Developing Students' Functional Thinking in Algebra through Different Visualisations of a Growing Pattern's Structure

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This design-based research project investigated the development of functional thinking in algebra for the upper primary years of schooling. Ten teachers and their students were involved in a sequence of five cycles of collaborative planning, team-teaching, evaluating and revising five lessons on functional thinking for their students over one year. This paper focuses on two aspects of the study related to developing students' functional thinking by visualising the structure of a growing pattern in different ways.

Spatial visualisation of patterns and moving from a language description of pattern to a symbolic equation are not skills that have been traditionally associated with the learning of algebra. There is much to be learned about the cognitive processes needed for students to be able to link patterns with multiple representations and reach generalisations (Lee & Freiman, 2004; Warren & Pierce, 2004). Blanton and Kaput (2004) conjectured that even very young students are capable of functional thinking. Hunter (2010) found that upper primary students benefited from "carefully designed tasks, specific pedagogical actions and extended discourse" that enabled them to develop their understanding of variables and to use algebraic notation successfully (p. 256). This study investigated upper primary students' visualisations of a growing pattern and their subsequent type and level of generalisation in creating functional relationships. It was a sub-project of *Contemporary Teaching and Learning Mathematics* (CTLM), a large-scale project conducted by the Australian Catholic University (2008-2012), in Victorian Catholic primary schools.

Background

The research literature highlights a variety of strategies for teaching pattern generalisation and developing students' functional thinking. "The interaction of context, multiple representational forms, and technological tools" is seen as a key strategy that supports functional understanding (Confrey & Smith, 1994, p. 32). Teachers help their students to generate functional relationships within a context and use multiple representations of them in exploring generalisation processes. This variety supports students in thinking about functions in diverse yet legitimate ways. Kaput (1999) also recommended the use of *familiar* contexts, for example, the heights of plants or people, temperatures, and numbers of people changing over time, and the cost of a product as a function of the number bought. He also advocated a multi-representational approach where students are encouraged to use a variety of diagrams, tables of values, language, equations and graphs to represent functional relationships. "The idea of a function embodies multiple instances, all collected within a single entity (e.g., a list, table, graph), a process that also involves generalizing – answering the question, 'What is it that all these instances have in common?'" (p. 146).

Research has found that teachers encouraged students' recognition of functional relationships by asking them first to describe the features of a geometric pattern verbally and then to express these algebraically (MacGregor & Stacey, 1995). Warren and Cooper (2008) used concrete materials to create growing patterns and found that they supported

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students' understanding of patterns and sequences. They also found that specific questioning highlighting the relationship between pattern item and its position number supported students' ability to generalise about unknown positions. There are two main types of rules which generalise quantifiable aspects of the pattern or sequence. Stacey (1989) referred to one type as "near generalisation" in which the next item in a pattern is found by using step-by-step drawing or counting. The other is "far generalisation" and involves finding a general rule for any item in a sequence (p. 150). Confrey and Smith (1994) referred to the same two approaches as *co-variation* and *correspondence* respectively. A co-variation approach describes the relationship between successive items in a pattern – also known as *recursive generalisation* or a *local rule* (Mason, 1996). A correspondence approach perceives the relationship between two quantities or variables, often the item/term position number in the pattern/sequence and a quantifiable aspect of the item/term itself such as the total number of shapes or pieces. This is also known as *explicit generalisation*.

Markworth (2010) researched Year 6 students' functional thinking to develop an empirically substantiated instruction theory about learning to generalise geometric growing patterns. She used a design-based research methodology with students who were anticipated to have had little prior experience with functional thinking or geometric growing patterns. Her subsequent learning trajectory was adapted to create the learning progression framework for use in the study as an assessment tool and professional learning resources for teachers. It is presented in Table 1.

Table 1
A Learning Progression Framework for the Development of Functional Thinking with Growing Patterns (Wilkie, 2013, adapted from Markworth, 2010, p. 253)

1. Extend a growing pattern by identifying its physical structure, features that change, and features that remain the same (*figural reasoning*).
2. Identify quantifiable aspects of items that vary in a geometric growing pattern.
3. Articulate the linear functional relationship between quantifiable aspects of a growing pattern by identifying the change between successive items in the sequence (*co-variation* or *recursive generalisation*).
4. Generalise the linear functional relationship between aspects of a growing pattern by:
    - 4.1 describing the relationship between a quantifiable aspect of an item and its position in the sequence (*correspondence* or *explicit generalisation*);
    - 4.2 using symbols or letters to represent variables; or
    - 4.3 representing the generalisation of a linear function in a full, symbolic equation.
5. Apply an understanding of linear functional relationships between variables to further pattern analysis and multiple representations.

Students can struggle to move from co-variation strategies (levels 2 and 3 in Table 1) to correspondence strategies (level 4) when generalising growing patterns (e.g., Jurdak & Mouhayar, 2014; Stacey, 1989). In addition to students learning to generalise growing patterns of sequential items, Kuchemann (2010) advocated experiences that focus on the structure of an *individual* item of a pattern, and treating it as a prototype, since this promotes students' use of correspondence strategies. He did, however, emphasise that both types of generalisation (co-variation and correspondence) complement each other, and that recursive (co-variation) strategies are helpful for exploring increments and the concept of slope or gradient of a function. With this in mind, some tasks in the lesson sequence for the study were designed to focus students' attention on finding the relationship between two quantifiable aspects of a non-sequential collection of figures. Kuchemann (2010) also recommended the exploration of non-linear functional relationships, as did Friel and Markworth (2009). This paper focuses on students' visualisations of the structure of a growing pattern and their subsequent ability to generalise the functional relationship, using the previously presented learning progression framework as an assessment tool.

Research Design

This study adopted a design-based methodology where teachers and researcher experience the project as a collective effort and where teacher learning and student learning are two joint goals (Gravemeijer & van Eerde, 2009). The three main features of this methodology are instructional design and planning, ongoing analysis of classroom events, and retrospective analysis (Cobb, 2000). Teachers and researcher inquire together "into the nature of learning in a complex system" to produce "useable knowledge" and "explanations of innovative practice" (Baumgartner, et al., 2003, pp. 7-8). Interactions between learners, teachers and materials in contexts of practice are enacted through continuous cycles to produce meaningful change (Baumgartner, et al., 2003). In this study, cycles of interaction involved collaborative planning, implementing, evaluating, and revising a sequence of lessons with the researcher and their year-level teaching teams (Hiebert & Stigler, 2000). Teachers were able to experiment iteratively with their teaching approaches "on the basis of conclusions they themselves draw from data from their own classrooms" (Gravemeijer & van Eerde, 2009, p. 523).

The researcher provided a set of instructional activities and materials drawn from the research literature on functional thinking An example of one of the tasks, which was used as an initial assessment task and discussed in this paper, is presented in the Appendix. Teachers and researcher co-analysed students' participation, work samples and inferred learning to revise the learning tasks and develop subsequent tasks. Each of 222 students' assessment tasks, which form the basis for this paper's results and discussion, were analysed and scored by the researcher and the class teacher, using the previously presented learning progression. The scores were compared and discussed in teaching team meetings. Both "careful review of the data and a reflection on the process of the teaching experiment" to understand more about "what induced the changes observed" were considered important (Gravemeijer & van Eerde, 2009, p. 514).

Teacher participants were recruited from among those involved in the previously mentioned CTLM project. Selected data from a larger case study of 10 teachers and their classes from two schools for a one-year period form the basis for the discussion in this paper. A sequence of five lessons in each teacher's class, with pre- and post-meetings in teaching teams and attended by the researcher, was timetabled. These meetings were audio-recorded and included discussion of: students' mathematical activity and work samples; classroom norms and mathematics practices; exploration of concepts using instructional materials; evaluation of the previous lesson, and planning for the next lesson. A researcher's journal was kept to document observations of student learning, student and teacher engagement and reflections from lessons and team meetings. The video-recording of lessons was not considered financially viable for this study.

At the beginning of the lesson sequence, students were given an assessment task to use as a starting point for teachers in planning future learning experiences (see Appendix). To ascertain the ways students visualised the structure of the growing pattern, the task asked them first to colour the leaves in different colours to show what they saw, and then to describe their thinking in words. These visualisations were then compared to their strategies in the second half of the task (parts c to e) which asked them to find the number of leaves for plants with a higher number of leaves (day #7, then #17, then any day number). The last question (part f) was used to elicit evidence of applied functional thinking in being able to find the day number given the number of leaves (assigned a score of 5 from the learning progression). The following section discusses how students' choice of visualisation for the growing pattern in the first part of the assessment task related to their ability to generalise it in the rest of the task using co-variation (recursive) or correspondence (explicit) approaches.

Discussion of Results and Implications

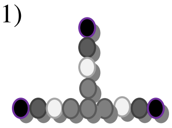
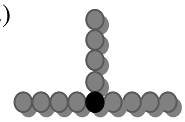
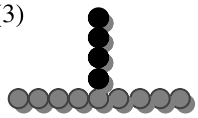
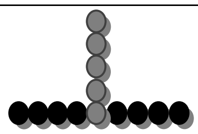
The students' colouring of the growing pattern as an indication of their initial visualisation were found to fall into four main categories and these were labelled Types 1 to 4. Some students' visualisations were unclear because they used only one colour, or did not follow a consistent colouring pattern, or did not colour in the pattern at all. These were categorised as *Unclear* but their subsequent solutions to parts b to e were analysed for the level of generalisation strategies. Some students coloured their growing pattern using one type of visualisation but then described a different type in writing or in solving the subsequent generalisation questions (parts c to e). These were categorised as *Transition* from one type to another.

The results of the visualisation categorisation and subsequent level of generalisation (using Table 1 as a rubric) are presented in Table 2.

Friel and Markworth (2009) presented three different ways to visualise the growing pattern used in the task described in this paper. These correspond to the first three types in Table 2 and were all evidenced by the students' responses to the colouring task. The results show that Type 1, the recursive visualisation of the growing pattern, was the most frequent, with nearly one third of the students colouring their growing pattern in a similar way. Most of these students (26 %) subsequently used co-variation approaches in tackling the generalisation tasks (levels 2 and 3 on the learning progression). Only 3 students were able to use this type of visualisation to generalise successfully using a correspondence approach (discussed further on in this sub-section.) Another 10 % initially coloured in their pattern using Type 1 but then employed a different visualisation in later parts of the task (*Transition between Types* in Table 2).

Nearly a third of students did not demonstrate their visualisation of the growing pattern in a recognisable or consistent way. A quarter of these students were not able to continue the growing pattern at all (by drawing plants for Day #4 and #5). A further quarter was able to continue the pattern but not describe its changing features. Less than 5 % of the students were able to generalise explicitly using a correspondence approach without previously demonstrating their visualisation of the structure. It is likely that the questions in the second half of the task promoted figural reasoning and resulted in some students re-visiting the growing pattern's structure to be able to answer parts d and e without having to draw large plants!

Table 2
Students' Type of Visualisation Strategy for the Upside-Down T Plant Growing Pattern and Subsequent Highest Level of Generalisation in Later Parts of the Task (n = 222)

	Highest level of generalisation demonstrated using learning progression								
	Incorrect	1	2	3	4.1	4.2	4.3	5	
Type of Visualisation	Students %	Students %	Students %	Students %	Students %	Students %	Students %	Students %	TOTAL (%)
Type (1)  Same structure as the previous day's plant with 3 additional leaves, one on each stem	*1.8*	3.6	10.8	15.3	0.5			0.5	**32.5**
Type (2)  Three stems each with the same number of leaves as the day number and a central leaf	*0.5*	4.5	0.9	0.9	4.5			2.2	**13.5**
Type (3)  One horizontal stem with twice the number of leaves as the day number and one extra, and a vertical stem with the same number of leaves as the day number		0.5	2.7	2.7					**5.9**
Type (4)  Two horizontal stems with the same number of leaves as the day number and a vertical stem with one extra leaf		1.4	0.5	0.9	0.5				**3.3**
UNCLEAR TYPE All leaves the same colour / no consistent way of colouring / no colour used	*8.6*	9.0	7.2	3.6	3.6			0.5	**32.5**
TRANSITION BETWEEN TYPES									
TYPE (1) -> (2)					5.9			2.7	**8.6**
TYPE (1) -> (3)		0.5		1.4					**1.9**
TYPE (1) -> (4)					0.5				**0.5**
TYPE (2) -> (4)				0.5					**0.5**
TYPE (3) -> (2)				0.5					**0.5**
TYPE (4) -> (2)								0.9	**0.9**

Type 2, a central leaf with three stems, was the second most frequent visualisation used by the students, and those who visualised the structure in this way were more likely to employ a correspondence approach to their generalisation. Perhaps surprisingly, over 5 % of students coloured their growing pattern in this way but did not make the connection between day number and number of leaves on each stem, instead relying on recursive strategies such as adding on three to the previous day's total. It is clear from this that students would benefit from explicit teaching and repeated opportunities to visualise the structure of different growing patterns and to make connections between variables.

Of the 28 students (13 %) who demonstrated a transition from one type to another, the most frequent shift was from Type 1 to Type 2; all of these students began with a recursive visualisation but later used a correspondence approach to create an explicit rule linked to their subsequent visualisation of 3 stems equalling the day number with an additional leaf in the centre. It seems likely that the need for a more efficient way of finding the total numbers of leaves for higher day numbers prompted students to investigate more closely the relationship between these two variables. Some growing patterns lend themselves more easily to students doing this and therefore it is important for students to learn to attend to the visual structure of patterns through well-sequenced experiences and well-chosen patterns.

Of the 6 % of students who visualised the structure as Type 3, all maintained a recursive approach to generalisation. Although it is possible to generate an explicit rule from this visualisation (e.g., number of leaves = day# + 1 + day# (long horizontal stem) + day# (vertical stem) or $t = (n + 1 + n) + n$, none of the students demonstrated this approach. If the relationship between two variables in a growing pattern is too well-hidden, students seem less likely to use correspondence strategies. This again emphasises the importance of pattern selection for supporting student's developing ability to generalise explicitly, as raised by Rivera (2010).

A type of visualisation additional to the three types described by Friel and Markworth (2009) was used by 11 students (5 %) in the study (Type 4 in Table 2). They coloured the full length of the central column in one colour and the two 'arms' in a different colour or used a generalisation strategy that demonstrated their grouping of one longer vertical column with two shorter arms. It is clear from this study that the upside-down T plant was a useful growing pattern for students to explore different visualisations of the structure and to use these for explicit generalisation. Some growing patterns may not lend themselves to such exploration as they elicit only one simplistic visualisation type. Some unusual explicit generalisation strategies used by students in the assessment task, and evidence of the ability to visualise the same growing pattern in multiple ways (from data collected on other tasks in the lesson sequence), are to be presented during the conference as space is limited in this paper.

Conclusion

This study has highlighted that particular growing patterns that can be visualised in different ways encourage students to focus on the structure of an item in the pattern and use this conceptually to generalise functional relationships using a correspondence approach. It has provided data on how students' initial visualisations lead to different types of generalisation or even to the use of a different visualisation.

Nearly 13 % students were found to have transitioned from one way of visualising the structure of the growing pattern (demonstrated in their colouring of the items) to another way, either in their description or in their solutions to the generalisation questions (parts c to e). Jurdak and Mouhayar (2014) found that students' levels of reasoning associated with strategy use were not uniform. This resonated with Siegler (2000) who asserted that students' "diverse strategies and ways of thinking co-exist over prolonged periods of time" and that "*experience* brings changes in relative reliance on existing strategies and ways of thinking" (p. 28, italics ours). The results of the task indicate that some students began by visualising the structure of the growing plant recursively and used a co-variation strategy initially, before shifting to a correspondence approach when dealing with larger plants (far generalisation).

Being given an *experience* that encourages such a shift through the choice of a particular growing pattern that promotes different types of visualisation, contributed to students' development of more sophisticated functional thinking. Students may use a mixture of strategies simultaneously in their development of functional thinking and targeted learning experiences are the key to students' progress in conceptual understanding. During *sharing time*, students were able to explain and demonstrate their different visualisations and methods of generalisation. Their considerable interest in each other's different strategies was noticeable and highlighted the value of these opportunities for enriching the students' functional thinking. Although there is much more to consider in developing students' functional thinking, including the use of multiple representations and non-linear relationships, this study has highlighted the exciting potential of growing patterns to be used in flexible and conceptually effective ways. Such patterns seem to also have the added benefit of piquing students' interest. One student reflected, "We did not muck around a lot. I found it very fun."

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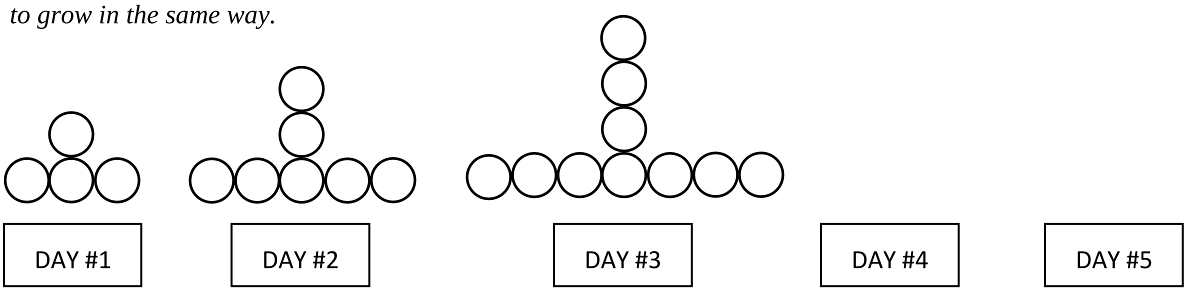
Appendix: Assessment task used at the beginning of the lesson sequence

The 'upside-down T' plant

In my garden one day, I saw a tiny plant with 4 leaves (Day #1). The next day it had grown and had more leaves (Day #2). On the following day, it had grown even more leaves. Each day I noticed that it continued to grow in the same way.



a) In the space above, add pictures of what the upside-down T plant will look like on each of the next 2 days (Day #4 and Day #5).

b) What do you notice about the structure of the plant and the way it grows each day? If you can, colour the leaves of the pictures above in different colours to show what you see, and explain your thinking below.

c) How many leaves will the plant have on Day #7? Explain / show how you obtained your answer.

d) How many leaves will the plant have on Day #17? Explain / show how you obtained your answer.

e) If someone gives you any day number, how do you find the number of leaves the plant will have on that day? Explain / show how you obtained your answer.

f) On what day number would the plant have 100 leaves? Explain / show how you obtained your answer.