Triviality results for probabilistic modals

Goldstein, Simon

TRIVIALITY RESULTS FOR PROBABILISTIC MODALS

Simon Goldstein
Lingnan University

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Abstract

In recent years, a number of theorists have claimed that beliefs about probability are transparent. To believe probably p is simply to have a high credence that p. In this paper, I prove a variety of triviality results for theses like the above. I show that such claims are inconsistent with the thesis that probabilistic modal sentences have propositions or sets of worlds as their meaning. Then I consider the extent to which a dynamic semantics for probabilistic modals can capture theses connecting belief, certainty, credence, and probability. I show that although a dynamic semantics for probabilistic modals does allow one to validate such theses, it can only do so at a cost. I prove that such theses can only be valid if probabilistic modals do not satisfy the axioms of the probability calculus.

1 Introduction

Ordinary language includes many claims that describe what the world is like: *it will rain later*, *the die will land on 6*, etc. A rational agent will have various attitudes towards these descriptive claims. These include coarse-grained attitudes such as belief and certainty. They also include fine-grained attitudes such as having a .2 credence that the die will land on 6.

But ordinary language also includes various claims that do not seem to directly describe the way the world is, but instead convey something about our epistemic relation to it. For example, we have conditional claims like *if the die is rolled, it will land on 6*, as well as probabilistic claims such as *there's a 20% probability that the die will land on 6*.

Ordinary agents adopt propositional attitudes towards these claims as well. Moreover, it seems that an agent's attitudes toward these claims should bear some relation to her attitudes toward the descriptive claims. For example, perhaps an agent should believe that there is a 50% probability that it will rain just in case she has a 50% credence that it will rain. Can we give systematic principles stating what the relation should be between our attitudes to these two sorts of claims? That's the topic I'll be exploring in this paper.

In what follows we will focus on principles governing the conditional *if…then…*, and the modals *probably* and *exactly n likely*. We will consider embedding these operators under three types of propositional attitudes: certainty, belief, and credence. For example, we will explore what it takes for an agent to believe that a fair die is exactly 50% likely to land odd. In addition, we will explore what it takes for an agent to believe that a fair die will probably land 1 through 5.

The assignment of credences to conditionals is well-studied. Stalnaker (1970) famously suggested that one's credence in the conditional *if A, then B* is always equal to one's conditional credence in B given A. Unfortunately, Lewis (1976) showed that this thesis led to triviality. If every credence function satisfies this equation, then one's credence in any claim is always independent of any other.

This paper will start by considering a variety of principles other than Stalnaker's original thesis concerning how we assign doxastic attitudes to both conditional and probabilistic claims. For example, we will consider the 'transparency' principle that an agent believes it is n likely that A just in case her credence in A is n. We will see that these other principles lead to structurally analogous triviality results.

Then we will turn to an increasingly popular strategy for responding to triviality results like that in Lewis 1976. According to this strategy, epistemic modal claims have a special kind of meaning that can be understood using the tools of dynamic semantics. We will see that this approach provides an opportunity for avoiding triviality results while retaining some plausible principles about believing modal claims.

In the final section, however, we will see that a dynamic response to triviality results comes with serious costs. The original triviality results reappear as impossibility results governing the logical properties of the probabilistic modal *it is n likely*. We will see in particular that these dynamic theories only avoid triviality on pain of making this modal, and the attitudes that embed it, violate the rules of the probability calculus.

2 Triviality Results

Before we start, we need some assumptions. In particular, throughout this section we will assume that modal claims, like ordinary descriptive claims, have sets of possible worlds as their meanings (in later sections we will give up this assumption). So we will assume that probabilities are assigned to sets of possible worlds, and then we will consider whether there could be a conditional operator $\rightsquigarrow$ on sets of worlds that satisfies various interesting constraints on credence:

Definition 2.1. Let W be the set of possible worlds. Let A, B, … be variables for subsets of W. Let a probability function Pr be a function on some Boolean algebra over W to $[0, 1]$ satisfying:

1. $Pr(A) \geq 0$ — NON-NEGATIVITY
2. $Pr(W) = 1$ — UPPER BOUND
3. If $A \cup B = \emptyset$, then $Pr(A \cup B) = Pr(A) + Pr(B)$ — ADDITIVITY

Let $Pr(B \mid A)$ represent an agent's credence that she inhabits a B world, conditional on inhabiting an A world, defined by the ratio $\frac{Pr(A \cap B)}{Pr(A)}$. Let $\rightsquigarrow$ be a two-place conditional propositional operator that takes two sets of worlds as input, and returns a new set of worlds.

Now that we've introduced a few tools for thinking about modal claims and probability, we can start to explore some theses about how agents assign propositional attitudes to modals. Let's start our discussion with the best explored thesis about assigning credences to conditionals. Stalnaker's Thesis, introduced in Stalnaker 1970, says that any rational agent's credence in $A \rightsquigarrow B$ is equal to her credence in B given A. Quantifying over such rational credence functions, we reach the following equation:

Definition 2.2 (Stalnaker's Thesis)**.** For any rational Pr, A, B:

$Pr(A \rightsquigarrow B) = Pr(B \mid A)$

For example, consider an agent's credence that if a fair die lands odd (odd), then the die will land 3 ($\mathsf{3}$). Stalnaker's Thesis says that this is equal to $Pr(\mathsf{3} \mid \mathsf{odd})$, which is $\frac{1}{3}$.

Lewis 1976 famously showed that Stalnaker's Thesis was untenable. In particular, he showed the following.

Fact 2.1 (Lewis)**.** If Stalnaker's Thesis holds, then $Pr(B \mid A) = Pr(B)$ for every rational Pr, A, B.

This conclusion is unacceptable. In a Bayesian framework an agent updates her credences by conditionalization.[1] So if Stalnaker's Thesis held, then an agent's credence in any proposition would remain the same when she learned any other claim.

The first thing we will see in this paper is that there are many other principles about epistemic modals besides Stalnaker's Thesis that lead to structurally similar triviality results. We will start by looking at another principle about conditionals, and then we will turn to probabilistic modals.

For our first principle, consider the following claim:

(1) If a fair die is rolled, then it will land greater than one.

According to Stalnaker's Thesis, a rational agent's credence in (1) should be $\frac{5}{6}$. For that is the agent's credence in the die landing greater than one, conditional on it being rolled. However, one natural response that an agent might make to (1) is simply to reject it. After all, the following response seems coherent:

(2) a. If the die is rolled, then it will land greater than one.
    b. That's not true; after all, it might land one.

Insofar as one finds (2) to be a natural reaction, one might be tempted by the following thesis. An agent is certain of the conditional $A \rightsquigarrow B$ just in

[1] As we will see below, here it is important that the set of rational credence functions be closed under conditionalization on a wide range of propositions, so that whenever Pr is a rational credence, $Pr(\cdot \mid A)$ is also a rational credence function. For some discussion of exactly what range of credence functions the above equations can apply to while avoiding triviality, see Lewis 1986 and Hájek and Hall 1994.

case she has a credence of 1 in B, given A. But if an agent's credence in B given A is less than 1, then she is certain that $A \rightsquigarrow B$ does not obtain. This theory can be thought of as broadly Ramseyan:

> If two people are arguing 'If p, then q?'...they are adding p hypothetically to their stock of knowledge and arguing on that basis about q.[2]

To represent Ramsey's Thesis precisely, let's assume that an agent assigns certainties to sets of worlds. Then Ramsey's Thesis says:

Definition 2.3 (Ramsey's Thesis)**.** For any A and B, where Pr represents an agent S's rational credence function:

1. S is certain of $A \rightsquigarrow B$ iff $Pr(B \mid A) = 1$
2. S is certain of $W - (A \rightsquigarrow B)$ iff $Pr(B \mid A) < 1$

Ramsey's Thesis is a bit structurally different than Stalnaker's Thesis. While Stalnaker's Thesis governed the connection between credence and conditional credence, Ramsey's Thesis instead governs the connection between certainty and conditional credence. On its own, Ramsey's Thesis is consistent.[3] But if we add further principles connecting certainty and credence, we run into trouble. For suppose that an agent is certain of A just in case her credence in A is 1:

Definition 2.4 (Certainty-Credence Link)**.** For any A, where Pr represents an agent S's rational credence function:

S is certain of A iff $Pr(A) = 1$.[4]

When we combine Ramsey's Thesis and Certainty-Credence Link, we reach the following principle:

Definition 2.5 (Credal Ramsey's Thesis)**.** For every rational Pr, A, B:
$$Pr(A \rightsquigarrow B) = \begin{cases} 1 & \text{if } Pr(B \mid A) = 1 \\ 0 & \text{otherwise} \end{cases}$$

But just like Stalnaker's Thesis, Credal Ramsey's Thesis leads to triviality:

Fact 2.2. If Credal Ramsey's Thesis holds, then $Pr(A) = 1$ or $Pr(A) = 0$ for every rational Pr, A.

(For proofs, see the appendix. This result is a corollary of a more general proof in Gardenförs 1988.)

This triviality result shows that Credal Ramsey's Thesis is untenable. For if Ramsey's Thesis held, then every agent would have maximally extreme credences; they could never have any uncertainty about any claim.

[2] Ramsey 1990

[3] Although see Gardenförs 1988 for a proof that if we add some other constraints to Ramsey's Thesis, then we run into trouble.

[4] Hájek 2013 has developed a variety of counterexamples to the right-to-left direction of this principle, involving infinite partitions. For example, imagine throwing an infinitely small dart at the real number line between 0 and 1. What is the probability it will land $\frac{1}{2}$? Seemingly zero. But one can't be certain that it won't. Nonetheless, for finite partitions both Ramsey's Thesis and Certainty-Credence Link appear plausible. I will show that they lead to triviality. It would be small consolation to retain Ramsey's Thesis merely in the case of infinite partitions.

We've now looked at two principles governing doxastic attitudes towards conditional claims, and seen that they generate surprising consequences. Now let's turn to probabilistic claims, like:

(3) The die will probably land two through six.

(4) The die is 50% likely to land odd.

A variety of recent work has investigated the meaning of natural language probabilistic modals, including *probably*, *likely*, and more precise expressions such as *it is n likely*. Many in the current literature advocate an elegant theory of what it takes to believe claims like (3) and (4), connecting belief with credence. On this 'transparency' thesis, all it is to believe probably ϕ is to have a sufficiently high credence that ϕ. All it is to believe that it is n likely that ϕ is to have a credence of n that ϕ.[5] It is not required that the agent have a belief about her high credence. It is not required that the agent have a belief about her evidence. All that is required is an attitude directed at the proposition that ϕ.

Now imagine we enrich our space of operators by including new probabilistic operators $\blacktriangle$ and $\blacktriangle_n$ that take a set of worlds A as input and returns the new sets of worlds $\blacktriangle A$ and $\blacktriangle_n A$. $\blacktriangle$ and $\blacktriangle_n$ will model the probabilistic modals *probably* and *n likely*. Our first transparency thesis says that an agent believes $\blacktriangle_n A$ just in case her credence in A is n:

Definition 2.6 (Belief Transparency for *n likely*)**.** For any A, where Pr represents an agent S's rational credence function:

S believes $\blacktriangle_n A$ iff $Pr(A) = n$.

In the case of the coarser modal *probably*, the analogous thesis says that an agent believes $\blacktriangle A$ just in case her credence in A is greater than $\frac{1}{2}$:

Definition 2.7 (Belief Transparency *probably*)**.** For any A, where Pr represents an agent S's rational credence function:

S believes $\blacktriangle A$ iff $Pr(A) > \frac{1}{2}$.[6]

Like Ramsey's Thesis, these principles on their own are consistent. But when we combine them with further principles connecting belief and credence, we run into trouble. In particular, imagine we accept a Lockean principle, on which belief amounts to credence above some threshold t. According to Lockeanism, we can move freely between claims about belief and claims about credence. In particular, the following inference pattern is predicted to be valid:

(5) a. Fred believes it's raining. $\Rightarrow$
b. Fred is fairly [/quite] confident that it's raining.

[5]See Yalcin 2007, 2012a, Swanson 2011, 2012, Rothschild 2012, and Moss 2015. For an analogous claim about possibility modals, see Stephenson 2007.

[6]Besides their simplicity and intuitive plausibility, there are a variety of arguments in the literature for transparency principles. For example, Bennett 2003 (90), Yalcin 2007, and Rothschild 2012 argue that one can have modal beliefs without having beliefs about one's own mental states. In addition, Yalcin 2007 uses transparency principles to explain why its incoherent to believe Moorean claims like *it's raining and its probably raining.*

More precisely, Lockeanism claims that there is some threshold t such that:

Definition 2.8 (Lockeanism)**.** For any A, where Pr represents an agent S's rational credence function:

S believes A iff $Pr(A) \geq t$.[7]

Once we accept Lockeanism, we can derive new transparency principles. While our old transparency principles connected belief in probabilistic claims with credence, our new principles only concern an agent's credence function. On these new principles, an agent's credence in $\blacktriangle_n A$ is greater than the Lockean threshold t just in case her credence in A is n, and her credence in $\blacktriangle A$ is greater than t just in case her credence in A is greater than $\frac{1}{2}$:

Definition 2.9 (Credal Transparency for *n likely*)**.** For any rational Pr, A:
$Pr(\blacktriangle_n A) \geq t$ iff $Pr(A) = n$

Definition 2.10 (Credal Transparency for *probably*)**.** For any rational Pr, A:
$Pr(\blacktriangle A) \geq t$ iff $Pr(A) > \frac{1}{2}$

These Credal Transparency theses follow from the earlier Belief Transparency theses and Lockeanism. For we reach Credal Transparency by taking Belief Transparency and replacing each occurrence of *believes* with *has sufficiently high credence*.

Unfortunately, it turns out that Credal Transparency leads to triviality. If it holds, then the probability of any claim is either 1 or 0:

Fact 2.3. If Credal Transparency holds for *n likely*, then $Pr(A) = 1$ or $Pr(A) = 0$ for any rational Pr, A.

A similar problem also occurs when we accept Credal Transprency for the modal *probably*. In that case, we reach the bizarre conclusion that the probability of any claim is either less than $\frac{1}{2}$ or greater than the Lockean threshold t:

Fact 2.4. If Credal Transparency holds for *probably*, then $Pr(A) \leq \frac{1}{2}$ or $Pr(A) \geq t$ for any rational Pr, A.[8]

We have now seen that Transparency principles are pretty much incompatible with Lockeanism. Once we accept both, we reach some absurd triviality results.[9] [10]

[7]For defenses of the Lockean view, see Foley 1993; Christensen 2005; Sturgeon 2008.

[8]This result, unlike the previous ones, relies on the assumption that the set of rational credences is closed under conditionalization on modal formulas (like $\triangle\phi \vee \neg\phi$). See Appendix 2 for discussion.

[9]There is one way to avoid the second result. When $t = \frac{1}{2}$, the result is trivial. So one might respond that the Lockean threshold t is simply $\frac{1}{2}$. Alternatively, one could reinterpret transparency so that one believes $\triangle\phi$ just in case one's credence in ϕ is greater than or equal to t. This avoids the problem.

Nonetheless, this doesn't solve all the problems. On the way to proving Fact 2.4, it can also be shown that every probability function assigns $\triangle\phi \vee \neg\phi$ a probability of 1. But only logical truths are assigned a probability of 1 by every probability function. Interestingly, however, no transparency theorist has so far accepted this claim. After all, from this proposal and reductio we reach the conclusion that ϕ and $\triangle\phi$ are logically equivalent. (Indeed, we get that anyways from the claim that for any Pr $Pr(\triangle\phi) = Pr(\phi)$). However, most transparency theorists instead say that ϕ is logically equivalent to $\Box\phi$; not to $\triangle\phi$.

[10]Of course, Lockeanism is a controversial thesis connecting belief and probability. Defenders

The results so far are interesting to compare to recent work in Russell and Hawthorne 2016.[11] Russell and Hawthorne 2016 show that two theses related to Credal Transparency lead to some strange results. Credal Transparency is a synchronic principle. It relates one's current credence in $\blacktriangle_n A$ to one's current credence in A, for example. Russell and Hawthorne consider some diachronic theses, which characterize what one's credence in A should be once one learns $\blacktriangle_n A$ or $\blacktriangle A$. In particular, they consider the claim that upon learning $\blacktriangle_n A$, one's credence in A should be n, and that one's credence in A should be greater than $\frac{1}{2}$ upon learning that $\blacktriangle A$. Russell and Hawthorne show that these results lead to some consequences similar to those above—for example that an agent must be certain of A after learning either $\blacktriangle_n A$ or $\blacktriangle A$. The theses Russell and Hawthorne consider are conceptually independent from Credal Transparency. On the other hand, since both theses lead to absurd results, they may ultimately entail each other.

In the face of their triviality results, Russell and Hawthorne 2016 join a growing tradition advocating a dynamic semantics for epistemic modals. In the next sections of the paper, I will consider to what extent a dynamic semantics for modals can save principles like Credal Transparency. Ultimately, I will part ways from Russell and Hawthorne 2016. In the later sections of the paper, I will show that a dynamic semantics for modals ultimately faces similar versions of the triviality results above. In particular, a dynamic semantics for modals can only salvage forms of Transparency at the cost of giving up the claim that the modal n *likely* satisfies the probability calculus. Here, I part ways from Russell and Hawthorne 2016, who not only suggest that dynamic semantics allows us to avoid the triviality results, but also argue that probabilistic principles like the law of additivity have little to do ultimately with avoiding triviality.[12] We will see that giving up some principles in the probability calculus is a necessary condition for embracing certain forms of Transparency.

3 Dynamic Semantics

In the last section, we saw that a variety of principles about certainty, belief, and credence lead to paradoxical results about epistemic modal claims. These results suggest that there is something paradoxical about

of Belief Transparency might just give it up. (For one recent critique of Lockeanism, see Ross and Schroeder 2014.) But saying just this isn't really enough to escape the problem. For suppose one rejects Lockeanism but still allows that $\triangle_n \phi$ can be assigned probability. In that context, it seems strange to accept Belief Transparency but deny Credal Transparency. For suppose contra Credal Transparency that the probability of $\triangle_n \phi$ can come apart radically from the probability of ϕ, but that Belief Transparency still holds. Then there may be a situation where one believes $\triangle_n \phi$, since the probability of ϕ is n, and yet the probability one assigns to $\triangle_n \phi$ is vanishingly small. Even enemies of Lockeanism should avoid this result.

[11] For another triviality result for epistemic modals, see Fuhrmann 1989, who reveals some paradoxical consequences of the thesis that an agent believes *might* ϕ just in case she does not believe $\neg\phi$.

[12] " The most promising way forward for the stalwart defender of such theses isn't any of the usual expressivist maneuvers (rejecting embeddings or standard probability theory) but rather something like this dynamic approach." (Russell and Hawthorne 2016 28).

assigning probabilities to epistemic modal claims in the first place. Indeed, proponents of principles like Belief Transparency above have tended to advocate a broadly 'nonfactual' semantics for epistemic modals. On this semantics, modal claims do not have as their meaning a set of possible worlds in which they are true, which can be straightforwardly assigned a probability by Pr. Rather than semantically expressing such a 'factual' condition, modal claims instead semantically contribute constraints on bodies of information. In this section, we will explore what such a semantics looks like, and then we will explore several ways in which this framework helps avoid the triviality results from the previous section.

In the previous section, we assigned probabilities directly to sets of worlds, and tried to model the conditional and probabilistic modals as operators on sets of worlds. In this section, we will instead define a formal language that we will semantically interpret. This language will have to be rich enough to represent all the claims we explored in this previous section. This means that in addition to symbols for the conditional ($\to$) and probabilistic modals ($\triangle$, $\triangle_n$), we will also need a variety of attitude verbs—enough to represent an agent's beliefs (B_S), certainties (C_S), and credences (Cr_S). Finally, in order to directly express claims about probabilities it will later be useful to have operations for addition, multiplication, and division.

Definition 3.1. Let $\mathscr{L}$ be a language containing atomic formulae $\alpha, \beta, \dots$, negation $\neg$, conjunction $\wedge$, a conditional operator $\to$, the probabilistic modal $\triangle$, a belief operator B_S, and a certainty operator C_S. In addition, $\mathscr{L}$ contains terms $n, m, \dots$ for the real numbers in $[0,1]$, the operations $\div$, $+$, and $\times$, the predicates $=$, $<$, and $>$ relating terms, and a quantifier $\exists$ over terms. Next, $\mathscr{L}$ contains the numerical modal $\triangle_n$, for any term n. Finally, for any sentence $\phi \in \mathscr{L}$, we have sentences of the form $Cr_S(\phi) = n$, $Cr_S(\phi) < n$, and $Cr_S(\phi) > n$ in $\mathscr{L}$ for any term. Let $\phi, \psi, \dots$ be arbitrary claims in $\mathscr{L}$. Let $\phi \vee \psi$ be defined as usual as an abbreviation of $\neg(\neg\phi \wedge \neg\psi)$. Let $\mathscr{L}_c$ be the classical fragment of $\mathscr{L}$—the sentences in $\mathscr{L}$ that do not contain $\to$, $\triangle$, or $\triangle_n$.

In the previous section, we considered whether there could be propositional operators $\rightsquigarrow$, $\blacktriangle$, and $\blacktriangle_n$ satisfying some constraints on assigning probabilities to sets of worlds using a probability function Pr. Now instead we will consider whether analogous constraints can be satisfied for sentences involving agent's attitudes towards modals, represented in our language with formulae like $C_S(\phi \to \psi)$, $B_S \triangle \phi$, $B_S \triangle_n \phi$.

We will focus on one particular implementation of a nonfactualist semantics: dynamic semantics.[13] According to dynamic semantics, the meaning of a sentence is not its truth conditions. Rather, the meaning of a sentence is its ability to change the context in which it is said:

> You know the meaning of a sentence if you know the change it brings about in the information state of anyone who accepts the news conveyed by it.[14]

[13] Stalnaker 1973; Karttunen 1974; Heim 1982; Heim 1983; Veltman 1985; Groenendijk and Stokhof 1990; Groenendijk and Stokhof 1991; and many others.

[14] Veltman 1996 221.

In this semantic framework, interpreting a language requires two steps. First, we need to know what a body of information or *context* is. Second, we need to provide an *update function* that assigns to each sentence in our language a *context-change potential*, a function that given a context returns the result of learning the original sentence in that context.

Here, I will outline one particular approach to these problems: a dynamic semantics developed in Yalcin 2012b to interpret probabilistic modals. On this proposal, a body of information has the structure of a Bayesian agent. At rock bottom, a body of information assigns credences to sets of possible worlds. Each possible world will assign a truth value to the atomic claims in $\mathscr{L}$. The credence function Pr in an information state will have the same structure we defined above in §2. But, importantly, we will also want the information structure to keep track of what is certain by the lights of a probability function. So we will let an information state σ contain not only a credence function Pr_σ but also a set of worlds s_σ assigned a probability of 1 by the credence function. Finally, we will assume that there is a privileged trivial information state $\top$ where $s_\top = W$ and $Pr_\top$ is some prior probability function.

Definition 3.2.
Let s be a set of worlds $w \in W$ that assign truth values $(w(\alpha))$ to the atomic claims α in $\mathscr{L}$. Let Pr be a probability function. Let a context $\sigma = \langle s_\sigma, Pr_\sigma \rangle$ be a pair of a set of worlds s_σ and a probability function Pr_σ, where $Pr_\sigma(s_\sigma) = 1$. Let $\top = \langle W, Pr_\top \rangle$.

With this definition of information in mind, the task is now to specify an update function $[\cdot]$ that maps each sentence ϕ in $\mathscr{L}$ to its context change potential $[\phi]$, itself a function from contexts to contexts.

Let's start by interpreting our classical fragment $\mathscr{L}_C$. In this framework, atomic sentences update a context by zooming in to the worlds where they are true, and conditionalizing the probabilities of the context on these worlds. The negation of a sentence updates the context in the opposite way as the original sentence. Conjunctions update the context in two steps; first with the left conjunct, second with the right conjunct:

Definition 3.3.

1. $\sigma[\alpha] = \langle \{w \in s_\sigma \mid w(\alpha) = 1\}, Pr_\sigma(\cdot \mid \{w \in s_\sigma \mid w(\alpha) = 1\}) \rangle$
2. $\sigma[\neg\phi] = \langle s_\sigma - s_{\sigma[\phi]}, Pr(\cdot \mid s_\sigma - s_{\sigma[\phi]}) \rangle$
3. $\sigma[\phi \wedge \psi] = \sigma[\phi][\psi]$.[15]

Sentences in $\mathscr{L}_C$ have an interesting property. They are all able to narrow down a context without becoming absurd. That is, all of the updates so far have the potential to be *informative*. Speakers can use them to transform their current context into a new, interesting state of information. Indeed, for any sentence ϕ in $\mathscr{L}_C$, updating with ϕ is equivalent to conditionalizing on the set of worlds where ϕ is true:

Definition 3.4. $[\![\phi]\!] = \{w \mid w \in s_{\top[\phi]}\}$

Fact 3.1. If $\phi \in \mathscr{L}_c$, then $\sigma[\phi] = \langle s_\sigma \cap [\![\phi]\!], Pr_\sigma(\cdot \mid [\![\phi]\!]) \rangle$

[15] Yalcin 2012b; for inspiration see Heim 1992, Veltman 1996, Jeroen Groenendijk and Veltman 1996.

Things become more complicated once we introduce epistemic modals into the language. To start, let's introduce our conditional operator $\to$. This conditional operator, like other modals, will be a *test*. This means that updating any context σ with a sentence $\phi \to \psi$ will always either return the original context σ, or will instead return an absurd body of information (represented by conditionalizing on $\emptyset$). In particular, following Gillies 2004, the conditional $\phi \to \psi$ will test a context σ to see that updating σ with ϕ will create a context in which updating with ψ has no effect:

Definition 3.5. $\sigma[\phi \to \psi] = \begin{cases} \sigma & \text{if } \sigma[\phi][\psi] = \sigma[\phi]; \\ \langle \emptyset, Pr_\sigma(\cdot \mid \emptyset)\rangle & \text{otherwise} \end{cases}$

In Yalcin 2012b's framework, probabilistic modals are also tests. These tests are essentially Bayesian: they explore properties of the probability function of the context. $\triangle\phi$ for example tests that ϕ is assigned a probability of greater than $\frac{1}{2}$ by the context's probability function, and $\triangle_n\phi$ tests that ϕ is assigned a probability of exactly n. Here, however, a bit of subtlety is required. For ϕ itself can be an epistemic modal claim. But modal claims are not themselves associated with a set of possible worlds. So $\triangle\phi$ cannot test that *the set of ϕ worlds* is assigned a certain probability. To avoid this problem, uttering $\triangle\phi$ in context σ performs a test on the probability of $s_{\sigma[\phi]}$, the set of worlds that remain in the context once one updates with ϕ. Abbreviating, we can say that $\triangle\phi$ and $\triangle_n\phi$ test the probability currently assigned to *the result of learning that ϕ*:

Definition 3.6.

1. $\sigma[\triangle\phi] = \begin{cases} \sigma & \text{if } Pr_\sigma(s_{\sigma[\phi]}) > \frac{1}{2} \\ \langle \emptyset, Pr(\cdot \mid \emptyset)\rangle & \text{otherwise} \end{cases}$
2. $\sigma[\triangle_n\phi] = \begin{cases} \sigma & \text{if } Pr_\sigma(s_{\sigma[\phi]}) = n \\ \langle \emptyset, Pr(\cdot \mid \emptyset)\rangle & \text{otherwise} \end{cases}$

In this semantics, epistemic modals are not directly associated with a set of worlds in which they are true. There is no single set of worlds, say $[\![\triangle\phi]\!]$, such that for any context σ updating σ with $\triangle\phi$ is equivalent to conditionalizing the credence function in σ on $[\![\triangle\phi]\!]$.[16]

Since meanings in this framework are not sets of possible worlds, it requires a bit of care to define semantic notions like entailment, that are often thought of in terms of truth at possible worlds. In a dynamic framework, it is common to at this point appeal to the notion of a body of information *supporting* or containing a sentence. In particular, we can say that a context σ supports ϕ when it 'already knows' ϕ: when updating with ϕ has no effect. Then entailment can be defined as preservation of support:

[16] We can interpret the arithmetic terms in $\mathscr{L}$ straightforwardly. First, any term n denotes the number n. Second, the operators $\div$, $+$, and $\times$ denote the relevant functions from pairs of numbers to numbers. Third, the predicates $=$, $>$, and $<$ denote functions from pairs of numbers to context change potentials. For example $\sigma[n = m]$ is either σ or $\langle \emptyset, Pr_\sigma(\cdot \mid \emptyset)\rangle$ depending on whether n is identical to m. Then we could also express the meaning of $[\triangle_n\phi]$ and $[Cr_S(\phi) = n]$ compositionally. So for example $[\triangle_{(\cdot)}\phi]$ would take $[n]$ as input and return the relevant test. Finally, we could interpret our existential quantifier over terms $\exists$ by enriching the above with assignment functions (see Yalcin 2015 for one implementation).

Definition 3.7. σ supports ϕ ($\sigma \models \phi$) iff $\sigma[\phi] = \sigma$. ϕ entails ψ ($\phi \models \psi$) iff for every σ if $\sigma \models \phi$ then $\sigma \models \psi$.[17]

4 Dynamic Semantics and Triviality Results

A variety of recent work has suggested that perhaps a dynamic semantics for epistemic modals can provide a solution to triviality results.[18] In this section, building on Starr Forthcoming, I will develop a systematic method by which a dynamic semantics for epistemic modals can avoid the triviality results we have explored. The dynamic response consists of several steps. In the first step, the semantics will validate the relevant modal bridge principle connecting belief or certainty in a modal claim with credence. In particular, the semantics we have looked at so far will validate Ramsey's Thesis and Belief Transparency. To avoid triviality, the semantics will invalidate the relevant non-modal bridge principle connecting belief or certainty with credence. So Lockeanism and Certainty-Credence Link will fail for modal claims. However, both principles will be preserved on the non-modal fragment of the language. This will allow the theory to invalidate Credal Ramsey's Thesis and Credal Transparency, both of which are trivializing.

Yet this alone is not much of a solution. All it has done is given up (albeit, in a careful way) one of the assumptions that led to triviality. To fully resolve the problem, a dynamic solution can go further. While giving up Credal Transparency and Credal Ramsey's Thesis, the dynamic solution can offer a variant in their place, as principles governing not credence but instead *probabilistic beliefs*—full beliefs whose complements contain the probabilistic modal $\triangle_n$.

4.1 Restricted Lockeanism

First, let's look at how a dynamic semantics can help give up Lockeanism and Certainty-Credence Link in a principled way: the principles will hold for non-modal claims; but not for modal claims. To achieve this solution, the semantics must be enriched to include attitude verbs like B_S and C_S, and Cr_S. So for example we will represent the claim that *S is certain that* ϕ with the sentence $C_S\phi$.

[17] The same points could also be made in what follows with a more dynamic 'update-to-test' notion of entailment, where $\phi \models \psi$ just in case for any context σ, $\sigma[\phi] \models \psi$.

[18] Yalcin 2012c; Russell and Hawthorne 2016; Starr Forthcoming. This dynamic solution contrasts interestingly with the 'no-truth-value' approach to modal claims, defended for example in Adams 1975, Edgington 1995, and Bennett 2003 §38. NTV frameworks famously respond to the original Lewis triviality results by giving up the claim that additivity applies to modal claims. We will see in what follows that dynamic solutions must accept similar consequences. However, NTV and dynamic theories reach these conclusions in different ways. According to NTV theories, modal claims function in a totally different way than ordinary, descriptive claims. This raises serious problems for understanding how claims from these two domains compositionally interact. By contrast, on the dynamic theory modal and non-modal claims traffic in the same kind of meaning: context change potentials.

To model attitude verbs in a dynamic approach, it is traditional to assign to any agent S at world w an information state σ_S^w with the same structure as a context. Then we can interpret attitude verbs as narrowing down the context to the set of worlds w where S's information state σ_S^w has some particular property.

Following a long tradition, we can model the sentence $C_S\phi$ as claiming that S's information supports ϕ. To do so, we can first define the set of worlds where S's information supports ϕ ($\mathscr{C}_S^\phi$). Then we can let the sentence $C_S\phi$ update a context by conditionalizing on this set of worlds:

Definition 4.1.

1. $\mathscr{C}_S^\phi = \{w \mid \sigma_S^w \models \phi\}$
2. $\sigma[C_S\phi] = \langle s_\sigma \cap \mathscr{C}_S^\phi, Pr_\sigma(\cdot \mid \mathscr{C}_S^\phi)\rangle$[19]

Following Beddor and Goldstein 2017, I propose a structurally analogous semantics for belief. $B_S\phi$ will again narrow down the context to the set of worlds where S's information has a structural property. But now S's information will not need to support ϕ. Rather, in a Lockean spirit S's information will need to probabilify ϕ. However, as we saw with the modal $\triangle\phi$ above, ϕ is not always associated with a set of worlds where it is true, which is assigned some probability. So instead we will say that $B_S\phi$ narrows down the context to the worlds where S's information probabilifies the result of updating with ϕ, the set of worlds that remain in her information state once she updates with ϕ ($s_{\sigma_S^w[\phi]}$). We will let $\mathscr{B}_S^\phi$ denote the set of worlds where A stands in this relation to ϕ, and then say that uttering $B_S\phi$ conditionalizes the context on this set of worlds:

Definition 4.2.

1. $\mathscr{B}_S^\phi = \{w \mid Pr_{\sigma_S^w}(s_{\sigma_S^w[\phi]}) \geq t\}$
2. $\sigma[B_S\phi] = \langle s_\sigma \cap \mathscr{B}_S^\phi, Pr_\sigma(\cdot \mid \mathscr{B}_S^\phi)\rangle$

All that's left is to give a semantics for Cr_S, the expression modeling our agent S's credence function. Here, there are two ways to proceed. We can either allow our agent to assign credence to modal claims, or not. Either way, there is a natural semantics to employ. First, suppose we have our agent only assign credence to non-modal claims. Then we can let $Cr_S\phi$ explore the probability that S currently assigns to $[\![\phi]\!]$, the set of worlds where the non-modal claim ϕ is true. Second, suppose we have agents also assign credence to modal claims. In that case, $Cr_S(\phi) = n$ can work similarly to $\triangle\phi$, by exploring the probability the agent assigns to the result of updating her current state with ϕ. Each semantics is worth exploring, and has its own set of pros and cons. The non-modal option will allow us to validate Ramsey's Thesis and Belief Transparency, but will give up Credal Ramsey's Thesis and Credal Transparency, since it doesn't assign credence to modal claims. The modal option will ultimately lead to a version of the triviality results we saw above. For now, I will focus on the non-modal option. Once we have explored this in detail, we will have the tools to think about the modal option more precisely. We will return to that towards the end of the paper.

[19] This was first proposed as a theory of *belief* by Hans Kamp, and is defended in Heim 1992; Zeevat 1992; and Willer 2013. Probabilistic versions are endorsed in Yalcin 2012b,c.

On the non-modal theory of credence, updating with the claim $Cr_S(\phi) = n$ is defined only if ϕ is associated with a set of worlds $[\![\phi]\!]$ such that updating any context σ is a matter of conditionalizing on ϕ.[20] In this case, $Cr_S(\phi) = n$ narrows down the context to the set of worlds where S's credence in $[\![\phi]\!]$ is n:

Definition 4.3.

1. $\sigma[Cr_S(\phi) = n]$ is defined only if $\exists p\ ([\![\phi]\!]) : \forall\sigma\ \sigma[\phi] = \langle s_\sigma \cap p, Pr_\sigma(\cdot \mid p)\rangle$.

   If defined, then:
2. $\mathscr{P}^n_S(\phi) = \{w \mid Pr_{\sigma^w_S}([\![\phi]\!]) = n\}$
3. $\sigma[Cr_S(\phi) = n] = \langle s_\sigma \cap \mathscr{P}^n_S(\phi), Pr_\sigma(\cdot \mid \mathscr{P}^n_S(\phi))\rangle$

This object language operator Cr_S is defined not over sets of possible worlds, but rather directly over claims in a language. In particular, it is an operator defined on the boolean, non-modal fragment of $\mathscr{L}$. On this fragment, Cr_S is a probability function, assigning each sentence in $\mathscr{L}_C$ a real number from 0 to 1 in a way that obeys the probability calculus. However, Cr_S never assigns modal claims a probability.

With this semantics in place, we can introduce sentences stating the conditional probability of ψ given ϕ using the Ratio Formula.

Definition 4.4. $Cr_S(\phi \mid \psi) = n := \exists m_1, m_2 : Cr_S(\phi \wedge \psi) = m_1 \wedge Cr_S(\phi) = m_2 \wedge n = \frac{m_1}{m_2}$

This semantics for Cr_S validates a form of Ramsey's Thesis and Belief Transparency. Whenever ϕ and ψ are non-modal, a context σ supports $C_S(\phi \to \psi)$ just in case the context supports $Cr_S(\psi \mid \phi) = 1$, and the context supports $C_S\neg(\phi \to \psi)$ if it supports $Cr_S(\psi \mid \phi) < 1$.

Fact 4.1 (Ramsey's Thesis)**.** Where ϕ and ψ are non-modal:

1. $C_S(\phi \to \psi) \dashv\models Cr_S(\psi \mid \phi) = 1$
2. $C_S\neg(\phi \to \psi) \dashv\models \neg(Pr_S(\psi \mid \phi) = 1)$.

In the same way, the current semantics preserves Belief Transparency:

Fact 4.2 (Belief Transparency for *n likely*)**.** Where ϕ is non-modal: $B_S\triangle_n\phi \dashv\models Cr_S(\phi) = n$.

Fact 4.3 (Belief Transparency for *probably*)**.** Where ϕ is non-modal: $B_S\triangle\phi \dashv\models Cr_S(\phi) > \frac{1}{2}$.

On the other hand, the semantics avoids triviality. To do so, it rejects Certainty-Credence Link and Lockeanism on the modal fragment of the language. After all, in the case where ϕ is modal $Cr_S(\phi) = n$ will not be defined. However, each principle remains valid on the non-modal fragment:

Fact 4.4 (Restricted Certainty-Credence Link)**.** Where ϕ is non-modal: $C_S\phi \dashv\models Cr_S(\phi) = 1$.

Fact 4.5 (Restricted Lockeanism)**.** Where ϕ is non-modal: $B_S\phi \dashv\models Cr_S(\phi) \geq t$.

[20] For this approach to definedness, see Heim 1983.

Since $Cr_S(\phi) = n$ is undefined for modal ϕ, this framework gives up Credal Transparency. So it can avoid the paradoxical results of that thesis.

So far, we've seen that dynamic semantics offers a way of avoiding triviality results, by modifying some of the traditional connections between belief, certainty, and credence for modal claims. Yet in itself, this solution is only so powerful. Lockeanism, Certainty-Credence Link are the sorts of claims we would like to validate. Further, the solution so far has entirely refrained from assigning credences to modal claims whatsoever. But this is somewhat surprising: one might have thought that just as one can assign full beliefs to modal claims, one can also assign partial beliefs to these claims. Finally, Credal Ramsey's Thesis and Credal Transparency were plausible principles in their own right. So if we give these principles up, it would be nice to have some explanation of why they seemed so plausible. In the next section, we will explore just such an explanation.

4.2 Probabilistic Beliefs

It turns out that the dynamic semantics we have explored provides some further interesting results that resolves the concerns from the last section. In particular, it turns out that the modal $\triangle_n$ itself can be a proxy for the operator Cr_S, when it comes to assigning credences to modal claims. While our semantics won't make sense of formulae like $Cr_S(\phi \to \psi) = n$, which are undefined, our semantics will make sense of formulae like $\triangle_n(\phi \to \psi)$ and $B_S\triangle_n(\phi \to \psi)$, which are perfectly well defined. In fact, it turns out that in this semantics, each of the principles discussed so far has a direct analogue in the logic of the operator $\triangle_n$, as a principle about how epistemic modals embed under $\triangle_n$. To see how, however, it is useful to introduce a conditional probability operator $\triangle_n(\psi \mid \phi)$, defined again using the ratio formula:

Definition 4.5. $\triangle_n(\psi \mid \phi) := \exists m_1, m_2 : \triangle_{m_1}(\phi \wedge \psi) \wedge \triangle_{m_2}\phi \wedge n = \frac{m_1}{m_2}$

In addition, it will be useful to introduce an operator $\triangle_{\geq n}$ which states that the probability of ϕ is greater than or equal to n, rather than exactly n:

Definition 4.6. $\triangle_{\geq n}\phi := \exists m : \triangle_m\phi \wedge m \geq n$

With these operators in place, we can salvage versions of Credal Transparency and Credal Ramsey's Thesis. Here, Credal Ramsey's Thesis will reappear as a principle governing the operator $\triangle_n$. We can substitute the operator C_S in the original formulation of Ramsey's Thesis with the operator $\triangle_1$, to reach the claim that it is 100% likely that $\phi \to \psi$ just in case it is 100% likely that ψ given ϕ, and otherwise it is 0% likely that $\phi \to \psi$. Here, however, we need a slight complication. Ramsey's Thesis originally ranged over any agent's information. Here, we mimic that type of quantification by quantifying over any claim χ that we could conditionalize $\triangle_n$ on:

Definition 4.7 (Embedded Ramsey's Thesis)**.** For any ϕ, ψ, χ:

1. $\triangle_1(\phi \to \psi \mid \chi) \dashv\models \triangle_1(\psi \mid \chi \wedge \phi)$
2. $\triangle_0(\phi \to \psi \mid \chi) \dashv\models \triangle_{<1}(\psi \mid \chi \wedge \phi)$.

Similarly, we replace Credal Transparency with a thesis about embedding $\triangle$ or $\triangle_n$ under $\triangle_n$ itself. Here we replace the operator B_S with the operator $\triangle_{\geq t}$. So we reach the claim that it is at least t% likely that $\triangle_n \phi$ just in case it is n likely that ϕ. Again, we mimic quantification over any agent by quantification over claims ϕ upon which we could conditionalize $\triangle_n$:

Definition 4.8 (Embedded Transparency for *n likely*)**.** For any ϕ, ψ: $\triangle_{\geq t}(\triangle_n \phi \mid \psi) \dashv\models \triangle_n(\phi \mid \psi)$.

Definition 4.9 (Embedded Transparency for *probably*)**.** For any ϕ, ψ: $\triangle_{\geq t}(\triangle \phi \mid \psi) \dashv\models \triangle(\phi \mid \psi)$.

It turns out that the semantics we explored above validates all of these principles:

Fact 4.6. Embedded Ramsey's Thesis, Embedded Transparency for *n likely* and Embedded Transparency for *probably* are all valid.

(See Appendix 1 for a discussion of the role of conditionalization in the formulation of these principles, and the extent to which it is eliminable.)

Besides their intuitive plausibility, these principles are interesting because of their consequences for belief. Each of these principles above has a direct analogue when each side of the principle is embedded under belief, as instances of single premise closure.[21] For example, we can now consider the principle that an agent believes it is 100% likely that if ϕ, then ψ just in case she believes it is 100% likely that ψ given ϕ. This principle, then, claims that $B_S \triangle_1(\phi \rightarrow \psi)$ is equivalent to $B_S \triangle_1(\psi \mid \phi)$.

Here's why principles like this are interesting. We saw in the previous section that our credence operator Cr_S was not defined for modal claims, on pain of triviality. But this left us no way of assigning anything like degrees of belief to modal claims. However, let's now make a distinction between *credence* and *probabilistic belief.*

Consider the claim that a fair die will land six. An agent can have a credence in this claim—say $\frac{1}{6}$. However, she can also have a full belief with a probabilistic content: that the die is $\frac{1}{6}$ likely to land six. In our language, letting six represent the proposition that the die will land six, we can thus distinguish between the formulae $Pr_S(\mathsf{six}) = \frac{1}{6}$ and $B_S \triangle_{\frac{1}{6}}(\mathsf{six})$. While credences cannot be directed at modal claims, we allow that probabilistic beliefs can. So instead of assigning a credence of n to a modal claim like $\phi \rightarrow \psi$, our system instead allows an agent to believe $\triangle_n(\phi \rightarrow \psi)$. That is, credence in a modal claim ϕ is replaced by a full belief about a probabilistic modal claim that operates on ϕ. In this system, it is crucial to distinguish credences in modal claims from probabilistic beliefs in modal claims.

We can now formulate each of our above principles as constraints on probabilistic belief:

Definition 4.10 (Ramsey's Probabilistic Belief Thesis)**.** For any ϕ, ψ, χ:

1. $B_S \triangle_1(\phi \rightarrow \psi \mid \chi) \dashv\models B_S \triangle_1(\psi \mid \chi \wedge \phi)$

[21] Interestingly, however, single premise closure for belief does not hold for the entire fragment. For example, while $\phi \models \triangle_1 \phi$, we do not have that $B_S \phi \models B_S \triangle_1 \phi$. Nonetheless, single premise closure holds for any ϕ and ψ that themselves have a modal operator at widest scope.

2. $B_S \triangle_0(\phi \to \psi \mid \chi) \dashv\models B_S \triangle_{<1}(\psi \mid \chi \wedge \phi)$.

Definition 4.11 (Probabilistic Belief Transparency for *n likely*)**.** For any ϕ, ψ:
$B_S \triangle_{\geq t}(\triangle_n \phi \mid \psi) \dashv\models B_S \triangle_n(\phi \mid \psi)$.

Definition 4.12 (Probabilistic Belief Transparency for *probably*)**.** For any ϕ, ψ:
$B_S \triangle_{\geq t}(\triangle \phi \mid \psi) \dashv\models B_S \triangle(\phi \mid \psi)$.

It turns out that each of these principles are valid in the semantics above:

Fact 4.7. Ramsey's Probabilistic Belief Thesis, Probabilistic Belief Transparency for *n likely* and Probabilistic Belief Transparency for *probably* are all valid.

This framework also allows us to provide analogues of Certainty-Credence Link and Lockeanism. For this semantics validates the following principles for the entire language:

Definition 4.13 (Certainty-Probabilistic Certainty Link)**.** For any ϕ:
$C_S \phi \dashv\models C_S \triangle_1 \phi$.

Definition 4.14 (Probabilistic Belief Lockeanism)**.** For any ϕ:
$B_S \phi \dashv\models B_S \triangle_{\geq t} \phi$.

Fact 4.8. Certainty-Probabilistic Certainty and Probabilistic Belief Lockeanism are both valid.

Again, Certainty-Credence Link and Lockeanism do not hold unrestrictly in this language, since credences are undefined for modal claims. But this does not mean that *probabilistic beliefs* are undefined for modal claims. So while an agent does not have a credence in $\phi \to \psi$ conforming to Lockeanism, she does have a probabilistic belief in $\phi \to \psi$ that conforms to Lockeanism. That is, she believes $\phi \to \psi$ iff she believes $\triangle_{\geq t}(\phi \to \psi)$.

We have now explored a variety of ways in which a dynamic semantics for modals can avoid the triviality results that emerge once we assign credences to modal claims. We saw two basic ways in which dynamic semantics avoids triviality results. First, we saw that dynamic semantics allows us to validate principles connecting certainty, belief, and credence in modals, once we introduce attitude verbs like C_S, B_S and Pr_S into our object language. But we could only do so by giving up Lockeanism and Certainty-Credence Link when it comes to modal claims. This was among other reasons because our operator Pr_S wasn't even defined for modal claims. To make some sense of assigning credences to modal claims, we then turned to the logical properties of $\triangle_n$. We saw that each principle about modal claims reemerged as a principle about how modal claims embed under $\triangle_n$, and then as a principle about how these more complex claims like $\triangle_n(\phi \to \psi)$ embed under B_S. That is, each of our principle about the connection between certainty, belief and credence reemerged as a connection between believing ϕ and believing $\triangle_n \phi$. We replaced connections between belief and credence with connections between belief and *probabilistic belief*, where probabilistic belief is full belief in a probabilistic modal content.

So far, things look very good for dynamic semantics as a way of avoiding triviality results. Unfortunately, the dynamic response we have examined so far has a serious problem. In the next section, we will see that the original triviality results we looked at have an analogues for our new principles. If $\triangle_n$ embeds epistemic modals in a way that satisfies Stalnaker's Thesis, Ramsey's Thesis or Transparency, then $\triangle_n$ will not satisfy the probability calculus. Similarly, if probabilistic beliefs satisfy Stalnaker's Thesis, Ramsey's Thesis or Transparency, then probabilistic beliefs will also violate the probability calculus.

5 Impossibility Results for Probabilistic Modals

In this section, I will argue that there is a serious problem for a dynamic response to triviality results. The introduction of formulas like $\triangle_n\phi$ into an object language allows for the formulation of the axioms of the probability calculus in that language. This will allow us to prove that each of our trivializing principles will entail that $\triangle_n$ violates the probability calculus. A nonfactualist can only validate Transparency at the cost of weakening the logic of numerical probability operators.

Our language now includes an operator $\triangle_n$ that models the exact probability of a claim. Now let's explore whether $\triangle_n$ itself obeys the probability calculus. First, we can ask whether for any claim ϕ, the claim *it is at least 0% likely that* ϕ is valid. Second, we can check whether for any tautology $\top$, the claim *it is 100% likely that* $\top$ is valid. Finally, we can check whether for any two inconsistent claims ϕ and ψ, the claim that it is *n likely that* $\phi \vee \psi$ is equivalent to the claim that *it is m% likely that* ϕ*, m'% likely that* ψ*, and n = m + m'*:

Definition 5.1 (Kolmogorov Axioms)**.**

1. $\models \triangle_{\geq 0}\phi$ NON-NEGATIVITY
2. $\models \triangle_1\top$ UPPER BOUND
3. If $\phi;\psi \models \bot$, then $\triangle_n(\phi \vee \psi) \dashv\models \exists m_1, m_2 : \triangle_{m_1}\phi \wedge \triangle_{m_2}\psi \wedge n = m_1 + m_2$ ADDITIVITY

For our purposes, it will be simpler to look at another consequence of the standard probability calculus: the Law of Total Probability. This principle says that the probability of any claim is a weighted sum of its probability conditional on each cell of a partition, weighted by the prior probability of that cell. As a claim about the logical properties of $\triangle_n$, the Law of Total Probability says:

Definition 5.2 (LTP)**.** $\triangle_n\phi \dashv\models \exists m_1, m_2, m_3, m_4 : \triangle_{m_1}\psi \wedge \triangle_{m_2}(\phi \mid \psi) \wedge \triangle_{m_3}\neg\psi \wedge \triangle_{m_4}(\phi \mid \neg\psi) \wedge n = m_1 \times m_2 + m_3 \times m_4$

LTP follows from Additivity given two assumptions:

Assumption 5.1. $\phi \dashv\models (\phi \wedge \psi) \vee (\phi \wedge \neg\psi)$

Assumption 5.2. $\phi \wedge \psi; \phi \wedge \neg\psi \models \bot$

Fact 5.1. Given Assumptions 5.1-5.2, the Kolmogorov Axioms imply the Law of Total Probability.

We now have the power to prove some strong results. We first saw that Credal Ramsey's Thesis and Credal Transparency lead to triviality results. Now we will see that analogous results apply to Embedded Ramsey's Thesis and Embedded Transparency. Let's start with Embedded Ramsey's Thesis. Again, this principle said that $\triangle_1(\phi \to \psi \mid \chi)$ is equivalent to $\triangle_1(\psi \mid \chi \wedge \phi)$, and that $\triangle_0(\phi \to \psi \mid \chi)$ is equivalent to $\triangle_{<1}(\psi \mid \chi \wedge \phi)$.

Unfortunately, it turns out that this principle is incompatible with $\triangle_n$ satisfying the Law of Total Probability. Together, they imply that every claim is either 100% or 0% likely:

Fact 5.2. Embedded Ramsey's Thesis and the Law of Total Probability jointly imply: $\models \triangle_1\phi \vee \triangle_0\phi$.

Fact 5.2 shows that if we validate an object language version of Ramsey's Thesis, then $\triangle_n$ can only behave probabilistically on pain of triviality. Together, these two assumptions imply that the probability of any claim is either 1 or 0.

We reach an analogous problem for our Embedded Transparency principles:

Fact 5.3. Embedded Transparency for *n likely* and the Law of Total Probability jointly imply: $\models \triangle_1\phi \vee \triangle_0\phi$

Fact 5.4. Embedded Transparency for *probably* and the Law of Total Probability jointly imply: $\models \triangle_{\leq \frac{1}{2}}\phi \vee \triangle_{\geq t}\phi$

Each of these results has a common structure. In each case, we start with our earlier triviality result for a principle involving credences. It turns out that each triviality result involves some application of the Law of Total Probability. So we can simply consider that application when it comes to the operator $\triangle_n$. If $\triangle_n$ itself obeys the Law of Total Probability, we can apply the analogous partition to reach the problem we started with. But instead of having a problem about assigning credence to a claim, we instead have a problem about how to embed that claim under the operator $\triangle_n$.

Let's now consider exactly how the semantics above engages with these results. The dynamic semantics we considered above validates each embedding principle. But it does not lead to triviality. So we know it must give up the Law of Total Probability. It turns out that the semantics gives up this principle in a somewhat strange way. First, it turns out that in the dynamic semantics above, the Law of Total Probability doesn't quite follow from the Kolmogorov Axioms. For in that system, ϕ is not logically equivalent to $(\phi \wedge \triangle_n\psi) \vee (\phi \wedge \neg\triangle_n\psi)$.[22] Nonetheless, the semantics invalidates not only the Law of Total Probability, but also Additivity. In addition, the semantics invalidate a weaker principle of *comparative entailment*, that whenever ϕ entails ψ, ψ is at least as likely as ϕ:

Definition 5.3. If $\phi \models \psi$, then $\triangle_n\phi \models \triangle_{\geq n}\psi$ COMPARATIVE ENTAILMENT

[22]Thanks to Jeremy Goodman for help here.

While Comparative Entailment, Additivity, and the Law of Total Probability hold on the non-modal fragment, they fail when $\triangle_n$ embeds epistemic modal claims.

Fact 5.5.

1. $\models \triangle_{\geq 0}\phi$ NON-NEGATIVITY
2. $\models \triangle_1 \top$ UPPER BOUND
3. $\phi; \psi \models \bot \not\Longrightarrow \triangle_n(\phi \vee \psi) \mathrel{=\!\mid\!\models} \exists m_1, m_2 : \triangle_{m_1}\phi \wedge \triangle_{m_2}\psi \wedge n = m_1 + m_2$ ~~ADDITIVITY~~
4. $\phi \models \psi \not\Longrightarrow \triangle_n \phi \models \triangle_{\geq n}\psi$ ~~COMPARATIVE ENTAILMENT~~

One open question from this, then, is whether it is possible to invalidate the Law of Total Probability while retaining Additivity or Comparative Entailment.

5.1 Probabilistic Belief

In the previous section, we saw that on pain of triviality, $\triangle_n$ cannot both be probabilistic, and embed epistemic modals in a transparent way. In this section, we will see that an analogous result extends to our principles about probabilistic belief, for example that an agent believes $\triangle_1(\phi \to \psi)$ iff she believes $\triangle_1(\psi \mid \phi)$, and otherwise believes $\triangle_0(\phi \to \psi)$. For one might have thought that probabilistic belief, just like credence, satisfies an analogue of the Kolmogorov Axioms. In particular:

Definition 5.4 (Kolmogorov Axioms for Probabilistic Belief)**.**

1. $\models B_A \triangle_{\geq 0}\phi$ NON-NEGATIVITY
2. $\models B_A \triangle_1 \top$ UPPER BOUND
3. If $\phi; \psi \models \bot$, then $B_A \triangle_n \phi \mathrel{=\!\mid\!\models} \exists m_1, m_2 : B_A \triangle_{m_1}\phi \wedge B_A \triangle_{m_2}\psi \wedge n = m_1 + m_2$ ADDITIVITY

Similarly, we can consider an analogue of the Law of Total Probability for probabilistic beliefs:

Definition 5.5 (LTP for Probabilistic Belief)**.** $B_A \triangle_n \phi \mathrel{=\!\mid\!\models} \exists m_1, m_2, m_3, m_4 : B_A \triangle_{m_1}\psi \wedge B_A \triangle_{m_2}(\phi \mid \psi) \wedge B_A \triangle_{m_3}\neg\psi \wedge B_A \triangle_{m_4}(\phi \mid \neg\psi) \wedge n = m_1 \times m_2 + m_3 \times m_4$

It turns out that our earlier triviality results about credences have direct parallels for probabilistic beliefs. In each case, we can show that if the Law of Total Probability for Probabilistic Belief holds, then each of Probabilistic Belief Stalnaker's Thesis, Probabilistic Belief Ramsey's Thesis and Probabilistic Belief Transparency lead to trivializing consequences:

Fact 5.6. Probabilistic Belief Ramsey's Thesis and the Law of Total Probability for Probabilistic Belief jointly imply: $\models B_S \triangle_1 \phi \vee B_S \triangle_0 \phi$.

Fact 5.7. Probabilistic Belief Transparency for *n likely* and the Law of Total Probability for Probabilistic Belief jointly imply: $\models B_S \triangle_1 \phi \vee B_S \triangle_0 \phi$

Fact 5.8. Probabilistic Belief Transparency for *probably* and the Law of Total Probability for Probabilistic Belief jointly imply: $\models B_S \triangle_{\leq \frac{1}{2}} \phi \vee B_S \triangle_{\geq t} \phi$

5.2 Modal Credence

When we first introduced a semantics for Cr_S, we restricted it to non-modal claims. Then we supplied a surrogate of Cr_S for modal claims: probabilistic belief. Let's now consider how our debate would look if we allowed Cr_S to embed modal claims directly.

In this case, it would be natural to interpret Cr_S with a structure analogous to $\triangle$ above. Cr_S could explore the probability that the agent currently assigns to $s_{\sigma_S^w[\phi]}$, the set of worlds that result from taking the agent's current certainties and updating them with ϕ. This generates the following definition:

Definition 5.6.

1. $\mathscr{P}_S^n(\phi) = \{w \mid Pr_{\sigma_S^w}(s_{\sigma_S^w[\phi]}) = n\}$
2. $\sigma[Cr_S(\phi) = n] = \langle s_\sigma \cap \mathscr{P}_S^n(\phi), Pr_\sigma(\cdot \mid \mathscr{P}_S^n(\phi))\rangle$

We are now in a good position to assess of the pros and cons of this kind of semantics. It turns out that on that semantics, credence in modal claims will be equivalent to probabilistic belief in modal claims $Cr_S(\phi) = n$ will be equivalent to $B_S\triangle_n\phi$:

Fact 5.9. Assume Definition 5.6. Then: $Cr_S(\phi) = n \mathrel{=}\!\!\mid\!\mid\!\!\models B_S\triangle_n\phi$.

It follows immediately that our modal-embedding semantics for Cr_S inherits all the pros and cons of the theory of probabilistic belief above. Credal Ramsey's Thesis and Credal Transparency will both hold. For example, the semantics will validate $Cr_S(\triangle_n\phi) \geq t \mathrel{=}\!\!\mid\!\mid\!\!\models Cr_S(\phi) = n$. But the cost of this proposal is that Cr_S will not obey the Kolmogorov Axioms. The Law of Total Probability will fail when applied to modal claims.

Summing up, we have a choice point when it comes to the interaction of credence and modals. Our theory can either assign credences to epistemic modal claims, or not. If the theory does assign credences to epistemic modal claims, the impossibility results above will highly constrain how these credences are assigned. These credences cannot obey Credal Ramsey's Thesis or Credal Transparency while remaining probabilistic. Alternatively, we can restrict the assignment of credence to non-modal claims. In that case, we will be able to retain Ramsey's Thesis and Belief Transparency; but we will not be able to accept the credal versions of these claims.

6 Conclusion

In this paper, we have seen that a variety of principles connecting certainty, belief, and credence lead to trivializing consequences. Then we considered one recently popular line of response to these sorts of triviality results—dynamic semantics. We saw that dynamic semantics offered two strategies for handling these triviality results. The first response was to reject Certainty-Credence Link and Lockeanism for modal claims, but retain them for non-modal claims. This allowed the framework to validate Belief Transparency and Ramsey's Thesis without accepting Credal Transparency and Credal Ramsey's Thesis. The second response was to accept all these principles not as connections between certainty, belief, and credence; but rather as principles connecting certainty, belief, and what I have called *probabilistic beliefs*—beliefs whose content contain the probabilistic modal $\triangle_n$. Unfortunately, we saw that this second sort of response led to problems of its own. It turned out that validating these principles about probabilistic belief requires giving up the thesis that either $\triangle_n$ or probabilistic belief obey the traditional principles of the probability calculus.

Perhaps this is not so surprising. In Lewis 1976's original paper on triviality results, Lewis considered whether giving up an assignment of truth values to conditionals might allow one to retain Stalnaker's Thesis:

> Merely to deny that probabilities of conditionals are probabilities of truth, while retaining all the standard laws of probability in suitably adapted form, would not yet make it safe to revive the thesis that probability of conditionals are conditional probabilities...whoever still wants to say [this] had better also employ a nonstandard calculus of "probabilities"...he might limit the law of additivity, refusing to apply it when the disjuncts A and B contain conditional conjuncts...But if it be granted that the "probabilities" of conditionals do not obey the standard laws, I do not see what is to be gained by insisting on calling them "probabilities".[23]

We have now seen this dialectic play out in a new domain - the logic of natural language probabilistic modals. Here, perhaps there is a significant difference. Since $\triangle_n$ is the natural language expression *probably*, it seems perfectly fair to call it a "probability".

All of this leaves open a further question. Is it possible to retain the first solution to the triviality results offered by dynamic semantics, without the second? That is, is there some semantics for epistemic modals and attitude verbs that validates Ramsey's Thesis and Belief Transparency, rejects Certainty-Credence Link and Lockeanism for modal claims, retains these principles for non-modal claims, and avoids triviality, without in addition validating the analogue of Embedded Transparency, Embedded Ramsey's Thesis, Probabilistic Belief Transparency, or Probabilistic Ramsey's Thesis, while preserving the validity of the Law of Total Probability and the Kolmogorov Axioms?[24]

This paper has focused on a particular group of principles about conditionals and probabilistic modals: Ramsey's Thesis, and Transparency principles. But we have not explored whether dynamic semantics offers a comparable solution to the problems with Stalnaker's Thesis. Starr Forthcoming has done just this, giving a dynamic semantics for conditionals that validates a version of Stalnaker's Thesis. In an appendix, I will review this work and then show that it faces analogous impossibility results to those we have already seen. Although dynamic semantics can rescue Stalnaker's Thesis, it does so at the cost of giving up the probabilistic nature of $\triangle_n$.

[23] Lewis 1976 304

[24] Thanks to David Black, Jeremy Goodman, Julia Staffel, Thony Gillies, Ernie Lepore, Andy Egan, Robert Beddor, Barteld Kooi, and the audience at FEW 2016.

Appendix 1: Propriety

The formulation of our Embedded Transparency and Ramsey principles has crucially relied on conditionalization. A simpler version of these principles goes as follows:

Definition 6.1 (Simplified Ramsey's Thesis)**.** For any ϕ, ψ:

- $\triangle_1(\phi \rightarrow \psi) \dashv\models \triangle_1(\psi \mid \phi)$
- $\triangle_0(\phi \rightarrow \psi) \dashv\models \triangle_{<1}(\psi \mid \phi)$

Definition 6.2 (Simplified Transparency for *n likely*)**.** For any ϕ:
$\triangle_{\geq t}\triangle_n\phi \dashv\models \triangle_n\phi$.

Definition 6.3 (Simplified Transparency for *probably*)**.** For any ϕ:
$\triangle_{\geq t}\triangle\phi \dashv\models \triangle\phi$.

This simplified principles are consequences of the embedded principles, since they follow from the original ones when we conditionalize on $\top$. The more interesting question is whether the embedded principles follow from the simplified principle. Given certain assumptions, they do follow. In particular, say that $\triangle_n$ is *proper* when $\triangle_n(\psi \mid \phi)$ is supported in a context σ just in case $\triangle_n\psi$ is supported in $\sigma[\phi]$:

Definition 6.4. $\triangle_n$ is *proper* iff:
$\sigma \models \triangle_n(\psi|\phi)$ iff $\sigma[\phi] \models \triangle_n\psi$.

In proper contexts, conditional probability claims are equivalent to updated probability claims. It turns out that the dynamic semantics we explored above does make $\triangle_n$ proper, since it defines conditional probability using $\wedge$, and $\wedge$ is defined dynamically using a sequencing operation.

Fact 6.1. $\triangle_n$ is proper.

Proof. $\sigma \models \triangle_n(\psi \mid \phi)$ iff $\sigma \models \frac{\triangle_m(\phi\wedge\psi)}{\triangle_{m'}(\phi)} \wedge n = \frac{m}{m'}$ iff $n = \frac{Pr_\sigma(s_{\sigma[\phi\wedge\psi]})}{Pr_\sigma(s_{\sigma[\phi]})}$ iff $n = \frac{Pr_{\sigma[\phi]}(s_{\sigma[\psi]})}{Pr_{\sigma[\phi]}(s_\sigma)}$ iff $n = Pr_{\sigma[\phi]}(s_{\sigma[\psi]})$ iff $\sigma[\phi] \models \triangle_n\psi$. □

Whenever $\triangle_n$ is proper, we can move freely between the simplified and Embedded forms of Ramsey's Thesis and Transparency. For whenever $\triangle_n$ is proper, a conditional likelihood claim expressed in one context is equivalent to an unconditional likelihood claim expressed in another context:

Fact 6.2. If $\triangle_n$ is proper, then:

- Embedded and Simplified Ramsey's Thesis are equivalent.
- Embedded and Simplified Transparency are equivalent.

Proof: Ramsey's Thesis. Suppose $\triangle_n$ is proper. The left to right direction is trivial, letting $\chi = \top$. For the right to left, suppose Simplified Ramsey's Thesis holds. Now take an arbitrary σ, ϕ, ψ, χ. We must show that $\sigma \models \triangle_1(\phi \rightarrow \psi \mid \chi)$ iff $\sigma \models \triangle_1(\psi \mid \chi \wedge \phi)$. From Simplified Ramsey's Thesis, we know that $\sigma[\chi] \models \triangle_1(\phi \rightarrow \psi)$ iff $\sigma[\chi] \models \triangle_1(\psi \mid \phi)$. By the propriety of $\triangle_n$, this entails that $\sigma \models \triangle_1(\phi \rightarrow \psi \mid \chi)$ iff $\sigma \models \triangle_1(\psi \mid \chi \wedge \phi)$. For the second clause of Simplified Ramsey's Thesis the proof is analogous. □

Proof: Transparency. Suppose $\triangle_n$ is proper. The left to right direction is trivial, letting $\chi = \top$. For the right to left, suppose Simplified Embedded Transparency for *n likely* holds. Now take an arbitrary σ, ϕ, ψ, χ. We must show that $\sigma \models \triangle_{\geq t}(\triangle_n\phi \mid \psi)$ iff $\sigma \models \triangle_n(\phi \mid \psi)$. From Simplified Transparency, we know that $\sigma[\psi] \models \triangle_{\geq t}\triangle_n\phi$ iff $\sigma[\psi] \models \triangle_n\phi$. By the propriety of $\triangle_n$, this entails that $\sigma \models \triangle_{\geq t}(\triangle_n\phi \mid \psi)$ iff $\sigma \models \triangle_n(\phi \mid \psi)$. For *probably*, the proof is analogous. □

We have seen that the Embedded principles lead to a variety of paradoxical results. One way to avoid these results may be to give up propriety, accept the simplified form of the principles, and reject the embedded form.[25]

Appendix 2: Triviality Results

Fact 2.2. If Credal Ramsey's Thesis holds, then $Pr(\phi) = 1$ or $Pr(\phi) = 0$ for every Pr, ϕ.

Proof. This follows from a more general triviality result in Gardenförs 1988. Here's an independent proof. The proof relies on the Law of Total Probability, which says that the probability of any claim is a weighted sum of its probability conditional on each cell of a partition, weighted by the prior probability of that cell of the partition:

Definition 6.5 (LTP)**.** If $\gamma_1 ... \gamma_n$ is a partition, then:
$Pr(\phi) = Pr(\gamma_1) \times Pr(\phi|\gamma_1) + ... + Pr(\gamma_n) \times Pr(\phi|\gamma_n)$

Here, the key is to apply the Law of Total Probability to the partition of the consequent of the conditional and its negation. Letting $Pr_\phi(\cdot)$ abbreviate $Pr(\cdot \mid \phi)$:

1. $Pr(\phi \to \psi) = Pr(\psi) \times Pr(\phi \to \psi \mid \psi) + Pr(\neg\psi) \times Pr(\phi \to \psi \mid \neg\psi)$ LTP
2. $Pr(\phi \to \psi) = Pr(\psi) \times Pr_\psi(\phi \to \psi) + Pr(\neg\psi) \times Pr_{\neg\psi}(\phi \to \psi)$ Pr_ϕ
3. $Pr_\psi(\psi \mid \phi) = 1$ & $Pr_{\neg\psi}(\psi \mid \phi) = 0$ Pr_ϕ
4. $Pr_\psi(\phi \to \psi) = 1$ & $Pr_{\neg\psi}(\phi \to \psi) = 0$ 3, Ramsey Test
5. $Pr(\phi \to \psi) = Pr(\psi) \times 1 + Pr(\neg\psi) \times 0$ 2, 4
6. $Pr(\phi \to \psi) = Pr(\psi)$ 5
7. $Pr(\psi) = 1$ or $Pr(\psi) = 0$ 6, Credal Ramsey's Thesis

□

Fact 2.3. If Credal Transparency holds for *n likely*, then $Pr(\phi) = 1$ or $Pr(\phi) = 0$ for any Pr, ϕ.

Proof. Here the key is to consider the claim $\triangle_n\phi$ and apply the Law of Total Probability to the partition $\{\phi, \neg\phi\}$. Take any n such that $0 < n < 1$:

[25] For example, see Bacon 2015 for a semantics for $\to$ that validates something like a simplified form of Stalnaker's Thesis, but not the Embedded form. See Russell and Hawthorne 2016 §6 for a semantics that validates something structurally similar to Transparency in its Simplified but not its Embedded form.

1. $Pr(\triangle_n\phi) = Pr(\phi) \times Pr(\triangle_n\phi \mid \phi) + Pr(\neg\phi) \times Pr(\triangle_n\phi \mid \neg\phi)$ LTP
2. $Pr(\triangle_n\phi) = Pr(\phi) \times Pr_\phi(\triangle_n\phi) + Pr(\neg\phi) \times Pr_{\neg\phi}(\triangle_n\phi)$ Pr_ϕ
3. $Pr_\phi(\phi) = 1$ & $Pr_{\neg\phi}(\phi) = 0$ Pr_ϕ
4. $Pr_\phi(\triangle_n\phi) < t$ & $Pr_{\neg\phi}(\triangle_n\phi) < t$ 3, Credal Transparency
5. $Pr(\triangle_n\phi) < Pr(\phi) \times t + Pr(\neg\phi) \times t$ 2, 4
6. $Pr(\triangle_n\phi) < t$ 5
7. $Pr(\phi) \neq n$ 6, Credal Transparency

□

Fact 2.4. If Credal Transparency holds for *probably*, then $Pr(\phi) < \frac{1}{2}$ or $Pr(\phi) \geq t$ for any Pr, ϕ.

Proof. To begin, I will show that Credal Transparency implies that for any claim ϕ, the probability of $\triangle\phi \vee \neg\phi$ is 1. This by itself is surprising, for in general only logical truths receive a probability of 1 on every probability function. Yet transparency theorists have not in general given a semantics for $\triangle$ on which $\triangle\phi \vee \neg\phi$ is a logical truth. With this intermediate result, I will go on to prove a stronger triviality result. Given the intermediate result, we will see that $Pr(\triangle\phi \mid \phi) = 1$. Using this, we will see that the probability of any claim in the language must be either $\frac{1}{2}$ or less, or greater than the Lockean threshold for belief.

First, I will show that any probability function satisfying Credal Transparency also satisfies the following equation: $Pr(\triangle\phi \vee \neg\phi) \geq \frac{1}{2}$.

1. $Pr(\phi) < \frac{1}{2} \vee Pr(\phi) \geq \frac{1}{2}$
2. Suppose $Pr(\phi) < \frac{1}{2}$
3. $Pr(\neg\phi) \geq \frac{1}{2}$ 2
4. $Pr(\triangle\phi \vee \neg\phi) \geq \frac{1}{2}$ 3, Additivity
5. Suppose $Pr(\phi) \geq \frac{1}{2}$
6. $Pr(\triangle\phi) \geq t$ Credal Transparency
7. $Pr(\triangle\phi) \geq \frac{1}{2}$ $t \geq \frac{1}{2}$
8. $Pr(\triangle\phi \vee \neg\phi) \geq \frac{1}{2}$ 7, Additivity
9. $Pr(\triangle\phi \vee \neg\phi) \geq \frac{1}{2}$ 1, 2-4, 5-8

Now say that a class of probability functions PR is *closed under conditionalization* just in case whenever $Pr \in PR$ and $Pr(\cdot \mid \phi)$ is defined, $Pr(\cdot \mid \phi) \in PR$:

Definition 6.6. PR is *closed under conditionalization* iff:
for any ϕ and any $Pr \in PR$, if $Pr(\cdot \mid \phi)$ is defined then $Pr(\cdot \mid \phi) \in PR$.

Whenever PR is closed under conditionalization and every probability function probabilifies ϕ above some threshold i greater than 0, every probability function in PR assigns ϕ a probability of 1. For suppose not. Then for some PR where $\forall Pr \in PR\ Pr(\phi) \geq i > 0$, PR contains a probability function Pr where $1 > Pr(\phi) \geq i > 0$. But in this case $Pr(\neg\phi) > 0$ and so Pr is defined. But $Pr(\phi \mid \neg\phi) = 0 < i$. Since PR

is closed under conditionalization, $Pr(\cdot \mid \neg\phi) \in PR$. So PR contains a probability function that assigns ϕ a value less than i. $\perp$.[26]

To complete our proof, we need one more assumption: that the set of rational credence functions is closed under conditionalization. In fact, we need only the weaker claim that the class of rational credence functions is closed under conditionalization on $\triangle\phi \vee \neg\phi$. With this last assumption, we reach the conclusion that if Credal Transparency holds, then for any claim ϕ and any probability function Pr, $Pr(\triangle\phi \vee \neg\phi) = 1$.

It follows immediately from this that $Pr(\triangle\phi \mid \phi) = 1$ wherever it is defined. Using this last fact[27], we can prove Fact 2.3. First, we will see that $Pr(\triangle\phi) = Pr(\phi)$:

1. $Pr(\triangle\phi) = Pr(\phi) \times Pr(\triangle\phi \mid \phi) + Pr(\neg\phi) \times Pr(\triangle\phi \mid \neg\phi)$ LTP
2. $Pr(\triangle\phi \mid \phi) = 1$ & $Pr(\triangle\phi \mid \neg\phi) = 0$
3. $Pr(\triangle\phi) = Pr(\phi) \times 1 + Pr(\neg\phi) \times 0$ 1, 2
4. $Pr(\triangle\phi) = Pr(\phi)$ 3

Now applying Credal Transparency again, we know that whenever $Pr(\phi) > \frac{1}{2}$, $Pr(\triangle\phi) \geq t$. Thus, whenever $Pr(\phi) > \frac{1}{2}$, $Pr(\phi) \geq t$. But ϕ is arbitrary. So no claim in the language has a probability greater than $\frac{1}{2}$ and less than t. □

One reasonable response to the proof above might be to deny the thesis that the set of rational credence functions is closed under conditionalization on claims of the form $\triangle\phi \vee \neg\phi$. After all, modal claims of this form are complex; perhaps these simply cannot be learned in the ordinary way.[28] This strategy is reasonable, and is related to other attempts to deflect triviality results (see Lewis 1986 and Hájek and Hall 1994). Nonetheless, this strategy would still leave one with the requirement that any rational agent's credence in $\triangle\phi \vee \neg\phi$ be greater than $\frac{1}{2}$. In addition, one would still need a different strategy to deal with related theses such as Belief Transparency for *n likely*.

Appendix 3: Impossibility Results

Fact 5.2. Embedded Ramsey's Thesis and the Law of Total Probability jointly imply: $\models \triangle_1\phi \vee \triangle_0\phi$

Proof. The proof is analogous to one above.

1. $\triangle_n(\phi \to \psi) \mathrel{=\!\mid\models} \triangle_m\psi \wedge \triangle_{m'}(\phi \to \psi \mid \psi) \wedge \triangle_l\neg\psi \wedge \triangle_{l'}(\phi \to \psi \mid \neg\psi) \wedge n = m \times m' + l \times l'$ LTP
2. $\models \triangle_1(\psi \mid \psi \wedge \phi) \wedge \triangle_0(\psi \mid \neg\psi \wedge \phi)$
3. $\models \triangle_1(\phi \to \psi \mid \psi) \wedge \triangle_0(\phi \to \psi \mid \neg\psi)$ 2, Embedded Ramsey's Thesis

[26] Thanks to David Black for help here.

[27] Charlow 2015 exploits a structurally similar strategy to prove some triviality results for conditionals. Charlow 2015's strategy is to find operators $O(\phi)$ such that $Pr(O(\phi)|\phi) = 1$.

[28] Thanks to an anonymous referee for suggesting this strategy.

4. $\triangle_n(\phi \to \psi) \dashv\models \triangle_m\psi \wedge \triangle_1(\phi \to \psi \mid \psi) \wedge \triangle_l\neg\psi \wedge \triangle_0(\phi \to \psi \mid \neg\psi) \wedge n = m \times 1 + l \times 0$ 1, 3
5. $\triangle_n(\phi \to \psi) \dashv\models \triangle_n\psi$ 4
6. $\models \triangle_1\phi \vee \triangle_0\phi$ 5, Embedded Ramsey's Thesis

□

Fact 5.3. Embedded Transparency for *n likely* and the Law of Total Probability jointly imply: $\models \triangle_1\phi \vee \triangle_0\phi$

Proof. The proof is analogous to one above. Consider some n where $0 < n < 1$:

1. $\triangle_n(\triangle_s\phi) \dashv\models \triangle_m\phi \wedge \triangle_{m'}(\triangle_s\phi \mid \phi) \wedge \triangle_l\neg\psi \wedge \triangle_{l'}(\triangle_s\phi \mid \neg\phi) \wedge n = m \times m' + l \times l'$ LTP
2. $\models \triangle_{<t}(\triangle_n\phi \mid \phi) \wedge \triangle_{<t}(\triangle_n\phi \mid \neg\phi)$ Embedded Transparency
3. $\models \triangle_n\phi \wedge \triangle_m\neg\phi \wedge n + m = 1$ Additivity, Upper Bound
4. $\models \triangle_{<t}\triangle_n\phi$ 1-3
5. $\models \neg\triangle_n\phi$ 4, Embedded Transparency

□

Fact 5.4. Embedded Transparency for *probably* and the Law of Total Probability jointly imply: $\models \triangle_{\leq\frac{1}{2}}\phi \vee \triangle_{\geq t}\phi$

Proof. The proof is analogous to the one for Fact 2.3. □

Fact 5.6. Probabilistic Belief Ramsey's Thesis and the Law of Total Probability for Probabilistic Belief jointly imply: $\models B_S\triangle_1\phi \vee B_S\triangle_0\phi$.

Fact 5.7. Probabilistic Belief Transparency for *n likely* and the Law of Total Probability for Probabilistic Belief jointly imply: $\models B_S\triangle_1\phi \vee B_S\triangle_0\phi$

Fact 5.8. Probabilistic Belief Transparency for *probably* and the Law of Total Probability for Probabilistic Belief jointly imply: $\models B_S\triangle_{\leq\frac{1}{2}}\phi \vee B_S\triangle_{\geq t}\phi$

Proof. The proofs are analogous to Facts 5.2-5, in each case prefixing the operator B_S to the outermost occurrence of $\triangle_n$ in each formula. □

Appendix 4: Properties of the Semantics

Fact 4.1 (Ramsey's Thesis)**.** Where ϕ is non-modal:

1. $C_S(\phi \to \psi) \dashv\models Cr_S(\psi \mid \phi) = 1$
2. $\neg C_S(\phi \to \psi) \dashv\models \neg(Cr_S(\psi \mid \phi) = 1)$.

Proof. For the first clause: $\sigma \models C_S(\phi \to \psi)$ iff for every $w \in s_\sigma$ $\sigma_S^w \models \phi \to \psi$ iff for every $w \in s_\sigma$ $\sigma_S^w[\phi] \models \psi$ iff for every $w \in s_\sigma$ $Pr_{\sigma_S^w}(\llbracket\psi\rrbracket \mid \llbracket\phi\rrbracket) = 1$. For the second clause: $\sigma \models C_S\neg(\phi \to \psi)$ iff for every $w \in s_\sigma$ $\sigma_S^w \not\models \phi \to \psi$ iff for every $w \in s_\sigma$ $\sigma_S^w[\phi] \not\models \psi$ iff for every $w \in s_\sigma$ $\sigma_S^w[\phi] \not\models \psi$ iff for every $w \in s_\sigma$ $Pr_{\sigma_S^w}(\llbracket\psi\rrbracket \mid \llbracket\phi\rrbracket) < 1$. □

Fact 4.2 (Belief Transparency for *n likely*)**.** Where ϕ is non-modal: $B_S \triangle_n \phi \dashv\models Cr_S(\phi) = n$.

Proof. $\sigma \models B_S \triangle_n \phi$ iff for every $w \in s_\sigma$ $Pr_{\sigma^w_S}(s_{\sigma^w_A[\triangle_n \phi]}) \geq t$ iff for every $w \in s_\sigma$ $\sigma^w_S \models \triangle_n \phi$ iff for every $w \in s_\sigma$ $Pr_{\sigma^w_S}([\![\phi]\!]) = n$. ☐

Fact 4.3 (Belief Transparency for *probably*)**.** Where ϕ is non-modal: $B_S \triangle \phi \dashv\models Cr_S(\phi) \geq t$.

Proof. $\sigma \models B_S \triangle \phi$ iff for every $w \in s_\sigma$ $Pr_{\sigma^w_S}(s_{\sigma^w_A[\triangle \phi]}) \geq t$ iff for every $w \in s_\sigma$ $\sigma^w_S \models \triangle \phi$ iff for every $w \in s_\sigma$ $Pr_{\sigma^w_S}([\![\phi]\!]) > \frac{1}{2}$. ☐

Fact 4.4 (Restricted Certainty-Credence Link)**.** When ϕ is non-modal: $C_S \phi \dashv\models Cr_S(\phi) = 1$.

Proof. $\sigma \models C_S \phi$ iff for every $w \in s_\sigma$ $\sigma^w_S \models \phi$ iff for every $w \in s_\sigma$ $Pr_{\sigma^w_S}([\![\phi]\!]) = 1$. ☐

Fact 4.5 (Restricted Lockeanism)**.** When ϕ is non-modal: $B_S \phi \dashv\models Cr_S(\phi) \geq t$.

Proof. $\sigma \models B_S \phi$ iff for every $w \in s_\sigma$ $Pr_{\sigma^w_S}(s_{\sigma^w_S[\phi]}) \geq t$ iff for every $w \in s_\sigma$ $Pr_{\sigma^w_S}([\![\phi]\!]) \geq t$. ☐

Fact 4.6. Embedded Ramsey's Thesis, Embedded Transparency for *n likely* and Embedded Transparency for *probably* are all valid.

Proof. Here we can focus on the simplified versions of the relevant theses (avoiding any mention of conditionalization) since we saw in appendix 1 that our semantics satisfies propriety. So let's start with the simplified form of Ramsey's Thesis: $\triangle_1(\phi \to \psi) \dashv\models \triangle_1(\psi \mid \phi)$ iff $\forall \sigma$ $\sigma \models \triangle_1(\phi \to \psi)$ iff $\sigma \models \triangle_1(\psi \mid \phi)$. $\sigma \models \triangle_1(\phi \to \psi)$ iff $Pr_\sigma(s_{\sigma[\phi \to \psi]}) = 1$ iff $\sigma[\phi \to \psi] = \sigma$ iff $\sigma[\phi] \models \psi$ iff $\frac{Pr_\sigma(s_{\sigma[\phi \wedge \psi]})}{Pr_\sigma(s_{\sigma[\phi]})} = 1$ iff $\sigma \models \triangle_1(\psi \mid \phi)$.

For simplified Transparency, let's consider $\triangle_n$. $\triangle_{\geq t}(\triangle_n \phi \mid \psi) \dashv\models \triangle_n(\phi \mid \psi)$ iff $\forall \sigma$ $\sigma \models \triangle_{\geq t}(\triangle_n \phi \mid \psi)$ iff $\sigma \models \triangle_n(\phi \mid \psi)$. $\sigma \models \triangle_{\geq t}(\triangle_n \phi \mid \psi)$ iff $Pr_{\sigma[\psi]}(s_{\sigma[\psi][\triangle_n \phi]}) \geq t$ iff $\sigma[\psi] \models \triangle_n \phi$ iff $Pr_{\sigma[\psi]}(s_{\sigma[\psi][\phi]}) = n$ iff $\sigma \models \triangle_n(\phi \mid \psi)$. The proof for $\triangle$ is analogous. ☐

Fact 4.7. Ramsey's Probabilistic Belief Thesis, Probabilistic Belief Transparency for *n likely* and Probabilistic Belief Transparency for *probably* are all valid.

Fact 4.8. Certainty-Probabilistic Certainty and Probabilistic Belief Lockeanism are both valid.

Fact 5.5. The semantics in Yalcin 2012b validates:

1. $\models \triangle_{\geq 0} \phi$ NON-NEGATIVITY
2. $\models \triangle_1 \top$ UPPER BOUND
3. $\phi; \psi \models \bot \not\Longrightarrow \triangle_n(\phi \vee \psi) \dashv\models \exists m_1, m_2 : \triangle_{m_1} \phi \wedge \triangle_{m_2} \psi \wedge n = m_1 + m_2$ ~~ADDITIVITY~~

4. $\phi \models \psi \not\Longrightarrow \triangle_n\phi \models \triangle_{\geq n}\psi$ ~~COMPARATIVE ENTAILMENT~~

Proof.

1. NON-NEGATIVITY: Follows from Pr being a probability function, plus $\sigma \models \triangle_n\phi$ iff $Pr_\sigma(s_{\sigma[\phi]}) = n$.
2. UPPER BOUND: $\top \models \phi$ iff $\forall\sigma\ \sigma[\phi] = \sigma$ iff $\forall\sigma\ Pr_\sigma(s_{\sigma[\phi]}) = 1$ iff $\forall\sigma\ \sigma \models \triangle_1\phi$.
3. ADDITIVITY: $\mathsf{B}; \neg(\mathsf{A} \to \mathsf{B}) \models \bot$. Suppose that $Pr_\sigma([\![\mathsf{B}]\!]) = \frac{1}{4}$ and $Pr_\sigma([\![\mathsf{B}]\!]|[\![\mathsf{A}]\!]) = \frac{1}{2}$. Then $\sigma \models \triangle_{\frac{1}{4}}\mathsf{B}$; $\sigma \models \triangle_1\neg(A \to B)$; and $\sigma \models \triangle_1(\mathsf{B} \vee \neg(A \to B))$.
4. COMPARATIVE ENTAILMENT: Suppose that A and B are mutually exclusive atomic formula. $\mathsf{A} \vee \mathsf{B} \models \neg\mathsf{A} \to \mathsf{B}$. Suppose that $Pr_\sigma([\![\mathsf{A}]\!]) = \frac{1}{3}$, and that $Pr_\sigma([\![\mathsf{B}]\!]) = \frac{1}{4}$. In this case, $\sigma \models \triangle_{\frac{7}{12}}(\mathsf{A} \vee \mathsf{B})$, since $Pr_\sigma(s_{\sigma[\mathsf{A}\vee\mathsf{B}]}) = Pr_\sigma([\![\mathsf{A}]\!] \cup [\![\mathsf{B}]\!]) = \frac{7}{12}$. But $\sigma \models \triangle_0(\neg\mathsf{A} \to \mathsf{B})$, since $Pr_\sigma(s_{\sigma[\neg\mathsf{A}\to\mathsf{B}]}) = Pr_\sigma(s_\bot) = 0$.

□

Appendix 5: Stalnaker's Thesis

In this appendix, I will review recent work Starr Forthcoming that uses a dynamic semantics for $\to$ and $\triangle_n$ to preserve a version of Stalnaker's Thesis. We will see that it runs into similar impossibility results to the semantics above.

The nonfactualist semantics for $\to$ of interest was proposed in Gillies 2004, and extended in Starr Forthcoming. On this proposal, the conditional $\phi \to \psi$ again tests a context σ to see whether updating σ with ϕ supports ψ. Starr Forthcoming adds the constraint that $\phi \to \psi$ presupposes that ϕ is possible, in that some world in the context supports ϕ. Note that any world w is itself a probability function—one that assigns each claim a probability of 1 or 0 depending on whether it is true at the world. Thus for any world w we can consider the context $\langle w, w\rangle$ to be a maximally informative context. Then for a claim ϕ to be possible in the context σ in the sense of Starr Forthcoming is for there to be some world in s_σ so that $\langle w, w\rangle$ supports ϕ:

Definition 6.7. $\sigma[\phi \to \psi] = \begin{cases} \sigma & \text{if } \exists w \in s_\sigma : \langle w, w\rangle \models \phi \ \&\ \sigma[\phi] \models \psi; \\ \langle \emptyset, Pr_\sigma(\cdot \mid \emptyset)\rangle & \text{if } \exists w \in s_\sigma : \langle w, w\rangle \models \phi \ \&\ \sigma[\phi] \not\models \psi; \\ \text{undefined} & \text{otherwise} \end{cases}$

This semantics for $\to$ can then be combined with a semantics for $\triangle_n$. On this semantics, $\triangle_n\phi$ only quantifies over possible worlds where ϕ is defined. In particular, it checks whether the set of worlds that support ϕ is assigned a probability of n once we zoom into the set of worlds where updating with ϕ is defined.[29]

Definition 6.8. $\sigma[\triangle_n\phi] = \begin{cases} \sigma & \text{if } Pr_\sigma(\{w \mid \langle w, w\rangle \models \phi\} \mid \{w \mid \langle w, w\rangle \models \phi \text{ or } \langle w, w\rangle \models \neg\phi\}) = n \\ \langle \emptyset, Pr_\sigma(\cdot \mid \emptyset)\rangle & \text{otherwise} \end{cases}$

[29]Here I've extrapolated from Starr Forthcoming's semantics for $\triangle$.

Finally, in this framework entailment can be defined as preservation of support among contexts in which the premises and conclusions are defined:

Definition 6.9. ϕ entails ψ ($\phi \models \psi$) iff for every σ such that $\sigma[\phi]$ and $\sigma[\psi]$ are defined, if $\sigma \models \phi$ then $\sigma \models \psi$.

This semantics validates embedded versions of Stalnaker's Thesis. On this proposal it is n likely that $\phi \to \psi$ just in case it is n likely that ψ given ϕ, and every agent's beliefs conform to this principle:

Definition 6.10 (Embedded Stalnaker's Thesis)**.** For any χ, ϕ, ψ:
$\triangle_n(\phi \to \psi \mid \chi) \dashv\models \triangle_n(\psi \mid \chi \wedge \phi)$

Definition 6.11 (Stalnaker's Probabilistic Belief Thesis)**.** For any χ, ϕ, ψ:
$B_S\triangle_n(\phi \to \psi \mid \chi) \dashv\models B_S\triangle_n(\psi \mid \chi \wedge \phi)$

Fact 6.3. Embedded Stalnaker's Thesis and Stalnaker's Probabilistic Belief Thesis are both valid.

As above, it will sometimes be useful to consider a simplified form of the above theses, where we do not conditionalize on χ. This is an innocent assumption, since:

Fact 6.4. If $\triangle_n$ is proper, then: Embedded and Simplified Stalnaker's Thesis are equivalent.

Proof. Suppose $\triangle_n$ is proper. The left to right direction is trivial, letting $\chi = \top$. For the right to left, suppose Simplified Stalnaker's Thesis holds. Now take an arbitrary σ, ϕ, ψ, χ. We must show that $\sigma \models \triangle_n(\phi \to \psi \mid \chi)$ iff $\sigma \models \triangle_n(\psi \mid \chi \wedge \phi)$. From Simplified Stalnaker's Thesis, we know that $\sigma[\chi] \models \triangle_n(\phi \to \psi)$ iff $\sigma[\chi] \models \triangle_n(\psi \mid \phi)$. By the propriety of $\triangle_n$, this entails that $\sigma \models \triangle_n(\phi \to \psi \mid \chi)$ iff $\sigma \models \triangle_n(\psi \mid \chi \wedge \phi)$. □

Unfortunately, the semantics above faces analogous impossibility results to those earlier in the paper. If the above principles are valid, then $\triangle_n$ and $B_A\triangle_n$ do not have probabilistic structure. For it turns out that Embedded Stalnaker's Thesis and the Law of Total Probability again lead to triviality:

Fact 6.5. Embedded Stalnaker's Thesis and the Law of Total Probability jointly imply: $\triangle_n(\psi \mid \phi) \dashv\models \triangle_n\psi$.

Proof. The proof is analogous to that in Lewis 1976.

1. $\triangle_n(\phi \to \psi) \dashv\models \triangle_m\psi \wedge \triangle_{m'}(\phi \to \psi \mid \psi) \wedge \triangle_l\neg\psi \wedge \triangle_{l'}(\phi \to \psi \mid \neg\psi) \wedge n = m \times m' + l \times l'$ LTP
2. $\models \triangle_1(\psi \mid \psi \wedge \phi) \wedge \triangle_0(\psi \mid \neg\psi \wedge \phi)$
3. $\models \triangle_1(\phi \to \psi \mid \psi) \wedge \triangle_0(\phi \to \psi \mid \neg\psi)$ 2, Embedded Stalnaker's Thesis
4. $\triangle_n(\phi \to \psi) \dashv\models \triangle_m\psi \wedge \triangle_1(\phi \to \psi \mid \psi) \wedge \triangle_l\neg\psi \wedge \triangle_0(\phi \to \psi \mid \neg\psi) \wedge n = m \times 1 + l \times 0$ 1, 3
5. $\triangle_n(\phi \to \psi) \dashv\models \triangle_n\psi$ 4
6. $\models \triangle_1\phi \vee \triangle_0\phi$ 5, Embedded Stalnaker's Thesis

□

That is, once we accept that $\triangle_n$ obeys a version of Stalnaker's Thesis and the Law of Total Probability, we will be forced to also accept that $\triangle_n$ makes any two claims probabilistically independent. Similarly for Probabilistic Belief:

Fact 6.6. Probabilistic Belief Stalnaker's Thesis and the Law of Total Probability for Probabilistic Belief jointly imply: $B_S\triangle_n(\psi \mid \phi) \dashv\models B_S\triangle_n\psi$.

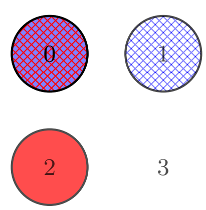
Like the semantics discussed in the paper, the semantics in Starr Forthcoming responds to these problems by giving up both Additivity and Comparative Entailment:

Fact 6.7.

1. $\models \triangle_{\geq 0}\phi$ NON-NEGATIVITY
2. $\models \triangle_1 \top$ UPPER BOUND
3. $\phi; \psi \models \bot \not\Longrightarrow \triangle_n(\phi \vee \psi) \dashv\models \exists m_1, m_2 : \triangle_{m_1}\phi \wedge \triangle_{m_2}\psi \wedge n = m_1 + m_2$ ~~ADDITIVITY~~
4. $\phi \models \psi \not\Longrightarrow \triangle_n\phi \models \triangle_{\geq n}\psi$ ~~COMPARATIVE ENTAILMENT~~

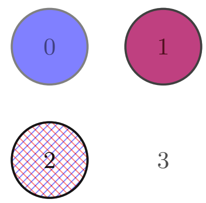
Proof.

1. NON-NEGATIVITY: Follows from Pr being a probability function, plus $\sigma \models \triangle_n\phi$ iff $Pr_\sigma(\{w|\ \langle w,w\rangle \models \phi\}|\{w|\langle w,w\rangle \models \phi$ or $\langle w,w\rangle \models \neg\phi\}) = n$.
2. UPPER BOUND: $\top \models \phi$ iff $\forall\sigma\ \sigma[\phi] = \sigma$ or $\sigma[\phi]$ is undefined only if $\forall w\ \langle w,w\rangle[\phi] = \{w\}$ or $\langle w,w\rangle[\phi]$ is undefined iff $\models \triangle_1\phi$.
3. ADDITIVITY: Consider the following model:



Let B be true in 0, false in 1, and undefined everywhere else (for example, $\mathsf{B} := (0 \vee 1) \rightarrow 0)$. Let R be true in 2, false in 0, and undefined everywhere else (for example, $\mathsf{R} := ((0 \vee 2) \rightarrow 2)$. Let b abbreviate $\neg\mathsf{B}$ and r abbreviate $\neg\mathsf{R}$. Suppose that $Pr(0) = \frac{1}{6}$, $Pr(2) = \frac{1}{3}$, $Pr(1) = \frac{1}{3}$, and $Pr(3) = \frac{1}{6}$.
$\triangle_n\mathsf{B}$ iff $n = Pr(\{w \in s|\ \langle w,w\rangle \models \mathsf{B}\} \mid \{w \in s_\sigma|\langle w,w\rangle \models \mathsf{B}$ or $\langle w,w\rangle \models \mathsf{b}\}) = Pr(\{0\} \mid \{0,1\}) = \frac{1}{3}$. $\triangle_n\mathsf{R}$ iff $n = Pr(\{w \in s|\ \langle w,w\rangle \models \mathsf{R}\} \mid \{w \in s_\sigma|\langle w,w\rangle \models \mathsf{R}$ or $\langle w,w\rangle \models \mathsf{r}\}) = Pr(\{0\} \mid \{0,2\}) = \frac{1}{3}$. $\triangle^{\#}(\mathsf{B} \vee \mathsf{R}) = Pr(\{w \in s|\ \langle w,w\rangle \models \mathsf{B} \vee \mathsf{R}\} \mid \{w \in s_\sigma|\langle w,w\rangle \models \mathsf{B} \vee \mathsf{R}$ or $\langle w,w\rangle \not\models \mathsf{B} \vee \mathsf{R}\}) = Pr(\{0\} \mid \{0\}) = 1$.
Two key assumptions: (i) $\phi \vee \psi$ is undefined whenever ϕ or ψ is; thus 1 and 2 are undefined for $\mathsf{B} \vee \mathsf{R}$. (ii) B and R are inconsistent since they are never true in the same place - even though they are sometimes undefined in the same place.

4. COMPARATIVE ENTAILMENT: consider the following model:



Let B be true in 0 and 1, false in 2, and undefined everywhere else (for example, $\mathsf{B} := (0 \vee 1 \vee 2) \rightarrow (0 \vee 1)$). Let R be true in 1, false in 2, and undefined everywhere else (for example, $\mathsf{R} := (1 \vee 2) \rightarrow 1$). Let b abbreviate $\neg\mathsf{B}$ and r abbreviate $\neg\mathsf{R}$. Suppose that $Pr(0) = Pr(1) = Pr(2) = Pr(3) = \frac{1}{4}$.

$\triangle_n\mathsf{B}$ iff $n = Pr(\{w \in s | \langle w, w\rangle \models \mathsf{B}\} \mid \{w \in s_\sigma | \langle w, w\rangle \models \mathsf{B}$ or $\langle w, w\rangle \models \mathsf{b}\}) = Pr(\{0, 1\} \mid \{0, 1, 2\}) = \frac{2}{3}$. $\triangle_n(\mathsf{B} \vee \mathsf{R})$ iff $n = Pr(\{w \in s | \langle w, w\rangle \models \mathsf{B} \vee \mathsf{R}\} \mid \{w \in s_\sigma | \langle w, w\rangle \models \mathsf{B} \vee \mathsf{R}$ or $\langle w, w\rangle \not\models \mathsf{B} \vee \mathsf{R}\}) = Pr(\{1\} \mid \{1, 2\}) = \frac{1}{2}$. So $\triangle_n\mathsf{B} \wedge \triangle_m(\mathsf{B} \vee \mathsf{R}) \wedge n > m$ But $\mathsf{B} \models \mathsf{B} \vee \mathsf{R}$.

□

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