider function from both perspectives.

It was inferred from TA1's concept map reference to 'algebra→predicting' that a process view of function is suggested. This view could be further inferred from inclusion of 'graphing→prediction' and also possibly 'graphing→interpretation'. There is no evidence of an object view of function in this concept map. The concept map of TB1 shows evidence of the development of a process view of function through the inclusion of 'algebra skills for inferences, solutions'; predict, utilize, interpolate' as uses of function. Indications of an object view of functions can be seen in comments such as 'conditional changes→ translators, rotators, reflectors'. Similarly, TB2 provides evidence of the need to develop both perspectives in secondary school mathematics. For process, she notes 'operations' and for an object perspective she includes 'connections (relationships) within/ between families of graphs'. For TB5, his referring to 'critical values' is suggestive of a process view of function. There is, however, no evidence of the need to develop an object view of function in his concept map. Similarly, TC1 shows no evidence of the development an object view. For a process view of function, she includes both 'substituting' and 'solving equations' on separate connectors between 'algebra rule' and 'interpreting'; she also includes 'finding key features'. TD1 indicates a process view when he refers to ‘use the relationship to make predictions’ and ‘algebraic manipulation of the rules of the relations’. Similarly the joint concept map of TD1 and TD2 includes ‘basic computational (numerical) skills and algebra’ and ‘basic graphing skills’, both implying a process view of function. Neither TD1’s concept map nor his joint concept map indicates an object view of function.

The perspective, process or object, presented in the concept maps constructed by the teachers are summarised in Table 2. All teachers included a process view of functions in their concept map for the understanding of function in secondary school mathematics, however, only two, TB1 and TB2, also included an object view of function.

Table 2
Process or Object View of Function in Secondary School Mathematics

	Teachers						
View of function	TA1	TB1	TB2	TB5	TC1	TD1	TD1&TD2
Process	✓	✓	✓	✓	✓	✓	✓
Object	×	✓	✓	×	×	×	×

Identification of the Importance of Working Within and Across Representations

The second important aspect of the Moschkovich, Schoenfeld, and Arcavi (1993) schema is with respect to representations. They argue the importance, not only that a process and an object perspective are required for deep understanding of function, as is the ability to move across these perspectives, but that this is necessary within a single representation and across representations. In this section, the concept maps are analysed with respect to this idea, working within and across representations.

In analysing the concept maps, an indication of the need to develop expertise in working within or across representations requires more than merely naming the representation. For example, TA1 includes ‘graphing’ and ‘algebra’ in her concept map, however, these are not linked to each other, nor is there any elaboration of within representation connections. This was not accepted as evidence for working within or across representations. In contrast, TB1 listed several representations or ‘format for received information’ with arrows between these labeled ‘dynamic movement between formats’. Instances of working within and across representations are presented in Table 3. This clearly indicates that the importance of working within and across representations for the development of understanding of functions was not evident in all of the concept maps.

Working *within* representations was addressed in only two concept maps. For TD1, the representation was algebraic, whereas for TB2 it was the graphical representation. None of the concept maps indicated working within the numerical representation. Instances of the importance of working *across* representations were more evident. Twenty-two instances of across representation notions were identified in the [5] concept maps. Six of these were in the joint concept map of TD1 and TD2, and six in the concept maps of TB1 and TB5. For the concept map of TD1 and TD2 and that of TB5 these instances are found in each of the possible six different cells in the table, and evidence by the general comment ‘recognise the interconnection among all these representations [table/ graph/ equation]’. In contrast, the six instances identified by TB1 were spread across only four of these possible cells. For both, ‘from algebraic to numerical’ and ‘from graphical to numerical’, TB1 includes two

Table 3
Instances of Movement Within and Across Representations

		To		
		Algebraic	Graphical	Numerical
From	Algebraic	TD1 [1]	TB1 (2 instances); TB2; TB5, TD1; TD1 & TD2 [6]	TB1; TB5; TD1 & TD2 [3]
	Graphical	TB1; TB2; TB5; TD1; TD1 & TD2 [5]	TB2 [1]	TB1(2 instances); TB5; TD1 & TD2 [4]
	Numerical	TB5, TD1 & TD2 [2]	TB5; TD1 & TD2 [2]	-

statements in his concept map that are indicative of these. In addition, he has one instance each of 'from graphical to algebraic', and 'from numerical to graphical'. Both TB2 and TD1's individual maps include only between the graphical and algebraic representations.

Clearly more attention was given to *across* rather than *within* representation notions, with 22 compared with 2 instances, albeit there are more possibilities for the former. The greatest number of instances, namely six, occurred from the algebraic to the graphical representation. Scanning the rows and columns sees the greatest number of instances, nine, of across representation movement occurring both for 'from algebraic, and 'from graphical'. In summary, not only was greater attention given to across, rather than within, representation movement—this was not evenly spread across representational pairs. Perhaps not surprisingly, connections between representations, where the initial representation was the numerical received the least attention.

Discussion

Conceptions of functions are complex and so too is the analysis of mathematics teacher constructed concept maps of the understanding of function developed by students during secondary school. With respect to the myriad notions related to the definition of functions, none of the concept maps included more than half the key notions of the framework based on the work of Tall (1992). Where teachers are working toward the development of a deep understanding of mathematics, it is of concern that these key notions are not at the forefront of the teachers' thinking and hence included in their function concept maps. Key notions identified by most teachers were: they can be represented in different ways (all concept maps), two groups are involved (six of the seven maps), and that a relationship may be described by a pattern (six of the concept maps). No concept maps indicated that different representations can contribute to making different aspects of a function transparent or the relationship more understandable. Only two of the concept maps indicated an object as well as a process approach. Given the claim of Sfard (1991) and others that being able to view a function as an object is at the higher level suggests this way of thinking about functions is essential for a deep understanding of the function concept.

Whilst the initial analysis showed that all concept maps indicated functions can be represented in different ways, a closer analysis focussing on identifying the importance of working both within and across representations highlights a worrying trend. Given the three representations being considered, only two of the concept maps included all the six ways of moving from one representation to a second. Two of the concept maps failed to indicate any links between or within the three representations. In addition, it was clear the algebraic and graphical representations received greater attention in their concept maps.

Fewer instances were identified where one of the representations was numerical. Given that each representation necessarily portrays only some aspects of function (e.g., Freidlander & Tabach, 2001) this lesser focus on the numerical representation may be restricting opportunities for students to gain the deep understanding of functions that is the aim of these teachers for their students. In conclusion, the concept maps did not indicate a broad range of key notions of functions, nor the importance of multiple representations and an object view of function. The numerical representation appeared to be valued less than other representations. These findings suggest a teaching emphasis that might not be supportive of students developing a deep understanding of functions.

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