When is Three Quarters not Three Quarters? Listening for Conceptual Understanding in Children's Explanations in a Fraction Interview

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Twelve students across Grades 2 to 6 were interviewed individually using a range of tasks, where the mathematical focus was a conceptual understanding of fractions. Careful listening established that despite giving a correct answer and appearing to have conceptual understanding, further probing sometimes revealed that the child had only a faulty procedural understanding. Similarly, success on one task did not guarantee success on a different but related task. Conversely, a task involving a continuous quantity enabled a child to move between discrete and continuous interpretations of fractional parts. This study supported the claimed advantages of one-to-one interviews over pen and paper tests, but also highlighted the importance of careful listening and the need for multiple tasks in the one mathematical domain in eliciting understanding.

The structured interview provides an opportunity to listen closely to children's explanations of their mathematical understandings. In the present study, a child successfully completed a reversibility task, finding the whole when given a fractional part, by stating that given the three frogs in the picture in front of her, and seeing that they were three quarters of the frogs in the pond (the others being underwater), then there were four frogs in the pond altogether. Her understanding, however, was much more complex than her straightforward explanation conveyed. She added, as an afterthought while answering a different question, "oh with the frogs right, if they're the same weight, length, height, everything, and then I draw a little frog, it's not a quarter."

This observation demonstrated that not only was her part whole knowledge of fractions stable but also that she had made connections to the measurement domain, in order to allude to the fact that rational numbers could involve more than a discrete whole number count. While three quarters could mean three out of four discrete items, in this case, frogs, three quarters could also mean a part of a whole that had continuous attributes like weight and length and height. This comment, an afterthought, was made possible by the one-to-one setting of a fraction interview, and the opportunity to respond to a range of tasks addressing similar content.

Research has indicated that fractions as rational numbers are more than just shorthand for a part whole relationship based on equal parts. Rational numbers encompass the subconstructs: decimals, equivalent fractions, ratio numbers, multiplicative operators, quotients, and measures or points on a number line (Kieren, 1976). With minor variations these are the terms used by subsequent researchers in the field of rational numbers (see, e.g., Behr, Post, Harel, & Lesh, 1993; Lamon, 1999; Pearn, 1996; Sowder, Bezuk, & Sowder, 1993). Others move beyond these subconstructs and re-position rational numbers as a subconstruct of the larger concept of proportional or multiplicative structures (Lamon, 1999; Vergnaud, 1988).

Kieren observed that children could perform computations involving fractions and use whole number knowledge without developing "concepts of part-whole relationships or proportionality" (p. 107). This is an example of instrumental understanding as opposed to relational understanding (Skemp, 1976). These terms can be mapped seamlessly onto the more widely used, procedural understanding and conceptual understanding. Skemp uses the example of the "turn it upside down and multiply" procedure for division by a fraction, which leads children to solve problems correctly, but leaves them with little if any conceptual understanding about what they have done. An example of a strategy based on conceptual understanding is the "residual" strategy used to compare two fractions such as 7/8 and 5/6. As both are only one "piece" away from a whole and eighths are smaller than sixths, 7/8 is closer to one (Post & Cramer, 2002). One procedural approach would be to turn both fractions into forty-eighths and compare numerators.

Conceptual understanding is sometimes equated with "concept stability," a term used by the researchers in the Rational Number Project. The ability of the child to reconstruct the unit when given a part, is an indicator of concept stability of a part whole understanding of fractions(Cramer, Behr, Post, & Lesh, 1997; Post, Cramer, Behr, Lesh, & Harel, 1993).

Whole number counting is by definition discrete, even when counting sets of objects such as baskets of eggs, because the attribute being measured is "oneness," regardless of other attributes such as size or length or weight. Rational numbers, on the other hand can be broken into infinitely smaller and smaller iterating units. So just like days can be broken into hours or minutes or seconds or micro seconds as the occasion warrants, so too can the concept of fraction density be applied to find an infinite number of fractions between two rational numbers.

Whole number procedures are often used when teaching fractions. Many children are introduced to fractions by being shown a set of elements, for example a rectangle divided into three equal pieces, some of which are shaded and told to count. The cardinal number of the count of shaded objects is the numerator, while the cardinal number of the count of shaded and unshaded elements becomes the denominator. And thus fractions become tied to counting and matching (Carrahar, 1996).

It is when children are asked to find the unit that they, drilled to look for the number of parts first, show up the weakness of procedural understanding. Children need to look for the unit first (Lamon, 1999). This process is called unitizing. A unit may be composed of many elements. Egg cartons (full) are an example where the child might count four of them but are counting 4[12 units]. If we move away from egg cartons and to a quantity that is not physically grouped but mentally grouped, we can re-unitise. This is an important strategy (Behr et al., 1993; Carpenter, Fennema, & Romberg, 1993; Sowder & Philipp, 1999).

The Present Study

The purpose of the study was to assemble a collection of informative tasks to be used in a one-to-one interview. Further research is intended to trial these tasks in order to elaborate a hypothetical learning trajectory (see, e.g., Simon, 1995) or growth points (see, e.g., Clarke, 2001) for developing fraction understanding. This paper focuses on using the explanations three children gave to five key questions and identifies whether they had conceptual or procedural knowledge of the fraction task.

The participants were 12 children from a low socio-economic, co-educational state primary school in suburban Melbourne. The students included two from Grade 2, seven from Grade 5 and three from Grade 6. Information letters and consent forms were sent home to approximately 30 students and all children whose parents returned consent forms were interviewed at the end of the school year. Each child was interviewed for around 45 minutes and the interview taped and transcribed. The children were not asked the same questions as there were 55 questions trialled.

The children were not told whether they were correct or incorrect with their initial answer, and were always asked to explain their reasoning. Some gave a correct answer but demonstrated faulty reasoning and so were deemed to have not successfully answered the question. Some initially gave an incorrect answer but subsequently self corrected while they were explaining their reasoning on the task. They were deemed to have completed the task successfully.

Many tasks in the original interview were gathered from the research literature or adapted from tasks contained in teacher reference books. They were collected in order to represent types of fractional understandings that had emerged from the literature, for example,

- part-whole, equivalence, decimal, ratio, quotient and operator understandings of fractions;
- discrete and continuous cases;
- unit and non-unit fractions;
- pictorial, concrete, regional area, numerical, story or verbal representations;
- perceptual distracters, unknown units, the need to re-unitize and reversibility in relation to the unit; and
- comparison problems using the same numerator, denominator, residual strategies or benchmarking.

Task 1 was chosen in order to assess concept stability of a part whole understanding of the complement fractions one quarter/three quarters using reversibility (Herscovics, 1996; Steffe & Olive, 2002).

Show child card with three frogs.
These are some of the frogs in a pond, but we can't see them all because some are underwater. This is three quarters of the frogs in a pond, how many frogs are there altogether? Please explain.

Task 2 was a based on a fraction density question developed by Sullivan and Lilburn (1997).

a) Please tell me a fraction that is really close to 1.
Prompt for closer number until fraction is smaller than 1/10th away from 1.

If answer to (a) is less than one, ask part (b)
b) Can you think of a fraction that is really close to one but is bigger than one?

How did you work that out?

Task 3 was similar to task two in that the conceptual understanding of 3/4 could be part-whole or as an operator.

a) *Place 24 plastic teddies on the table.*
Here are 24 teddies, Please show me 3/4 of 24.

If incorrect and not modelled ask part (b):
b) Could you use the teddies to help you work that out?

If incorrect to (a) or (b) ask part (c):
c) Please show me 1/4 of 24.

Task 4 was a task that was discussed in different research studies and national assessments (Cramer et al., 1997; Kieren, 1988; Post, 1989; Wearne & Hiebert, 1988)

Show child fraction equation cards one at a time.
I want you to estimate the answer to these problems. You don't have to work out the answer exactly, just tell me the closest whole number. So your answer can be 0 or 1 or 2 or 5 or 20, any whole number. Round about what is the answer to this?

$\frac{7}{8}+\frac{12}{13}=$ $\qquad\qquad$ $\frac{3}{8}+\frac{5}{12}=$

How did you work that out?

Results and Discussion

In Task 1, Christy, a Grade 5 child answered "four" when asked for how many frogs altogether after being shown three frogs and told that they were three quarters if the frogs in a pond. When asked to explain, she added that "because when you said there were three quarters I thought of a circle, three quarters coloured in, so four." Christy moved seamlessly between a discrete model, frogs in a pond, and a continuous model, a regional area representation. When she answered the next question, Task 2, however, the breadth of her conceptual understanding about Task 1 was revealed. In order to describe a fraction close to one she drew a diagram of a circle divided into eighths and then added an inner concentric circle. This observation of non-congruent parts appeared to prompt her to add a comment about Task 1, "oh, with the frogs right, if they're the same weight, length, height, everything, and then I draw a little frog, it's not a quarter."

She used the conceptual understanding that the fractional parts must be the same in a designated attribute, and in Task 2 this attribute was area. In order to explain that her regional area model for Task 2 needed to have equivalent but not necessarily congruent parts, she was struck by the fact that the frogs in the previous question needed to be equivalent too. The revealing thing was that she could move between discrete and continuous attributes of the frogs and still hold onto the task of identifying the complement fraction, three quarters/one quarter. So when is three quarters not three quarters? When the last frog is too light, too short or too small.

Jake, another Grade 5 child, used a different approach to the frog task. He quite correctly identified that the unit was "four", "because if it's three it needs more, it means it's not equal, so if I put one more frog there, it will be four, one whole, not three quarters."

He had the correct answer and demonstrated conceptual understanding. It was his answer to other questions involving three quarters and two thirds that showed he was using a counting strategy that worked in Task 1 but one that could not be generalised to other tasks.

When asked to draw a circle around two thirds of an array of 15 dots, a task that was used in the Rational Number Project (Cramer et al., 1997), Jake drew a line around all but the bottom right dot and explained, "because it's two thirds, not like the whole thing. Two thirds you need one less so I just left one." Jake, like Christy, had the picture of the circle divided into parts in his mind. His understanding of it was a quite literal part-whole model. Two thirds was one piece not shaded in. He had overgeneralised this to apply to sets of items and had drawn a circle around all but one of the dots, because two thirds was, in Jake's reasoning, the same as one left over. He could not see 2/3 as an operator on the set of 15 items, he could only see part whole relationships, and they were based on additive rather than proportional relationships.

Jake did not have a conceptual understanding of the fractions 2/3 or 3/4 despite being successful at Task 1 where he demonstrated that he knew the complement fractions one quarter/three quarters. In fact, his response to Task 3, modelling 3/4 of 24 teddies, showed that he was in danger of applying a procedure, an unsuccessful algorithm, to a whole class of questions. When asked to show three quarters of the twenty four teddies, he showed 23 teddies and explained that "that means take everyone except left one….because it's the same with two thirds, with three quarters that means there's three on the top, four on the bottom, so you take everything, it's almost to a whole but it's not. Take one away from it, it's approximately one. So I took one away to make it three. And that's four. And if I take one away that's three, four." Just like 3 frogs were 3/4 of 4, 2/3 of 15 was 14, and 3/4 of 24 was 23. They were all one less. Not only did he not have conceptual understanding but because he used an unsuccessful algorithm, he did not have procedural understanding either.

He did not have a part-whole conceptual understanding, or an understanding of 2/3 or 3/4 as operators. He could not re-unitize 1[24 unit] into 4[6units]. He was successful at modelling one quarter of 24 by mentally halving, " there's twenty four here, half it makes twelve, halve it again, that makes six." But he did not see that a ratio relationship should exist between the 6 teddies for a quarter and his one teddy for three quarters.

Task 4 shows how carefully the interviewer has to listen to distinguish between a correct answer with full conceptual understanding and a correct answer with little conceptual understanding and, in this case, faulty procedural knowledge. Maddison, a grade 5 child, was asked to estimate the answer to 7/8 + 12/13. This task could be characterized as an equivalence problem as the child was being asked to see both 7/8 and 12/13 as a near equivalence to one. It could also be described as a benchmarking task, with the child using 1 as a benchmark and reasoning that a little less than one added to a little less than one had to be just less than two.

Maddison answered, "two," the correct answer. Her reasoning appeared modest but faultless, "I just guessed. That's seven bits of eight. Twelve bits of thirteen." This would seem a successful part-whole approach. Her next comment was strange but seemingly unrelated to her answer of two, "And I just added eight and thirteen." Well obviously, she didn't add eight and thirteen and come up with two, so she was scored as correct and asked the next question to see if she could use half as a benchmark. Here her answer was "twelve".

Her reasoning threw light on her cryptic comment in the previous question, "But eight and twelve are twenty and the three and the five are covering a bit of it and so I took it away." In other words, adding the numerators and taking away that from the sum of the denominators was her procedure. If you applied that whole number procedure to the previous question she could get an answer of two; 8 + 13, which is what she said she did, was 21. 21 – (7 +12) was 2.

In fact any residual pair of fractions (fractions where the numerator is one less than the denominator) would give the answer of two using Maddison's method, so the question can't simply have different fractions. This question has been used as a standardized test. Only 25% of 13 year olds, on a multiple-choice test chose correctly the answer 2 (Wearne & Hiebert, 1988). If Maddison had been doing this pen and paper test she would have been one of the successful 25%, when in fact she had little conceptual understanding of the question.

Summary and Implications

In order to continue to refine tasks that may prove useful in establishing fraction growth points in later studies, close attention must be paid to the children's explanations of their answers to tasks. It is important that they are not credited with conceptual understanding when it is unstable, such as Jake's idea of three quarters being one less that any unit. Nor should they be seen as having conceptual understanding because they produced the correct answer with faulty reasoning, such as in Maddison's "subtract the sum of the numerators from the sum of the denominators" counting procedure.

There is no question that well-chosen tasks can elicit informative explanations from children, giving insight into different aspects of their fraction knowledge. But caution must be exercised when making generalizations about a child's conceptual understanding from a single task. It would be a mistake to say that Jake understands three quarters, but that would be the assumption if he had only been asked the whole from the part, frogs in the pond question.

Conversely, Christy showed that she could move comfortably between a discrete model and a continuous model when discussing the frogs in the pond. By listening closely to the explanations as well as the answers that children give to structured interviews, researchers can make more accurate judgments about the extent and interaction of a child's whole number knowledge and their rational number knowledge.

References

Behr, M., Post, T., Harel, G., & Lesh, R. E. (1993). Rational numbers: Towards a semantic analysis. In T. P. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 13-47). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Carpenter, T. P., Fennema, E., & Romberg, T., A. (1993). Toward a unified discipline of scientific enquiry. In T. P. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 1-11). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Carrahar, D. W. (1996). Learning about fractions. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 241-266). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Clarke, D. M. (2001). Understanding, assessing and developing young children's mathematical thinking: Research as powerful tool for professional growth. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 9-26). Sydney: MERGA.

Cramer, K., Behr, M., Post, T., & Lesh, R. (1997). *Rational number project: Fraction lessons for the middle grades - Level 2*. Retrieved 18/11/2003, 2003, from http://education.umn.edu/rationalnumberproject/97_3.html

Herscovics, N. (1996). The construction of conceptual schemes in mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 351-379). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. E. Lesh (Ed.), *Number and measurement; Papers from a research workshop*. Athens, Georgia: Georgia Centre for the Study of Learning and Teaching Mathematics, University of Georgia.

Kieren, T. E. (1988). Personal knowledge of rational numbers; its intuitive and formal development. In J. A. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). National Council of the Teachers of Mathematics, Reston, Virginia: Lawrence Erlbaum Associates

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah, New Jersey: Lawrence Erlbaum Associates.

Pearn, C. (1996). Young children's strategies in solving rational number tasks. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning*. Adelaide, S.A.: The Australian Association of Mathematics Teachers Inc.

Post, T. (1989). One point of view- Fractions and other rational numbers. *The Arithmetic Teacher, 37*(1), 3-28.

Post, T., & Cramer, K. A. (2002). Children's strategies in ordering rational numbers. In D. L. Chambers (Ed.), *Putting research into practice in the elementary grades* (pp. 141-144). Reston, Virginia: National Council of the Teachers of Mathematics.

Post, Cramer, K. A., Behr, M., Lesh, R. E., & Harel, G. (1993). Curriculum implications of research on the learning, teaching and assessing of rational number concepts. In T. P. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 327-362). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114-145.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

Sowder, J., Bezuk, N., & Sowder, L. K. (1993). Using principles from cognitive psychology to guide rational number instruction for prospective teachers. In T. P. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 239-259). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Sowder, J., & Philipp, R. (1999). Promoting learning in middle-grades mathematics. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 89-108). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Steffe, L. P., & Olive, J. (2002). The problem of fractions in the elementary school. In D. L. Chambers (Ed.), *Putting research into practice in the elementary grades* (pp. 128-132). Reston, Virginia: National Council of the Teachers of Mathematics.

Sullivan, P., & Lilburn, P. (1997). *Open-ended maths activities: Using "good" questions to enhance learning*. South Melbourne: Oxford University Press.

Vergnaud, G. (1988). Multiplicative structures. In J. A. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). National Council of the Teachers of Mathematics, Reston, Virginia

Wearne, D., & Hiebert, J. A. (1988). Constructing and using meaning for mathematical symbols; the case of decimal fractions. In J. A. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 220-235). National Council of the Teachers of Mathematics, Reston, Virginia: Lawrence Erlbaum Associates